Canad. Math. Bull. Vol. 63 (2), 2020 pp. 393–404
http://dx.doi.org/10.4153/S0008439519000493

Dense Orderings in the Space of Left-orderings of a Group

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Abstract. Every left-invariant ordering of a group is either discrete, meaning there is a least element greater than the identity, or dense. Corresponding to this dichotomy, the spaces of left, Conradian, and bi-orderings of a group are naturally partitioned into two subsets. This note investigates the structure of this partition, specifically the set of dense orderings of a group and its closure within the space of orderings. We show that for bi-orderable groups, this closure will always contain the space of Conradian orderings—and often much more. In particular, the closure of the set of dense orderings of the free group is the entire space of left-orderings.

1 Introduction

A group G is *left-orderable* if there is a strict total ordering $<$ of its elements such that $g < h$ implies $fg < fh$ for all $f, g, h \in G$. Stronger than the notion of left-orderability is Conradian left-orderability: a left-ordering of a group G is said to be *Conradian* if for every pair of elements $g, h \in G$ with $1 < g, h$ there exists $n > 0$ such that $1 < g^{-1}hg^n$. This turns out to be equivalent to requiring that $1 < g^{-1}hg^2$ for all such pairs of elements [9]. Stronger still is the requirement that G admit a left-ordering such that $g < h$ implies $gf < hf$ for all $f, g, h \in G$, in which case $<$ is called a *bi-ordering* and G is called *bi-orderable*. It is straightforward to see that every bi-ordering is a Conradian left-ordering. Given a left-ordering $<$ of G (resp. a Conradian ordering or bi-ordering), the pair $(G, <)$ will be called a *left-ordered group* (resp. *Conradian ordered* or *bi-ordered*).

Every left-ordering of G can be uniquely identified with its *positive cone* $P = \{g \in G \mid g > 1\}$, which is a subset of G satisfying

(1) $P \sqcup P^{-1} \sqcup \{1\} = G$,

(2) $P \cdot P \subset P$.

Conversely, every subset of G satisfying the two properties above determines a left-ordering via the prescription $g < h$ if and only if $g^{-1}h \in P$. A positive cone P is the positive cone of a Conradian left-ordering if, in addition to the two properties above, it satisfies

(3a) If $g, h \in P$, then $g^{-1}hg^2 \in P$.

A positive cone P of a left-ordering is the positive cone of a bi-ordering if it satisfies

(3b) $gPg^{-1} \subset P$ for all $g \in G$.

Received by the editors April 22, 2019; revised July 24, 2019.
Published online on Cambridge Core December 17, 2019.
Adam Clay was partially supported by NSERC grant RGPIN-2014-05465.
AMS subject classification: **06F15**.
Keywords: ordered group, space of left-orderings, bi-orderable group.

For a fixed group G, if we denote the collections of all positive cones of left-orderings, Conradian orderings and bi-orderings of G by $\mathrm{LO}(G)$, $\mathrm{CO}(G)$, and $\mathrm{BO}(G)$, respectively, then we have $\mathrm{BO}(G) \subset \mathrm{CO}(G) \subset \mathrm{LO}(G)$. Each of these sets can be topologized so as to become a totally disconnected compact Hausdorff space, as follows.

Let $\mathcal{P}(G)$ denote the power set of G and observe that $\mathrm{LO}(G) \subset \mathcal{P}(G)$. The power set can be identified with $\{0,1\}^G$, and thus can be equipped with the product topology. This makes $\mathcal{P}(G)$ into a totally disconnected Hausdorff space, which is compact by Tychonoff's theorem. One checks that properties (1) and (2) above define a closed subset of $\mathcal{P}(G)$ (similarly for (3a) and (3b)), so that $\mathrm{LO}(G) \subset \mathcal{P}(G)$ is closed, and hence compact, when equipped with the subspace topology. See the beginning of Section 2 for a description of a subbasis for the topology on $\mathrm{LO}(G)$. Similarly, each of $\mathrm{CO}(G)$ and $\mathrm{BO}(G)$ are closed and hence compact.

We call a left-ordering $<$ of a group G *discrete* if every element in $(G,<)$ has an immediate predecessor and successor, which is equivalent to its positive cone $P = \{g \in G \mid g > 1\}$ having a smallest element. A left-ordering of a group G that is not discrete is *dense*, in the sense that whenever $g, h \in G$ satisfy $g < h$ there exists $f \in G$ with $g < f < h$. Equivalently, the positive cone of the ordering does not have a least element. Throughout this note, the set of positive cones of dense left-orderings of the group G will be denoted $\mathrm{D}(G)$.

Thus, each of the spaces $\mathrm{LO}(G)$, $\mathrm{CO}(G)$, and $\mathrm{BO}(G)$ admits a decomposition into two subsets: the set of dense orderings and the set of discrete orderings. Our work investigates how the nesting $\mathrm{BO}(G) \subset \mathrm{CO}(G) \subset \mathrm{LO}(G)$ behaves with regards to this dichotomy. We show the following theorem.

Theorem 1.1 *If G is a bi-orderable group that is not isomorphic to the integers, then* $\mathrm{CO}(G) \subset \overline{\mathrm{D}(G)}$.

In fact, we show something much stronger, which proves that (in many situations) this containment is proper; see Theorem 3.7 and the subsequent examples. When G is nilpotent, it is known that $\mathrm{LO}(G) = \mathrm{CO}(G)$, and so this yields the following corollary.

Corollary 1.2 *Suppose G is a torsion-free nilpotent group that is not isomorphic to the integers. Then* $\mathrm{LO}(G) = \overline{\mathrm{D}(G)}$.

Leveraging the full strength of Theorem 3.7 also allows for an analysis if the space of orderings of a free group. We show that every left-ordering of a nonabelian free group is an accumulation point of orderings whose Conradian souls[1] are nontrivial, noncyclic subgroups. From this we conclude the following theorem.

Theorem 1.3 *Suppose that F is a free group having $n \geq 2$ generators or countably infinitely many generators. Then* $\mathrm{LO}(F) = \overline{\mathrm{D}(F)}$.

Our motivation behind these considerations is as follows. The spaces $\mathrm{LO}(G)$, $\mathrm{CO}(G)$, and $\mathrm{BO}(G)$ are all totally disconnected, compact Hausdorff spaces—in fact,

[1]See the discussion preceding Theorem 3.7 for an explanation of the Conradian soul of an ordering.

they are metrizable when G is countable. Therefore, when G is countable, each space is homeomorphic to the Cantor set if and only if it is perfect. As a result there has been a considerable amount of effort in the literature devoted to identifying isolated points and accumulation points in $\mathrm{LO}(G)$ (*e.g.*, [7,12]).

This effort can be viewed as an initial step towards a more general problem. Recall that if X is a topological space, X' denotes the set of all accumulation points of X. Set $X^{(0)} = X$, and for each ordinal number α define $X^{(\alpha+1)} = (X^{(\alpha)})'$ and $X^{(\lambda)} = \bigcap_{\alpha<\lambda} X^{(\alpha)}$ if λ is a limit ordinal. Define the *Cantor–Bendixson rank* of X to be the smallest ordinal α such that $X^{(\alpha+1)} = X^{(\alpha)}$; such an α always exists for cardinality reasons. These notions were used to great success, for example, in showing that every group admits either finitely many or uncountably many left-orderings [8].

Viewed through the lens of Cantor–Bendixson ranks and derived subsets, the question of whether or not $\mathrm{LO}(G)$ admits any isolated points becomes a question of whether or not the Cantor–Bendixson rank of $\mathrm{LO}(G)$ is larger than zero. For example, the spaces of orderings of the braid groups and of various free products with amalgamation admit isolated points and so have Cantor–Bendixson rank larger than zero [6, 7]; on the other hand, the spaces of orderings of free groups, free products with amalgamation, and torsion-free abelian groups admit no isolated points and thus have Cantor–Bendixson rank zero [12, 13]. For countable groups, there is a well-known upper bound on the Cantor–Bendixson rank: since $\mathrm{LO}(G)$ is Polish its Cantor–Bendixson rank is at most a countable ordinal, by the Cantor–Bendixson theorem.

These matters are connected to the notions of discrete and dense orderings as follows. One of the main results of [2] is that under mild hypotheses[2] on the group G, we have $\mathrm{D}(G) \subset \mathrm{D}(G)'$. From this, we conclude $\mathrm{D}(G) \subset \mathrm{LO}(G)^{(\alpha)}$ for all α, and thus $\overline{\mathrm{D}(G)} \subset \mathrm{LO}(G)^{(\alpha)}$ for all α. Our study of $\overline{\mathrm{D}(G)}$, therefore, is an attempt to understand the structure of the sets $\mathrm{LO}(G)^{(\alpha)}$ for large α and ultimately determine the Cantor–Bendixson rank of $\mathrm{LO}(G)$ for G in some nontrivial class of groups. Specifically, the question motivating our work is the following.

Question 1.4 Let G be a bi-orderable group. Can G admit a non-isolated point $P \in \mathrm{LO}(G)$ with $P \notin \overline{\mathrm{D}(G)}$?

If the answer to this question is "no", it would follow that whenever G is bi-orderable, the Cantor–Bendixson rank of $\mathrm{LO}(G)$ must be either 1 or 0.

1.1 Organization

We organize our arguments as follows. In Section 2, we prepare some preliminary results concerning torsion-free abelian groups and the distribution of dense and discrete orderings in their spaces of orderings. In Section 3, we apply these results in the study of bi-orderable groups, and discuss several illustrative examples. Section 4 deals with the case of free groups.

[2]Namely that every rank one abelian subgroup of G be isomorphic to the integers.

https://doi.org/10.4153/S0008439519000493 Published online by Cambridge University Press

2 Discrete Orderings of Abelian Groups

When A is a torsion-free abelian group, it is known that $\mathrm{LO}(A)$ has no isolated points unless A is rank one abelian. When A is not rank one abelian but is finitely generated, the set of dense orderings $\mathrm{D}(A) \subset \mathrm{LO}(A)$ is fairly well understood.

Theorem 2.1 ([2, Proposition 4.3]) *Suppose that A is an abelian group. Then $\overline{\mathrm{D}(A)} = \mathrm{LO}(A)$ if and only if A is not isomorphic to the integers.*

Question 4.6 of [2] then asks the natural question: What can be said of the set of discrete orderings in $\mathrm{LO}(A)$? We give a partial answer below by mirroring the proof of [2, Proposition 4.3]. We will need this result (specifically Corollary 2.3) for later.

Our main tool in the proof that follows, which we use repeatedly below and elsewhere in this note, is the procedure of "changing an ordering on a convex subgroup". For an ordered group $(G, <)$, a subgroup $C \subset G$ is called *convex relative to* $<$ if whenever $f \in G$ and $g, h \in C$, the inequalities $g < f < h$ imply $f \in C$. In this case, if P is the positive cone of the ordering $<$, then one can check that $P' = P \setminus (P \cap C) \cup Q$ is the positive cone of a left-ordering of G for every $Q \in \mathrm{LO}(C)$. That is, one can replace the portion of P that lies in C with any other positive cone in C. Note also that an ordering $<$ of a group G is discrete with smallest positive element g if and only if $\langle g \rangle$ is a convex subgroup; a property which we also use often in this paper.

We also recall that if X is either $\mathrm{LO}(G)$, $\mathrm{CO}(G)$ or $\mathrm{BO}(G)$, a subbasis for the topology on X is given by the family of sets $U_g = \{P \in X \mid g \in P\}$, where g ranges over all nonidentity elements of G.

Proposition 2.2 *Suppose that $k \geq 2$ and that $E \subset \mathrm{LO}(\mathbb{Z}^k)$ is the set of discrete left-orderings. Then $\overline{E} = \mathrm{LO}(\mathbb{Z}^k)$.*

Proof For contradiction, suppose $k > 1$ is the smallest k for which the claim fails, and choose a nonempty basic open set $\bigcap_{i=1}^{n} U_{g_i}$ in $\mathrm{LO}(\mathbb{Z}^k)$, say it contains the positive cone P (here $g_i \in \mathbb{Z}^k$ for $i = 1, \dots, n$). Note that we may assume that none of the g_i's are scalar multiples of one another. Suppose this basic open set contains no discrete orderings.

Extend the ordering $<$ defined by P to an ordering of $\mathbb{Q}^k$ by declaring $v_1 < v_2$ for $v_1, v_2 \in \mathbb{Q}^k$ if $mv_1 < mv_2$ whenever $mv_1, mv_2 \in \mathbb{Z}^k$. Let $H \subset \mathbb{R}^k$ be the subset of elements $x \in \mathbb{R}^k$ where every Euclidean neighbourhood of x contains both positive and negative elements. One can check that H is a hyperplane that divides $\mathbb{R}^k$ into two components H_- and H_+, where H_- contains only negative elements of $\mathbb{Q}^k$ and H_+ contains only positive elements of $\mathbb{Q}^k$. Thus, the elements of $\{g_1, \dots, g_n\}$ must lie in either H_+ or H. There are three cases to consider.

Case 1. Two or more elements of $\{g_1, \dots, g_n\}$ lie on H. In this case, $H \cap \mathbb{Z}^k = \mathbb{Z}^m$ for some $1 < m < k$. By assumption, the positive cone $P_H = P \cap (H \cap \mathbb{Z}^k) \subset \mathbb{Z}^m$ is an accumulation point of discrete orderings. Enumerate the g_i's so that $g_i \in P_H$ for $i \leq r$. There exists a positive cone $P'_H \in \bigcap_{i=1}^{r} U_{g_i}$ corresponding to a discrete ordering. Note that relative to the ordering defined by P, the subgroup $H \cap \mathbb{Z}^k$ is a convex subgroup

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of $\mathbb{Z}^k$. Thus, $P' = (P \setminus P_H) \cup P'_H$ defines a new positive cone on $\mathbb{Z}^k$, which is the positive cone of a discrete ordering since P'_H defines a discrete ordering on $H \cap \mathbb{Z}^k$. By construction, $P' \in \bigcap_{i=1}^n U_{g_i}$, a contradiction.

Case 2. Exactly one of the g_i's, say g_1, lies in H. In this case, P itself defines a discrete ordering of $\mathbb{Z}^k$, since $P \cap H = \langle g_1 \rangle$ is a convex subgroup of the ordering of $\mathbb{Z}^k$. In particular, $\bigcap_{i=1}^n U_{g_i}$ contains a discrete ordering, a contradiction.

Case 3. None of the g_i's are contained in H. Suppose H has normal vector $\vec{v} = (v_1, \dots, v_k)$. Let $\epsilon > 0$ and choose $\vec{w} = (w_1, \dots, w_k) \in \mathbb{Q}^k$ with $\|\vec{v} - \vec{w}\| < \epsilon$. Then choose $(y_1, \dots, y_k) \in \mathbb{Z}^k$ such that $y_i w_i \in \mathbb{Z}$ for each $i = 1, \dots, k$. Choose $j \in \{1, \dots, k-1\}$ and let

$$m_1 = \sum_{i=1}^{j} y_i w_i \quad \text{and} \quad m_2 = \sum_{i=j+1}^{k} y_i w_i.$$

Then

$$\vec{x} = (m_2 y_1, \dots, m_2 y_j, -m_1 y_{j+1}, \dots, -m_1 y_k) \in \mathbb{Z}^k$$

satisfies $\vec{w} \cdot \vec{x} = 0$. Thus, the hyperplane H' with normal vector $\vec{w}$ satisfies $H' \cap \mathbb{Z}^k = \mathbb{Z}^m$ for some $m > 0$, and since we can choose $\epsilon > 0$ as small as we please, we may suppose that the g_i's all lie to one side of H'. By equipping $\mathbb{Z}^m$ with a discrete ordering, we can lexicographically define a discrete ordering P' on $\mathbb{Z}^k$ with each g_i positive. ∎

Recall that if G is a group with subgroup H, then the *isolator* of H in G is

$$I_G(H) = \{g \in G \mid \exists k \in \mathbb{Z} \text{ such that } g^k \in H\}.$$

A subgroup H of G is called *isolated* in G if $I_G(H) = H$. In general, $I_G(H)$ is a subset of G properly containing H that is not a subgroup unless additional hypotheses are imposed on the group G. For instance, if G is abelian (or even nilpotent, see [11]), then $I_G(H)$ is a subgroup; see also Lemma 3.2.

Corollary 2.3 *Suppose that A is a torsion-free abelian group, and for each $P \in \mathrm{LO}(A)$, let C_P denote the smallest nontrivial convex subgroup of the ordering corresponding to P. Then the set*

$$\{P \in \mathrm{LO}(A) \mid C_P \text{ is rank one abelian}\}$$

is dense in $\mathrm{LO}(A)$.

Proof Let $P \in \mathrm{LO}(A)$ be given, and suppose $\bigcap_{i=1}^n U_{a_i}$ is a basic open neighbourhood of P. Let $H = \langle a_1, \dots, a_n \rangle$, and let $Q' = P \cap H$. By Theorem 2.1, there exists a positive cone $Q \subset H$ with $\{a_1, \dots, a_n\} \subset Q$ with $Q \neq Q'$ that corresponds to a discrete ordering of H, say with smallest positive element $h \in Q$.

Observe that the positive cone Q extends uniquely to a positive cone $\overline{Q}$ of the subgroup $I_A(H)$, by declaring that $a \in \overline{Q}$ if and only if there exists $k > 0$ such that $a^k \in Q$. One can check that if $C \subset H$ is a convex subgroup of the ordering induced by Q, then $I_A(C) \subset I_A(H)$ is a convex subgroup of the ordering induced by $\overline{Q}$. Thus, $I_A(\langle h \rangle)$ becomes the smallest nontrivial convex subgroup of $I_A(H)$ relative to the ordering induced by $\overline{Q}$.

Now using the short exact sequence

$$1 \longrightarrow I_A(H) \xrightarrow{i} A \xrightarrow{q} A/I_A(H) \longrightarrow 1$$

one can equip $A/I_A(H)$ with an arbitrary positive cone R and set $P' = i(\overline{Q}) \cup q^{-1}(R)$. By construction, $P' \in \bigcap_{i=1}^{n} U_{a_i} \subset \mathrm{LO}(A)$ and the smallest nontrivial convex subgroup of the corresponding ordering is $I_A(\langle h \rangle)$, which is rank one abelian. ■

3 Dense Orderings of Bi-orderable Groups

In this section, we use the property of bi-orderability of G to give sufficient flexibility in the construction of left-orderings of G that we can approximate any Conradian ordering by dense orderings. In what follows, we will use K to denote the Klein bottle group $\langle x, y \mid xyx^{-1} = y^{-1} \rangle$.

Lemma 3.1 *Suppose that G is a group that does not contain a copy of the Klein bottle group, and that P is the positive cone of a discrete Conradian ordering of G. Suppose that h is the least element of P, that h generates the proper normal cyclic subgroup $H \cong \mathbb{Z}$, and that G/H is abelian. Then $P \in \mathrm{LO}(G)$ is an accumulation point of dense orderings.*

Proof First, observe that for every $g \in G$, we have $[g, h] = 1$. To see this, note that since $H = \langle h \rangle$ is normal in G, every element $g \in G$ satisfies $ghg^{-1} = h^{\pm 1}$. In particular, if $ghg^{-1} = h^{-1}$, one can check that G would contain K, which we assume is not possible. From this, it follows that if G/H were rank one abelian, then G itself would be abelian, as H is cyclic. Thus, G is a torsion-free rank two abelian group, and so the result follows from Theorem 2.1.

On the other hand, suppose G/H is torsion free abelian of rank larger than two. Let $g_1, \dots, g_n$ be finitely many elements of P. We will produce a positive cone Q corresponding to a dense ordering that contains $g_1, \dots, g_n$.

Since there is a short exact sequence with H convex, we have

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{q} G/H \longrightarrow 1,$$

and P is constructed, as $P = \{h^k\}_{k>0} \cup q^{-1}(P')$ for some positive cone $P' \subset G/H$. Suppose that $g_1, \dots, g_n$ are enumerated so that $g_1, \dots, g_r$ are powers of h and $g_{r+1}, \dots, g_n$ lie in $q^{-1}(P')$, meaning $q(g_i) \in P'$ for $r < i \le n$.

By Proposition 2.3, we can choose a positive cone Q' of G/H containing $q(g_i)$ for $r < i \le n$ that produces an ordering with rank one abelian convex subgroup $C \subset G/H$. The subgroup $q^{-1}(C)$ is abelian of rank two and convex in the ordering whose positive cone is $R = \{h^k\}_{k>0} \cup q^{-1}(Q')$. The positive cone $R \cap q^{-1}(C)$ contains the elements $g_1, \dots, g_s$ for some $s \ge r$. By Theorem 2.1, there is a cone $R' \subset q^{-1}(C)$ containing $g_1, \dots, g_s$ that is different from R, and which defines a dense ordering of $q^{-1}(C)$. Now set $S = R' \cup q^{-1}(Q' \setminus (C \cap Q'))$, which is the positive cone of a dense ordering of G that contains $g_1, \dots, g_n$ by construction. ■

We need two lemmas concerning isolators of abelian subgroups before moving on to our main theorem.

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Lemma 3.2 *Suppose that G is a bi-orderable group and A is an abelian subgroup. Then $I_G(A)$ is an abelian subgroup.*

Proof First, observe that all elements of $I_G(A)$ commute, because if $[g^n, h^m] = 1$ for some $g, h \in G$, then $[g, h] = 1$ by bi-orderability. It then follows that $I_G(A)$ is a subgroup, since $g^k = a \in A$ and $h^\ell = b \in A$ implies $(gh)^{k\ell} = a^\ell b^k \in A$, and closure under taking inverses is obvious. ∎

Lemma 3.3 ([3, Lemma 3.2]) *Suppose that A is an isolated abelian subgroup of a bi-orderable group G. Then A is relatively convex in G.*

Proposition 3.4 *Every bi-orderable group that is not isomorphic to the integers admits a dense left-ordering.*

Proof Let $(G, <)$ be a bi-ordered group. If $<$ is dense, we are done. Otherwise, let $g \in G$ be the least positive element of $<$, and observe that g is central. Since $1 < g$, we know that $1 < hgh^{-1}$ for all $h \in G$. If h does not commute with g, this forces $g < hgh^{-1}$, since g is the least positive element. But then conjugation yields $h^{-1}gh < g$, a contradiction.

Thus, g is central, and since G is not infinite cyclic, there exists $h \in G$ that is not a power of g. Then $\langle g, h\rangle$ is a rank two abelian subgroup of G, by Lemma 3.2, $I_G(\langle g, h\rangle)$ is an isolated abelian subgroup; one can check it also has rank two. By Lemma 3.3, $I_G(\langle g, h\rangle)$ is relatively convex. Every rank two abelian group admits a dense ordering, so we are done. ∎

Note that bi-orderability is essential in the previous proposition. The finitely generated Tararin groups

$$T_n = \langle x_1, \dots, x_n \mid x_i x_{i-1} x_i^{-1} = x_{i-1}^{-1} \text{ for } i = 2, \dots n\rangle$$

satisfy $|\mathrm{LO}(T_n)| = 2^n$, and all the orderings are discrete. It is also possible to construct groups having uncountably many orderings, all of them discrete, such as the so-called infinite Tararin group $\langle x_i, i \in \mathbb{N} \mid x_i x_{i-1} x_i^{-1} = x_{i-1}^{-1} \text{ for } i \in \mathbb{N}_{>1}\rangle$. None of these groups are bi-orderable, as each contains an element that is conjugate to its own inverse.

Proposition 3.5 *Suppose that G is a bi-orderable group that is not isomorphic to the integers and that P is the positive cone of a Conradian ordering of G. Then $P \in \overline{\mathrm{D}(G)}$.*

Proof Suppose that P is the positive cone of a discrete Conradian ordering, and that $P \in \bigcap_{i=1}^n U_{g_i}$. Let $h > 1$ denote the least element of P.

First, suppose that there exists a convex subgroup C such that $(\langle h\rangle, C)$ is a convex jump. Assume that $g_1, \dots, g_n$ are enumerated so that $g_1, \dots, g_r \in C$ and $g_{r+1}, \dots, g_n \notin C$. By Lemma 3.1, there exists a positive cone $Q \in \mathrm{LO}(C)$ such that $Q \neq P \cap C$ and $g_1, \dots, g_r \in Q$. Then $P' = (P \setminus (P \cap C)) \cup Q$ contains $g_1, \dots, g_n$, is different from P and is the positive cone of a dense ordering.

On the other hand, suppose that there is no convex subgroup C such that $(\langle h\rangle, C)$ is a convex jump. Suppose further that $g_1, \dots, g_n$ are enumerated so that $g_1, \dots, g_r$ are powers of h and $g_{r+1}, \dots, g_n$ are not; suppose also that g_{r+1} is the smallest element

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that is not in $\langle h\rangle$. Then g_{r+1} determines a convex jump (C, D); note that $g_j \notin C$ for all $j > r$ and that the containment $\langle h\rangle \subset C$ is proper. To complete the proof, it suffices to observe that C can be equipped with a dense ordering by Proposition 3.4. Thus, we can choose a positive cone $Q \subset C$ with $h \in Q$ and set $P' = (P \smallsetminus (P \cap C)) \cup Q$ as before. ∎

Corollary 3.6 *If G is a torsion-free nilpotent group that is not isomorphic to the integers, then* $\mathrm{LO}(G) = \overline{\mathrm{D}(G)}$.

Proof Every torsion-free nilpotent group is bi-orderable, and all left-orderings of every torsion-free nilpotent group are Conradian [1]. ∎

We can extend the previous proposition so that it applies to certain orderings of non-bi-orderable groups. Indeed, as Example 3.8 shows, the group G need not even be locally indicable for our generalized result to apply. For the statement of our theorem below, recall that the *Conradian soul* of an ordering $<$ of a group G is the largest convex subgroup $C \subset G$ such that the restriction of $<$ to C is Conradian.

Theorem 3.7 *Suppose that P is the positive cone of a left-ordering of a bi-orderable group G. If the Conradian soul of the ordering corresponding to P is bi-orderable, non-trivial, and not isomorphic to* $\mathbb{Z}$*, then* $P \in \overline{\mathrm{D}(G)}$.

Proof With P and G as in the statement of the theorem, suppose that $P \in \bigcap_{i=1}^{n} U_{g_i}$ where $g_1, \dots, g_n \in G$. Let C denote the Conradian soul of the ordering corresponding to P, and suppose that the g_i's are enumerated so that $g_1, \dots, g_r \in C$ and $g_i \notin C$ for $i = r+1, \dots, n$. By Proposition 3.5, there is a positive cone $Q \subset C$ containing $g_1, \dots g_r$ whose corresponding ordering of C is dense. Then $P' = P \smallsetminus (P \cap C) \cup Q$ is the positive cone of a dense ordering of G, and $P' \in \bigcap_{i=1}^{n} U_{g_i}$ by construction. ∎

With this generalization, it is straightforward to construct orderings of non-biorderable groups that are accumulation points of dense orderings.

Example 3.8 Recall that

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if } |i-j| = 1 \text{ and } \sigma_i\sigma_j = \sigma_j\sigma_i \text{ otherwise}\rangle.$$

The positive cone P_D of the Dehornoy ordering $<_D$ of B_n is defined as follows. Given a word w in the generators σ_i, we say that w is i-positive if it contains no occurrences of σ_j for $j < i$, and all occurrences of σ_i (of which there must be at least one) occur with positive exponent. A braid $\beta \in B_n$ lies in P_D if and only if it admits a representative word w that is i-positive for some i.

Fix $n > 4$ and consider B_n. The convex subgroups of $<_D$ are precisely the subgroups $\langle \sigma_r, \dots, \sigma_{n-1}\rangle$ with $r \geq 1$ [5]. In particular, $\langle \sigma_{n-2}, \sigma_{n-1}\rangle \cong B_3$ is a proper convex subgroup. Equip this copy of B_3 with any left-ordering whose Conradian soul is contained in $[B_3, B_3] \cong F_2$ and is not infinite cyclic. Extend this ordering to B_n using the Dehornoy ordering outside of $\langle \sigma_{n-2}, \sigma_{n-1}\rangle$. By Theorem 3.7, the resulting ordering is an accumulation point of dense orderings of B_n. However, B_n itself is

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not bi-orderable—in fact, not even Conradian left-orderable, since $[B_n, B_n]$ is finitely generated and perfect for $n \geq 5$. ∎

Remark 3.9 A family of left-orderings of B_n of particular interest are the *Nielsen-Thurston orderings*. These are the orderings that arise from considering the action of B_n, thought of as a mapping class group, on the boundary of the universal cover of the n-punctured disk equipped with a hyperbolic metric (see [5, Chapter XIII] for more details). Such orderings are either of finite or infinite type, depending on how a certain geodesic which describes the ordering cuts up the n-punctured disk. The authors of [10] show that the Nielsen–Thurston orderings of infinite type are dense, while those of finite type have Conradian soul isomorphic to $\mathbb{Z}^k$ for $k \geq 1$. When $k > 1$ such orderings are obviously an accumulation point of dense orderings, but when $k = 1$ the picture is not so clear (though it is known that these orderings are not isolated points). It may be of some interest to determine whether or not the Nielsen–Thurston orderings with Conradian soul isomorphic to $\mathbb{Z}$ lie in $\overline{\mathrm{D}(B_n)}$, as this would imply that all Nielsen–Thurston orderings lie in $\overline{\mathrm{D}(B_n)}$.

4 Free Groups

Let F_n denote the free group on generators $\{x_1, \ldots, x_n\}$. In this section, we show that $\mathrm{LO}(F_n) = \overline{\mathrm{D}(F_n)}$, which will follow as a corollary of the following theorem.

Theorem 4.1 *Let $n \geq 2$ and suppose that $P \in \bigcap_{i=1}^{m} U_{g_i} \subset \mathrm{LO}(F_n)$ for some collection of nonidentity elements $g_1, \ldots, g_m \in F_n$. Then there exists $Q \in \bigcap_{i=1}^{m} U_{g_i}$ and a subgroup $C \subset F_n$ with $g_i \notin C$ for all i, satisfying the following:*

(i) *C is convex relative to the ordering of F_n determined by Q;*
(ii) *C is nontrivial and not isomorphic to $\mathbb{Z}$.*

Some of the details of the proof are a special case of computations done in [12], and so are omitted here for clarity of exposition.

Proof Corresponding to the positive cone P, there is a left-ordering $<$ of F_n with $1 < g_i$ for $i = 1, \ldots, m$. Let $\rho\colon F_n \to \mathrm{Homeo}_+(\mathbb{R})$ denote a dynamic realization of $<$, which is a representation satisfying $1 < g$ if and only if $\rho(g)(0) > 0$ for all $g \in F_n$. For the rest of this proof, let B_k denote the k-ball in F_n relative to the generating set $\{x_1, \ldots, x_n\}$.

We will show how to construct, for each $k \geq 1$, a representation $\rho_k\colon F_n \to \mathrm{Homeo}_+(\mathbb{R})$ satisfying the following:

(1) $\rho_k(w)(0) = \rho(w)(0)$ for all $w \in B_k$;
(2) there exist nonidentity elements $h_1, h_2 \in F_n$ such that $\langle h_1, h_2 \rangle$ is not cyclic and $\rho_k(h_i)(0) = 0$ for $i = 1, 2$.

Having constructed such representations, the theorem follows. First choose an enumeration of the rationals $\{r_0, r_1, r_2, \ldots\}$ with $r_0 = 0$ and define a left-ordering $\prec$ of $\mathrm{Homeo}_+(\mathbb{R})$ according to the rule $f_1 \prec f_2$ if and only if $f_1(r_i) < f_2(r_i)$, where r_i is the first rational in the enumeration $\{r_0, r_1, r_2, \ldots\}$ with $f_1(r_i) \neq f_2(r_i)$. The stabilizer of 0, $\mathrm{Stab}(0)$, is a convex subgroup in this left-ordering. Now choose $k \geq 1$ such

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that $g_1, \dots, g_m \in B_k$. Then with the representation ρ_k constructed as above, consider the short exact sequence $1 \to \ker(\rho_k) \to F_n \to \rho_k(F_n) \to 1$, and lexicographically order F_n using the restriction of $<$ to $\rho_k(F_n)$ and whatever ordering one pleases on $\ker(\rho_k)$. Then $C = \langle h_1, h_2 \rangle$ is not cyclic and is contained in $\rho_k^{-1}(\mathrm{Stab}(0))$, which is a convex subgroup relative to the resulting ordering of F_n. Moreover, if we use Q to denote the positive cone of this ordering of F_n, then by our choice of ρ_k, we have $\rho_k(g_i)(0) = \rho(g_i)(0) > 0$ for all $i = 1, \dots, m$, and hence $Q \in \bigcap_{i=1}^m U_{g_i}$.

Thus, we fix $k \geq 1$ and focus on constructing ρ_k as above. Let $g^+ = \max B_k$ and $g^- = \min B_k$, where the maximum and minimum are taken relative to the ordering $<$ of F_n restricted to B_k. Since the dynamic realization ρ satisfies $\rho(h)(0) > 0$ if and only if $h > 1$, the assignment $h \mapsto \rho(h)(0)$ is order-preserving. We conclude that $\rho(w)(0) \in [\rho(g^-)(0), \rho(g^+)(0)]$ for all $w \in B_k$. From this, it follows by induction on the length of w, that if ρ_k satisfies $\rho_k(x_i)(y) = \rho(x_i)(y)$ for $i = 1, \dots, n$ and for all $y \in [\rho(g^-)(0), \rho(g^+)(0)]$, then $\rho_k(w)(0) = \rho(w)(0)$ for all $w \in B_k$ (this is a special case of [12, Lemma 1.9]).

Now for each $j = 1, \dots, n$, choose $\epsilon_j = \pm 1$ such that $x_j^{\epsilon_j} g^+ > g^+$, and choose j_0 such that $x_{j_0}^{\epsilon_{j_0}} g^+ = \min\{x_1^{\epsilon_1} g^+, \dots, x_n^{\epsilon_n} g^+\}$. To simplify notation, set $a = x_{j_0}^{\epsilon_{j_0}}$. Since $n \geq 2$, we may choose $\ell \neq j_0$, and set $b = x_\ell^{\epsilon_\ell}$.

For ease of notation in the arguments below, in place of $\rho(h)(x)$ we simply write $h(x)$ whenever $h \in F_2$ and $x \in \mathbb{R}$. Define order-preserving homeomorphisms $f_1, f_2\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$f_1(x) = \begin{cases} a(x) & \text{if } x \leq g^+(0), \\ \left(\frac{bg^+(0) - ag^+(0)}{ag^+(0) - g^+(0)}\right)(x - ag^+(0)) + bg^+(0) & \text{otherwise.} \end{cases}$$

Then noting that $f_1(bg^+(0)) > bg^+(0)$, set

$$f_2(x) = \begin{cases} b(x) & \text{if } x \leq g^+(0), \\ \left(\frac{f_1(bg^+(0)) - bg^+(0)}{bg^+(0) - g^+(0)}\right)(x - bg^+(0)) + f_1(bg^+(0)) & \text{otherwise.} \end{cases}$$

See Figures 1 and 2 for graphical explanations of these functions; note that $g^+(0) < ag^+(0) < bg^+(0)$ and $g^+(0) < bg^+(0) < f_1(bg^+(0))$ follow from our choices of j_0, ϵ_{j_0}, and ϵ_ℓ.

Define $\rho_k\colon F_n \to \mathrm{Homeo}_+(\mathbb{R})$ as follows. For $i \notin \{j_0, \ell\}$, set $\rho_k(x_i) = x_i$, and set $\rho_k(a) = f_1$ and $\rho_k(b) = f_2$. Next set $h_1 = (bg^+)^{-1} a^2 g^+$ and $h_2 = (abg^+)^{-1} b^2 g^+$. Observe that bg^+ and ag^+ are reduced words, since the exponents ϵ_{j_0} and ϵ_ℓ are chosen so that $bg^+(0), ag^+(0) \notin [g^-(0), g^+(0)]$. Since $j_0 \neq \ell$, it follows that h_1 and h_2 are reduced words in the generators $\{x_1, \dots, x_n\}$, and we conclude that h_1, h_2 do not represent the identity. Moreover, there are no integers s, t such that $h_1^s = h_2^t$, because the commutator $[h_1, h_2]$ is not the identity, so $\langle h_1, h_2 \rangle$ is not cyclic.

Lastly, by using the facts that (1) $\rho_k(g^+)^{\pm 1}(0) = (g^+)^{\pm 1}(0)$ and (2) $\rho_k(a)(x) = f_1(x)$ and $\rho_k(b)(x) = f_2(x)$ for all $x \in \mathbb{R}$, one computes that $\rho_k(h_1)(0) = 0$ and $\rho_k(h_2)(0) = 0$. This completes the proof. ∎

Corollary 4.2 *If* $n \geq 2$, *then* $\mathrm{LO}(F_n) = \overline{\mathrm{D}(F_n)}$.

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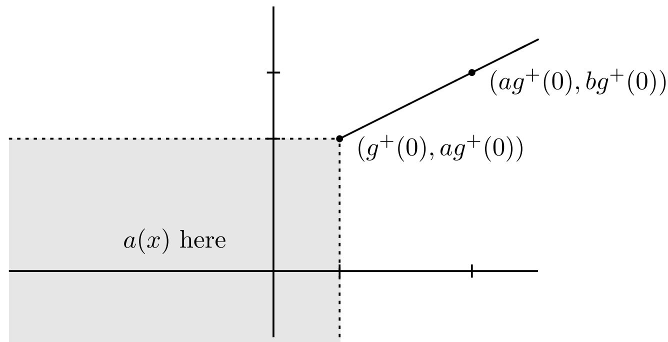

Figure 1: The function $f_1(x)$.

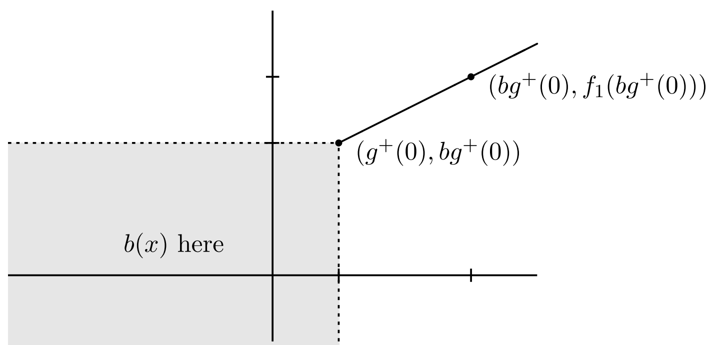

Figure 2: The function $f_2(x)$.

Proof Suppose that P is the positive cone of a left-ordering of F_n, and that $P \in \bigcap_{i=1}^m U_{g_i}$ for some $g_1, \ldots, g_m \in F_n$. Choose $Q \in \bigcap_{i=1}^m U_{g_i}$ with corresponding subgroup $C \subset F_n$ as in the conclusion of Theorem 4.1. Choose a bi-ordering of C with positive cone R, and set $Q' = Q \setminus (Q \cap C) \cup R$. Then $Q' \in \bigcap_{i=1}^m U_{g_i}$, and Q' corresponds to a left-ordering of F_n whose Conradian soul contains C. In particular, Theorem 3.7 implies that $Q' \in \overline{\mathrm{D}(F_n)}$.

It follows that the positive cone P is an accumulation point of elements of $\overline{\mathrm{D}(F_n)}$, so $P \in \overline{\mathrm{D}(F_n)}$. ■

While the previous proof can be modified to handle the case of F_∞ (the free group with countably infinitely many generators), the space $\mathrm{LO}(F_\infty)$ can also be analyzed directly as below.

Example 4.3 Let F_∞ denote the free group on countably many generators $\{x_i\}_{i \in \mathbb{N}}$. Then $\mathrm{LO}(F_\infty)$ is homeomorphic to the Cantor set. If $P \in \bigcap_{i=1}^n U_{g_i} \subset \mathrm{LO}(F_\infty)$, choose k large enough that x_i for $i \geq k$ does not occur in any reduced word representing $g_1, \ldots, g_n$. Then the automorphism $\phi\colon F_\infty \to F_\infty$ defined by $\phi(x_i) = x_i$ for $i \neq k$ and $\phi(x_k) = x_k^{-1}$ yields a positive cone $\phi(P) \neq P$ that contains $g_1, \ldots, g_n$.

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We can in fact approximate such a positive cone P by dense orderings of $\mathrm{LO}(F_\infty)$. With k as above, consider the map $h\colon F_\infty \to \langle x_1, \dots, x_{k-1}\rangle \cong F_{k-1}$ given by $h(x_i) = x_i$ for $i < k$ and $h(x_i) = 1$ for $i > k$. Equip F_{k-1} with the positive cone $P \cap F_{k-1}$, and the normal closure $\langle\langle x_k, x_{k+1}, \dots\rangle\rangle$ with any positive cone Q corresponding to a dense ordering of $\langle\langle x_k, x_{k+1}, \dots\rangle\rangle$. Note that since free groups are bi-orderable [4, Section 3.2], such an ordering exists by Proposition 3.4. Now using the short exact sequence

$$1 \longrightarrow \langle\langle x_k, x_{k+1}, \dots\rangle\rangle \xrightarrow{i} F_\infty \xrightarrow{h} F_{k-1} \longrightarrow 1,$$

we lexicographically order F_∞ using the positive cone $P' = i(Q) \cup h^{-1}(P \cap F_{k-1})$. The result is a positive cone $P' \in \bigcap_{i=1}^n U_{g_i}$ whose corresponding ordering is dense. We conclude that $\overline{\mathrm{D}(F_\infty)} = \mathrm{LO}(F_\infty)$. ∎

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