ON QUADRUPLE SYSTEMS

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1. Introduction. Given a set E of n elements we denote by $S(l, m, n)$, $(l \leqslant m \leqslant n)$ a system[1] of subsets of E, having m elements each, such that every subset of E having l elements is contained in exactly one set of the system $S(l, m, n)$.

It is clear **(3)**, that a necessary condition for the existence of $S(l, m, n)$ is that

$$\binom{n-h}{l-h} \Big/ \binom{m-h}{l-h} = \text{integer}, \qquad (h = 0, 1, \ldots, l-1). \tag{1}$$

$$\binom{n}{l} \Big/ \binom{m}{l}$$

is the number of elements of $S(l, m, n)$ and

$$\binom{n-h}{l-h} \Big/ \binom{m-h}{l-h}$$

is the number of those elements of $S(l, m, n)$ which contain h fixed elements of E.

It is known that condition (1) is not sufficient for $S(l, m, n)$ to exist. It has been proved that no finite projective geometry exists with 7 points on every line.[2] This implies non-existence of $S(2, 7, 43)$.

There arises a problem of finding a necessary and sufficient condition for the existence of $S(l, m, n)$, or more precisely, of finding—for given values of l and m—all values of n for which $S(l, m, n)$ exists.

Already in 1852 Steiner **(6)** (see also **(4)**) raised the following problem:

(a) For what integer N is it possible to form triples, out of N given elements, in such a way that every pair of elements appears in exactly one triple?

(b) Assuming (a) solved we require the further possibility of forming quadruples so that any three elements, not already forming a triple, should appear in exactly one quadruple and that no quadruple should contain a triple. Does this impose new conditions on the number N?

(c) Assuming (a) and (b) solved, can we moreover form quintuples so that any four elements, neither forming a quadruple nor containing a triple,

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[1]The term "family" would be more appropriate, but for historical reasons we shall use the term "system."

[2]This follows from the proof by G. Tarry that the "36 Offiziere" problem of Euler has no solution **(8)**.

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should appear in exactly one quintuple and that no quintuple should contain a quadruple or a triple? Does this impose further restrictions on the number N?

Steiner carries on stating analogous problems for sixtuples, septuples, etc.

The problem of Steiner is essentially equivalent to that of $S(m-1, m, n)$ systems. The special case of $S(2, 3, n)$ constitutes Steiner's famous triple problem (a), and $S(3, 4, n)$ are equivalent to Steiner's quadruple problem (b) with $n = N + 1$. It can easily be seen that by adding an additional element to the system described in (b) and by joining it to every existing triple, a $S(3, 4, N + 1)$ system evolves.

The present form of the problem is due to Moore **(3)**.

So far the problem has been solved completely only for $l = 2$, $m = 3$, that is, for triple systems: from (1) it follows that a necessary condition for the existence of $S(2, 3, n)$ is

$$n = 1 \quad \text{or} \quad 3 \pmod 6; \tag{2}$$

on the other hand, Reiss **(5)** and later independently Moore **(2)** proved that this condition is also sufficient. Other results are limited to special values of l, m, n. A list of systems $S(l, m, n)$ which are known to exist can be found in **(7)**.

In the case of $l = 3$, $m = 4$, that is, of quadruple systems, it follows from (1) that a necessary condition for the existence of $S(3, 4, n)$ is $n \equiv 2$ or $4 \pmod 6$. The object of the present paper is to prove that this condition is also sufficient.

2. Definitions and notation.

2.1. *The systems $P_\alpha(m)$.* Given a set of $2m$ elements $0, 1, \ldots, 2m-1$, we decompose the $m(2m-1)$ unordered pairs $[r, s]$ formed from them into $2m-1$ systems $P_\alpha(m)$, $(\alpha = 0, 1, \ldots, 2m-2)$, each containing m mutually disjoint pairs.[3]

For $m \equiv 0 \pmod 2$ we form the systems $P_\alpha(m)$ as follows:

$$P_{2\beta}(m) = \{[2a, 2a + 2\beta + 1] \colon a = 0, 1, \ldots, m-1\},$$

$$P_{2\beta+1}(m) = \{[2a, 2a - 2\beta - 1] \colon a = 0, 1, \ldots, m-1\},$$

$$(\beta = 0, 1, \ldots, \tfrac{1}{2}(m-2));$$

$$P_{m+\gamma}(m) = \left\{\begin{array}{l} [b, 2\gamma - b] \colon b = 0, 1, \ldots, \gamma - 1 \\ [c, 2m + 2\gamma - c - 2] \colon c = 2\gamma + 1, 2\gamma + 2, \ldots, m + \gamma - 2 \\ \left[2m - \dfrac{3}{2} - (-1)^\gamma \tfrac{1}{2}, \gamma\right], \left[2m - \dfrac{3}{2} + (-1)^\gamma \tfrac{1}{2}, m + \gamma - 1\right] \end{array}\right\},$$

$$(\gamma = 0, 1, \ldots, m-2).$$

For $m \equiv 1 \pmod 2$ we put:

[3]Other systems of pairs may be found in **(5)** and **(4)**, but in the sequel we shall need the systems $P_\alpha(m)$ as defined here.

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$$P_{2\beta}(m) \quad = \{[2a, 2a + 2\beta + 1]: a = 0, 1, \ldots, m - 1\},$$
$$P_{2\beta+1}(m) \quad = \{[2a, 2a - 2\beta - 1]: a = 0, 1, \ldots, m - 1\},$$
$$(\beta = 0, 1, \ldots, \tfrac{1}{2}(m - 3));$$

$$P_{m-1+\gamma}(m) = \left\{\begin{array}{l}[b, 2\gamma - b]: b = 0, 1, \ldots, \gamma - 1 \\ {[c, 2m + 2\gamma - c]: c = 2\gamma + 1, 2\gamma + 2, \ldots, m + \gamma - 1} \\ {[\gamma, m + \gamma]}\end{array}\right\},$$
$$(\gamma = 0, 1, \ldots, m - 1).$$

It can be easily verified that the pairs in every system are mutually disjoint and that no pair appears twice. As the number of pairs in the systems is $m(2m - 1)$ it follows that every pair appears in some system.

2.2. *The systems $\bar{P}_\xi(m)$.* In the sequel we shall also need another decomposition of pairs formed from $2m$ elements, namely into $2m$ systems $\bar{P}_\xi(m)$, $(\xi = 0, 1, \ldots, 2m - 1)$ such that each of the m systems $\bar{P}_\eta(m)$, $(\eta = 0, 1, \ldots, m - 1)$ should contain $m - 1$ mutually disjoint pairs not containing the elements 2η and $2\eta + 1$, and each of the other m systems should contain m mutually disjoint pairs.

We shall form the systems $\bar{P}_\xi(m)$ using the systems $P_\alpha(m)$ defined in the preceding section.

If $m \equiv 0 \pmod 2$ it can easily be seen that

$$[2\mu, 4\mu + 1] \in P_{2\mu}(m),$$
$$[2m - 2 - 2\mu, 2m - 1 - 4\mu] \in P_{2\mu-1}(m), \ (\mu = 1, 2, \ldots, \tfrac{1}{2}(m - 2));$$
$$[2m - 2, 0] \in P_m(m); \ [2m - 1, 1] \in P_{m+1}(m).$$

Clearly, these pairs are mutually disjoint. We remove them from their respective systems and form from them a new system.

Performing the following permutation of the elements

$$\begin{pmatrix} 2\mu, & 4\mu + 1, & 2m - 2 - 2\mu, & 2m - 1 - 4\mu, & 2m - 2, & 0, & 2m - 1, & 1 \\ 4\mu, & 4\mu + 1, & 4\mu - 2, & 4\mu - 1, & 1, & 0, & 2m - 1, & 2m - 2 \end{pmatrix}$$
$$(\mu = 1, 2, \ldots, \tfrac{1}{2}(m - 2))$$

we obtain by a suitable reordering of the systems the new systems $\bar{P}_\xi(m)$.

In the case $m \equiv 1 \pmod 2$ we have

$$[2\mu, 4\mu + 1] \in P_{2\mu}(m),$$
$$[2m - 2 - 2\mu, 2m - 3 - 4\mu] \in P_{2\mu+1}(m), \ (\mu = 0, 1, \ldots, \tfrac{1}{2}(m - 3));$$
$$[m - 1, 2m - 1] \in P_{2m-2}(m).$$

These pairs are again mutually disjoint.

By the permutation

$$\begin{pmatrix} 2\mu, & 4\mu + 1, & 2m - 2 - 2\mu, & 2m - 3 - 4\mu, & m - 1, & 2m - 1 \\ 4\mu, & 4\mu + 1, & 4\mu + 2, & 4\mu + 3, & 2m - 2, & 2m - 1 \end{pmatrix},$$
$$(\mu = 0, 1, \ldots, \tfrac{1}{2}(m - 3)),$$

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of the elements and using the same procedure as in the case $m \equiv 0 \pmod 2$ we obtain the systems $\bar{P}_\xi(m)$.

2.3. *The quadruples.* Let there be given a set F of f elements $0, 1, \ldots, f-1$. If a system $S(3, 4, f)$ exists we say that it is possible to form a quadruple system from F and we write $F \in Q$ and also $f \in Q$.

If $F \in Q$, we shall in the sequel denote by $\{x, y, z, t\} \subset F$ any quadruple in F, that is, an element of $S(3, 4, f)$. The number of quadruples will be denoted by $q(f)$:

$$q(f) = \frac{1}{24} f(f-1)(f-2).$$

If $f + 1 \in Q$, then $F \cup \{A\} \in Q$ where A is some additional element. The quadruples which contain A will be denoted by $\{A, u, v, w\}$ and their number by $p(f)$:

$$p(f) = \frac{1}{6} f(f-1).$$

The other quadruples will be denoted by $\{x, y, z, t\}$ and their number by $q'(f)$:

$$q'(f) = q(f+1) - p(f) = \frac{1}{24} f(f-1)(f-3).$$

2.4. *The elements.* Elements of the sets used in this paper will often be denoted by a pair of numbers (i, j), $(i = 0, 1, \ldots, g-1; j = 0, 1, \ldots, f-1)$ i will then be called the first index and j the second index of the element.

For the sake of uniformity we shall denote sometimes elements by (A, h), $(h = 0, 1, \ldots, e-1)$ instead of the more commonly used notation A_h.

In the above notation we shall also include elements (a, b), (A, c) with a, b, and c not necessarily restricted to $a < g$, $b < f$, and $c < e$. In these cases the indices are to be taken modulo g, f, and e respectively.

2.5. *Checking of systems.* In order to show that some given family of quadruples formed from the elements of a set N (having n elements) are a system it must be proved that:

(i) Every subset of N having 3 elements is contained in some quadruple.

(ii) The intersection of every two quadruples has 2 elements at most.

Evidently (i) will imply (ii) if it can be verified that the total number of the quadruples is $q(n)$ and we shall use this method of checking the systems in the sequel.

3. Theorem. *A system $S(3, 4, n)$ exists if and only if $n \equiv 2$ or $4 \pmod 6$.*

Proof. The necessity has been proved in § 1. The proof of sufficiency will be given by induction. Evidently $4 \in Q$. We shall show that if $n \equiv 2$ or $4 \pmod 6$ and if for every $g < n$ satisfying $g \equiv 2$ or $4 \pmod 6$, $g \in Q$ holds, then also $n \in Q$. The proof will be given separately for each of the following cases which evidently exhaust all the possibilities:

3.1. $n \equiv 4$ or $8 \pmod{12}$,

3.2. $n \equiv 4$ or $10 \pmod{18}$,

3.3. $n \equiv 34 \pmod{36}$,

3.4. $n \equiv 26 \pmod{36}$,

3.5. $n \equiv 2$ or $10 \pmod{24}$, $(n > 2)$,

3.6. $n \equiv 14$ or $38 \pmod{72}$.

3.1.[4] $n \equiv 4$ or $8 \pmod{12}$. We put $n = 2f$, where $f \equiv 2$ or $4 \pmod 6$ and by the assumption of the induction $f \in Q$. Denote $F = \{j : j = 0, 1, \ldots, f-1\}$, $N = \{(i, j) : i = 0, 1; j = 0, 1, \ldots, f-1\}$. Further let $\{x, y, z, t\}$ be any quadruple in F; the number of such quadruples is $q(f)$.

Form the following quadruples in N:

$L_1 : (a_1, x)(a_2, y)(a_3, z)(a_4, t),\ a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod 2;$

$L_2 : (0, j)(0, j')(1, j)(1, j'),\ (j = 0, 1, \ldots, f-1; j' = 0, 1, \ldots, f-1; j \neq j')$

In the quadruples L_1 three of the indices a_1, a_2, a_3, a_4 can be chosen freely from the numbers 0 and 1, and accordingly the number of these quadruples is $8q(f)$. The number of quadruples L_2 is evidently $\frac{1}{2}f(f-1)$. The total number of quadruples is therefore $8q(f) + \frac{1}{2}f(f-1) = q(n)$.

By 2.5, it remains to be shown that every subset of N containing 3 elements $\{(i_1, j_1)(i_2, j_2)(i_3, j_3)\}$ is included in some quadruple. This is, however, evident as:

(a) if $j_1 \neq j_2 \neq j_3 \neq j_1$ it is included in some L_1;

(b) otherwise it is included in some L_2.

Consequently $n \in Q$.

3.2. $n \equiv 4$ or $10 \pmod{18}$. Put $n = 3f + 1$, where $f = 1$ or $3 \pmod 6$. Thus $f + 1 \in Q$. Denote $F = \{j : j = 0, 1, \ldots, f-1\}$, $N = \{(A); (i, j) : i = 0, 1, 2; j = 0, 1, \ldots, f-1\}$. (See 2.3 for definitions of $\{(A), u, v, w\}$ and $\{x, y, z, t\}$.)

Form the following quadruples in N:		their number being:
$L_1 : (a_1, x)(a_2, y)(a_3, z)(a_4, t),$	$a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod 3;$	$27q'(f)$
$L_2 : (A)(b_1, u)(b_2, v)(b_3, w),$	$b_1 + b_2 + b_3 \equiv 0 \pmod 3;$	$9p(f)$
$L_3 : (i, u)(i, v)(i+1, w)(i+2, w),$		$9p(f)$
$L_4 : (i, j)(i, j')(i+1, j)(i+1, j'),$	$j \neq j';$	$3 \cdot \frac{1}{2}f(f-1)$
$L_5 : (A)(0, j)(1, j)(2, j);$		f
	totalling	$q(n)$.

[4]This part of the proof is not new. It is well known that from $f \in Q$ follows $2f \in Q$, (see, for example, **(1)** and **(7)**).

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Now every subset T of N containing three elements is contained in one of the quadruples, namely:

(a) if $T = \{(A)(i_1, j_1)(i_2, j_2)\}$ and

(aa) if $j_1 \neq j_2$, in some L_2;

(ab) if $j_1 = j_2$, in some L_5;

(b) if $T = \{(i_1, j_1)(i_2, j_2)(i_3, j_3)\}$ and

(ba) if $j_1 \neq j_2 \neq j_3 \neq j_1$ and
(baa) if j_1, j_2, j_3 form a $\{u, v, w\}$, in L_2 or in L_3;
(bab) otherwise, in L_1;

(bb) if $j_1 = j_2 \neq j_3$ and
(bba) if $i_1 \neq i_2 \neq i_3 \neq i_1$, in L_3;
(bbb) otherwise, in L_4

(bc) if $j_1 = j_2 = j_3$, in L_5.

It is thus proved that $n \in Q$.

3.3. $n = 34 \pmod{36}$. Put $n = 3f + 4$, where $f \equiv 10 \pmod{12}$ and denote $f = 12k + 10$. Here $f + 4 \in Q$. Denote $F = \{j : j = 0, 1, \ldots, f - 1\}$, $N = \{(i, j); (A, h) : i = 0, 1, 2; j = 0, 1, \ldots, f - 1; h = 0, 1, 2, 3\}$. We have $\tilde{F} = F \cup \{(A, h) : h = 0, 1, 2, 3\} \in Q$. By $\{x, y, z, t\}$ we denote quadruples in $\tilde{F}$, one of them being $\{(A, h) : h = 0, 1, 2, 3\}$.

Form the following quadruples in N: their number being:

$L_1 : (A, 0)(A, 1)(A, 2)(A, 3)$; $\quad 1$

$L_2 : (i, x)(i, y)(i, z)(i, t)$,[5] $\quad 3[q(f + 4) - 1]$
quadruple L_1 excluded;

$L_3 : (A, a_1)(0, a_2)(1, a_3)(2, a_4)$, $\quad 4f^2$
$a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod{f}$;

$L_4 : (i + 2, b_3)(i, b_1 + 2k + 1 + i(4k + 2) - d)(i, b_1 + 2k + 2 + i(4k + 2) + d)(i + 1, b_2)$, $\quad 3(2k + 1)f^2$
$b_1 + b_2 + b_3 \equiv 0 \pmod{f}$,
$d = 0, 1, \ldots, 2k$;

$L_5 : (i, r_\alpha)(i, s_\alpha)(i + 1, r_\alpha')(i + 1, s_\alpha')$, $\quad 3(8k + 7)(\tfrac{1}{2}f)^2$
$[r_\alpha, s_\alpha]$ and $[r_\alpha', s_\alpha']$ are (equal or different) pairs in $P_\alpha(6k + 5)$, (see 2.1.),
$\alpha = 4k + 2, 4k + 3, \ldots, 12k + 8$;

totalling $\quad q(n)$

[5]Whenever for x, y, z, or t appears (A, h), omit the first index i.

Again, every subset T of N containing three elements is contained in some quadruple:

(a) if $T = \{(A, h_1)(A, h_2)(A, h_3)\}$, in L_1;
(b) if $T = \{(A, h_1)(A, h_2)(i_1, j_1)\}$, in L_2;
(c) if $T = \{(A, h_1)(i_1, j_1)(i_2, j_2)$ and
  (ca) if $i_1 = i_2$, in L_2;
  (cb) otherwise, in L_3;
(d) if $T = \{(i_1, j_1)(i_2, j_2)(i_3, j_3)\}$ and
  (da) if $i_1 \neq i_2 \neq i_3 \neq i_1$ and
    (daa) if $j_1 + j_2 + j_3 \equiv f - 3, f - 2, f - 1$, or $0 (\text{mod} f)$, in L_3;
    (dab) otherwise, in L_4;
  (db) if $i_1 = i_2 \neq i_3$ and
    (dba) if $|j_2 - j_1| \equiv 1 (\text{mod } 2)$ and $|j_2 - j_1| \leqslant 4k + 1$, in L_4;
    (dbb) otherwise, in L_5;
  (dc) if $i_1 = i_2 = i_3$, in L_2.

Thus $n \in Q$.

3.4. $n \equiv 26 (\text{mod } 36)$. Here $n = 3f + 2$, $f \equiv 8 (\text{mod } 12)$, or $f = 12k + 8$; $f + 2 \in Q$. Denote $F = \{j : j = 0, 1, \ldots, f - 1\}$, $N = \{(i, j); (A, h) : i = 0, 1, 2; j = 0, 1, \ldots, f - 1; h = 0, 1\}$. We have $\tilde{F} = F \cup \{(A, h) : h = 0, 1\} \in Q$. By $\{x, y, z, t\}$ denote quadruples in $\tilde{F}$.

Form the following quadruples in N: their number being:

$L_2 : (i, x)(i, y)(i, z)(i, t)$;[6] $\qquad 3q(f + 2)$

$L_3 : (A, a_1)(0, a_2)(1, a_3)(2, a_4)$, $\qquad 2f^2$
$$a_1 + a_2 + a_3 + a_4 \equiv 0 (\text{mod} f);$$

$L_4 : (i + 2, b_3)(i, b_1 + 2k + 1 + i(4k + 2) - d)(i, b_1 + 2k + 2 + i(4k + 2) + d)(i + 1, b_2)$, $\qquad 3(2k + 1)f^2$
$$b_1 + b_2 + b_3 \equiv 0 (\text{mod} f),$$
$$d = 0, 1, \ldots, 2k;$$

$L_5 : (i, r_\alpha)(i, s_\alpha)(i + 1, r_\alpha')(i + 1, s_\alpha')$, $\qquad 3(8k + 5)(\tfrac{1}{2}f)^2$
$[r_\alpha, s_\alpha]$ and $[r_\alpha', s_\alpha']$ are pairs in $P_\alpha(6k + 4)$,
$$\alpha = 4k + 2, 4k + 3, \ldots, 12k + 6;$$

totalling $\qquad q(n)$

Checking that every subset of N containing 3 elements is contained in some quadruple is made in the same way as in the preceding section with the only difference that: the case (a) is omitted and (daa) reads: if $j_1 + j_2 + j_3 \equiv f - 1$, or $0 (\text{mod} f)$ a.s.o. Consequently $n \in Q$.

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[6] *Ibid.*

3.5. $n \equiv 2$ or $10 \pmod{24}$, $n > 2$. Put $n = 4f + 2$, $f \equiv 0$ or $2 \pmod 6$, $(f > 0)$; $f = 2k$; by the assumption of the induction $f + 2 \in Q$. Denote $F = \{j : j = 0, 1, \ldots, f - 1\}$, $N = \{(h, i, j); (A, l) : h = 0, 1; i = 0, 1; j = 0, 1, \ldots, f - 1; l = 0, 1\}$. By $\{x, y, z, t\}$ denote quadruples in $F \cup \{(A, l) : l = 0, 1\} \in Q$.

Form the quadruples in N: their number being:

$L_1 : (h, i, x)(h, i, y)(h, i, z)(h, i, t)$;[7] $\quad 4q(f + 2)$

$L_2 : (A, l)(0, 0, 2c_1)(0, 1, 2c_2 - \epsilon)(1, \epsilon, 2c_3 + l)$, $\quad f^2$
$c_1 + c_2 + c_3 \equiv 0 \pmod k$,
$\epsilon = 0, 1$;

$L_3 : (A, l)(0, 0, 2c_1 + 1)(0, 1, 2c_2 - 1 - \epsilon)(1, \epsilon, 2c_3 + 1 - l)$; $\quad f^2$

$L_4 : (A, l)(1, 0, 2c_1)(1, 1, 2c_2 - \epsilon)(0, \epsilon, 2c_3 + 1 - l)$; $\quad f^2$

$L_5 : (A, l)(1, 0, 2c_1 + 1)(1, 1, 2c_2 - 1 - \epsilon)(0, \epsilon, 2c_3 + l)$; $\quad f^2$

$L_6 : (h, 0, 2c_1 + \epsilon)(h, 1, 2c_2 - \epsilon)(h + 1, 0, \bar{r}_{c3})(h + 1, 0, \bar{s}_{c3})$, $\quad (k - 1)f^2$
$[\bar{r}_{c3}, \bar{s}_{c3}]$ are pairs in $\bar{P}_{c3}(k)$, (see 2.2),
$c_3 = 0, 1, \ldots, k - 1$;

$L_7 : (h, 0, 2c_1 - 1 + \epsilon)(h, 1, 2c_2 - \epsilon)(h + 1, 1, \bar{r}_{c3})(h + 1, 1, \bar{s}_{c3})$; $\quad (k - 1)f^2$

$L_8 : (h, 0, 2c_1 + \epsilon)(h, 1, 2c_2 - \epsilon)(h + 1, 1, \bar{r}_{k+c3})(h + 1, 1, \bar{s}_{k+c3})$; $\quad kf^2$

$L_9 : (h, 0, 2c_1 - 1 + \epsilon)(h, 1, 2c_2 - \epsilon)(h + 1, 0, \bar{r}_{k+c3})(h + 1, 0, \bar{s}_{k+c3})$; $\quad kf^2$

$L_{10} : (h, 0, r_\alpha)(h, 0, s_\alpha)(h, 1, r_\alpha')(h, 1, s_\alpha')$, $\quad 2(f - 1)k^2$
$[r_\alpha, s_\alpha]$ and $[r_\alpha', s_\alpha']$ are pairs
in $P_\alpha(k)$, (see 2.1.),
$\alpha = 0, 1, \ldots, f - 2$;

totalling $\quad q(n)$

It will now be checked that every triple T in N is contained in some quadruple.

(a) If T is of the form $\{(A, 0)(A, 1)(h_1, i_1, j_1)\}$, it is contained in some L_1;

(b) if $T = \{(A, l_1)(h_1, i_1, j_1)(h_2, i_2, j_2)\}$ and

(ba) if $h_1 = h_2$ and

(baa) if $i_1 = i_2$, in L_1;

(bab) if $i_1 \neq i_2$ say $i_1 = 0$, $i_2 = 1$ and

(baba) if $j_1 \equiv 0 \pmod 2$, in L_2 or L_4;
(babb) if $j_1 \equiv 1 \pmod 2$, in L_3 or L_5;

[7] *Ibid.*

(bb) if $h_1 \neq h_2$ say $h_1 = 0$, $h_2 = 1$ and

(bba) if $i_1 = i_2 = 0$ and

(bbaa) if $j_1 + j_2 + l_1 \equiv 0 \pmod 2$, in L_2 or L_3, $(\epsilon = 0)$;

(bbab) if $j_1 + j_2 + l_1 \equiv 1 \pmod 2$, in L_4 or L_5, $(\epsilon = 0)$;

(bbb) if $i_1 = i_2 = 1$ and

(bbba) if $j_1 + j_2 + l_1 \equiv 0 \pmod 2$, in L_4 or L_5, $(\epsilon = 1)$;

(bbbb) if $j_1 + j_2 + l_1 \equiv 1 \pmod 2$, in L_2 or L_3, $(\epsilon = 1)$;

(bbc) if $i_1 = 0$, $i_2 = 1$, and

(bbca) if $j_1 + j_2 + l_1 \equiv 0 \pmod 2$, in L_2 or L_3, $(\epsilon = 1)$;

(bbcb) if $j_1 + j_2 + l_1 \equiv 1 \pmod 2$, in L_4 or L_5, $(\epsilon = 0)$;

(bbd) if $i_1 = 1$, $i_2 = 0$ and

(bbda) if $j_1 + j_2 + l_1 \equiv 0 \pmod 2$, in L_2 or L_3, $(\epsilon = 0)$;

(bbdb) if $j_1 + j_2 + l_1 \equiv 1 \pmod 2$, in L_4 or L_5, $(\epsilon = 1)$;

(c) if $T = \{(h_1, i_1, j_1)(h_2, i_2, j_2)(h_3, i_3, j_3)\}$ and

(ca) if $h_1 = h_2 = h_3$ and

(caa) if $i_1 = i_2 = i_3$, in L_1;

(cab) otherwise, in L_{10};

(cb) if $h_1 = h_2 \neq h_3$ and

(cba) if $i_1 = i_2 = 0$, in L_6 or L_9;

(cbb) if $i_1 = i_2 = 1$, in L_7 or L_8;

(cbc) if $i_1 \neq i_2$, say $i_1 = 0$, $i_2 = 1$ and

(cbca) if $i_3 = 0$ and

(cbcaa) if $j_1 + j_2 \equiv 1 \pmod 2$, in L_9;

(cbcab) if $j_1 + j_2 \equiv 0 \pmod 2$ and

(cbcaba) if $j_1 + j_2 + j_3 \not\equiv 0, 1 \pmod f$, in L_6;

(cbcabb) if $j_1 + j_2 + j_3 \equiv 0$ or $1 \pmod f$, then
if $h_1 = h_2 = 0$, in L_2 or L_3, $(\epsilon = 0)$;
if $h_1 = h_2 = 1$, in L_4 or L_5, $(\epsilon = 0)$;

(cbcb) if $i_3 = 1$ and

(cbcba) if $j_1 + j_2 \equiv 0 \pmod 2$, in L_8;

(cbcbb) if $j_1 + j_2 \equiv 1 \pmod 2$ and

(cbcbba) if $j_1 + j_2 + j_3 \not\equiv 0, f - 1 \pmod f$, in L_7;

(cbcbbb) if $j_1 + j_2 + j_3 \equiv 0$ or $f - 1 \pmod f$, then
if $h_1 = h_2 = 0$, in L_2 or L_3, $(\epsilon = 1)$;
if $h_1 = h_2 = 1$, in L_4 or L_5, $(\epsilon = 1)$.

This proves that $n \in Q$.

3.6. $n \equiv 14$ or $38 \pmod{72}$. Here $n = 12f + 2$, $f \equiv 1$ or $3 \pmod 6$ and $f + 1 \in Q$. We shall prove that $n \in Q$.

We begin proving that $14 \in Q$. We take as elements the 14 symbols:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D};$$

and form $q(14) = 91$ quadruples as follows:

0125	038D	1236	157C	24AC	3579	479C
$013B$	$039A$	1247	1589	$24BD$	$358B$	5678
0146	0459	$128B$	$15BD$	$257B$	$35AC$	$569B$
0178	$047B$	$129A$	1679	$258A$	$367B$	$56CD$
$019D$	$048A$	$12CD$	$168D$	$259C$	3689	$59AD$
$01AC$	$04CD$	1345	$16BC$	$267C$	$36AD$	$68AC$
0234	$057D$	$137D$	$17AB$	$269D$	$3BCD$	$789A$
0268	$058C$	$138A$	$235D$	$26AB$	$457A$	$78BC$
0279	$05AB$	$139C$	$237A$	$278D$	$458D$	$79BD$
$02AD$	$067A$	$148C$	$238C$	$346C$	$45BC$	$7ACD$
$02BC$	$069C$	$149B$	$239B$	3478	$467D$	$89CD$
0356	$06BD$	$14AD$	2456	$349D$	$468B$	$8ABD$
$037C$	$089B$	$156A$	2489	$34AB$	$469A$	$9ABC$

It can be easily checked that every three elements are included in some quadruple and consequently these quadruples form a $S(3, 4, 14)$.

We now form the set $N' = \{(i, j); (A, h) : i = 0, 1, 2; j = 0, 1, \ldots, 11; h = 0, 1\}$ having 38 elements and we will show that $N' \in Q$. The system of quadruples in N' will be constructed so that it will contain all the quadruples in $\{(i, j); (A, h); j = 0, 1, \ldots, 11; h = 0, 1\}$ for $i = 0, 1, 2$. By $\{x', y', z', t'\}$ denote quadruples in $\{j; (A, h) : j = 0, 1, \ldots, 11; h = 0, 1\}$.

Form the following quadruples: their number being:

$L_1 : (i, x')(i, y')(i, z')(i, t')$;[8] 273

$L_2 : (A, h)(0, b_1)(1, b_2)(2, b_3 + 3h)$, 288

$$b_1 + b_2 + b_3 \equiv 0 \pmod{12},$$
$$h = 0, 1;$$

$L_3 : (i, b_1 + 4 + i)(i, b_1 + 7 + i)(i + 1, b_2)(i + 2, b_3)$; 432

$L_4 : (i, j)(i + 1, j + 6\epsilon)(i + 2, 6\epsilon - 2j + 1)(i + 2, 6\epsilon - 2j - 1)$, 72

$$\epsilon = 0, 1;$$

$L_5 : (i, j)(i + 1, j + 6\epsilon)(i + 2, 6\epsilon - 2j + 2)(i + 2, 6\epsilon - 2j - 2)$; 72

$L_6 : (i, j)(i + 1, j + 6\epsilon - 3)(i + 2, 6\epsilon - 2j + 1)(i + 2, 6\epsilon - 2j + 2)$; 72

$L_7 : (i, j)(i + 1, j + 6\epsilon + 3)(i + 2, 6\epsilon - 2j - 1)(i + 2, 6\epsilon - 2j - 2)$; 72

$L_8 : (i, j)(i, j + 6)(i + 1, j + 3\epsilon)(i + 1, j + 6 + 3\epsilon)$; 36

[8] *Ibid.*

$L_9 : (i, 2g + 3\epsilon)(i, 2g + 6 + 3\epsilon)(i', 2g + 1)(i', 2g + 5),$ 72
$i' \neq i,$
$g = 0, 1, 2, 3, 4, 5;$

$L_{10} : (i, 2g + 3\epsilon)(i, 2g + 6 + 3\epsilon)(i', 2g + 2)(i', 2g + 4);$ 72

$L_{11} : (i, j)(i, j + 1)(\mathrm{i} + 1, j + 3e)(i + 1, j + 3e + 1),$ 144
$e = 0, 1, 2, 3;$

$L_{12} : (i, j)(i, j + 2)(i + 1, j + 3e)(i + 1, j + 3e + 2);$ 144

$L_{13} : (i, j)(i, j + 4)(i + 1, j + 3e)(i + 1, j + 3e + 4);$ 144

$L_{14} : (i, r_\alpha)(i, s_\alpha)(i', r_\alpha')(i', s_\alpha')$ 216
$[r_\alpha, s_\alpha]$ and $[r_\alpha', s_\alpha']$ are pairs
in $P_\alpha(6)$, (see 2.1),
$\alpha = 4, 5;$

totalling $2109 = q(38)$

Checking that every triple T' in N' is contained in some quadruple is carried out as follows:

(a) if $T' = \{(A, 0)(A, 1)(i_1, j_1)\}$, it is contained in L_1;

(b) if $T' = \{(A, h_1)(i_1, j_1)(i_2, j_2)\}$ and
- (ba) if $i_1 = i_2$, in L_1;
- (bb) if $i_1 \neq i_2$, in L_2;

(c) if $T' = \{(i_1, j_1)(i_2, j_2)(i_3, j_3)\}$ and
- (ca) if $i_1 = i_2 = i_3$, in L_1;
- (cb) if $i_1 = i_2 \neq i_3$, we may assume that $1 \leqslant (j_2 - j_1)(\mathrm{mod}\ 12) \leqslant 6$. Now
  - (cba) if $j_1 + j_2 + j_3 \not\equiv 0 \pmod 3$ and
    - (cbaa) if $j_2 - j_1 \equiv 1 \pmod{12}$, in L_{11};
    - (cbab) if $j_2 - j_1 \equiv 2 \pmod{12}$, in L_{12};
    - (cbac) if $j_2 - j_1 \equiv 3 \pmod{12}$, in L_3;
    - (cbad) if $j_2 - j_1 \equiv 4 \pmod{12}$, in L_{13};
    - (cbae) if $j_2 - j_1 \equiv 5 \pmod{12}$, in L_{14};
    - (cbaf) if $j_2 - j_1 \equiv 6 \pmod{12}$, in L_9 or L_{10};
  - (cbb) if $j_1 + j_2 + j_3 \equiv 0 \pmod 3$ and
    - (cbba) if $j_2 - j_1 \equiv 1 \pmod{12}$ and
      - (cbbaa) if $j_1 \equiv 0 \pmod 2$, in L_7;
      - (cbbab) if $j_1 \equiv 1 \pmod 2$ in L_6;
    - (cbbb) if $j_2 - j_1 \equiv 2 \pmod{12}$ and
      - (cbbba) if $j_1 \equiv 0 \pmod 2$, in L_{10};
      - (cbbbb) if $j_1 \equiv 1 \pmod 2$, in L_4;

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(cbbc) if $j_2 - j_1 \equiv 3 \pmod{12}$, in L_3;
(cbbd) if $j_2 - j_1 \equiv 4 \pmod{12}$ and
(cbbda) if $j_1 \equiv 0 \pmod 2$, in L_5;
(cbbdb) if $j_1 \equiv 1 \pmod 2$, in L_9;
(cbbe) if $j_2 - j_1 \equiv 5 \pmod{12}$, in L_{14};
(cbbf) if $j_2 - j_1 \equiv 6 \pmod{12}$, in L_8;

(cc) if $i_1 \neq i_2 \neq i_3 \neq i_1$ and
(cca) if $j_1 + j_2 + j_3 \equiv 0$ or $3 \pmod{12}$, in L_2;
(ccb) if $j_1 + j_2 + j_3 \equiv 4, 5, 6, 7, 8,$ or $9 \pmod{12}$, in L_3;
(ccc) if $j_1 + j_2 + j_3 \equiv 1, 2, 10$ or $11 \pmod{12}$ it is evident that two of the second indices, say j_1 and j_2 must be $j_2 - j_1 \equiv 0 \pmod 3$. Now
(ccca) if $j_2 - j_1 \equiv 0 \pmod 6$ and
(cccaa) if $j_1+j_2+j_3 \equiv 1$ or $11 \pmod{12}$, T' is contained in L_4;
(cccab) if $j_1 + j_2 + j_3 \equiv 2$ or $10 \pmod{12}$, in L_5;
(cccb) if $j_2 - j_1 \equiv 3 \pmod 6$ and
(cccba) if $j_1 + j_2 + j_3 \equiv 1$ or $2 \pmod{12}$, in L_7;
(cccbb) if $j_1 + j_2 + j_3 \equiv 10$ or $11 \pmod{12}$, in L_6.

Thus $38 \in Q$ is proved.

We are now able to prove the case $n \equiv 14$ or $38 \pmod{72}$, (that is, $f \equiv 1$ or $3 \pmod 6$) generally. We introduce an auxiliary element B and obtain $\tilde{F} = \{j; B : j = 0, 1, \ldots, f-1\} \in Q$. (The quadruples $\{B, u, v, w\}$ and $\{x, y, z, t\}$ in $\tilde{F}$ are defined in § 2.3.) Denote $N = \{(i, j); (A, h) : i = 0, 1, \ldots, f-1; j = 0, 1, \ldots, 11; h = 0, 1\}$ and form the quadruples in N:

their number being:

$M_1 : (i, x')(i, y')(i, z')(i, t')$;[9] $\qquad 91.f$

$$M_2 : \begin{cases} (A, h)(u, b_1)(v, b_2)(w, b_3 + 3h), \\ \qquad\qquad b_1 + b_2 + b_3 \equiv 0 \pmod{12}; \\ (u, \alpha_1)(v, \alpha_2)(w, \alpha_3)(w, \alpha_4); \\ (i, \beta_1)(i, \beta_2)(i', \beta_3)(i', \beta_4) \\ \qquad\qquad i' \neq i. \end{cases} \qquad (2109\text{-}273) \,.\, p(f)$$

α_ν, β_ν, $(\nu = 1, 2, 3, 4)$ are to be replaced by the second indices of $L_3 - L_{14}$ corresponding to the first indices 0, 1, 2 for u, v, w respectively. It should be noted that i and i' define uniquely a $\{u, v, w\}$ in which they are contained and therefore they may be considered as two indices from this $\{u, v, w\}$.

$M_3 : (x, a_1)(y, a_2)(z, a_3)(t, a_4), a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod{12}$; $\qquad 1728.q'(f)$

totalling $\qquad q(n)$

[9] *Ibid.*

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It is easy to see that every triple T in N is contained in some quadruple:

(a) if $T = \{(A, 0)(A, 1)(i_1, j_1)\}$ it is contained in M_1;

(b) if $T = \{(A, h_1)(i_1, j_1)(i_2, j_2)\}$ and
 (ba) if $i_1 = i_2$, it is contained in M_1;
 (bb) if $i_1 \neq i_2$, in M_2;

(c) if $T = \{(i_1, j_1)(i_2, j_2)(i_3, j_3)\}$ and
 (ca) if $i_1 = i_2 = i_3$, in M_1;
 (cb) if $i_1 = i_2 \neq i_3$, in M_2;
 (cc) if $i_1 \neq i_2 \neq i_3 \neq i_1$, and
  (cca) if i_1, i_2, i_3 form a $\{u, v, w\}$, in M_2;
  (ccb) otherwise in M_3.

Consequently in this case again $n \in Q$, and the proof is herewith completed.

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