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CONTENT ALGEBRAS

BY
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Let R be a commutative ring with identity and let X be an indeterminate. In [5] and [16] it was shown that if $f, g \in R[X]$, then for some integer $n \geq 1$, $c(f)^{n+1}c(g) = c(f)^n c(fg)$, where $c(h)$ denotes the additive subgroup of R generated by the coefficients of $h \in R[X]$. Actually the statements in [5] and [16] are not so general as this; however, the proofs are. Specifically, in [16] Mertens considers only the case that R is a polynomial ring over the integers, but this gives the result for any ring by specializing the coefficients of f and g. In [5] Dedekind considers only the case that R is a ring of algebraic integers, but his proof is completely general. Further, the above formula then holds if one lets $c(h)$ denote the S-submodule of R generated by the coefficients of $h \in R[X]$, S a subring of R, and it is in this form that it usually appears, especially the case $S = R$. Dedekind's very elegant proof is reproduced in [15, p. 9, Lemma 6.1]. Other proofs can be found in [21, p. 24], and in [2], [4, p. 562, Exercise 21], [19], [9, pp. 343–347], where it is extended to any number of variables. This formula is extended to certain semigroup rings in [19], and to certain power series in [13, Theorem 3.6]. Many results in commutative ring theory have this formula, or an immediate consequence of it, as a basic ingredient, e.g. [15, p. 9], [14, p. 128], [10], [9, Section 28], [11], [12], [17, Proposition 4.5], [20], [21, Theorem 6.5], [26]. In this note we indicate further the prominence of this formula by noting a few of the results on polynomial rings which hold for R-algebras which have a content formula as above. This is useful, for instance, because it precludes the need to reprove for some group algebras, many of the polynomial ring results, as well as giving the results for wider classes of R-algebras [19, 21]. (See Section 1.) Also, due to the greater generality of the content algebra property, it is much more stable than being a polynomial or group algebra.

In the first section of this note we collect some facts about R-algebras which are content R-modules, define weak content R-algebras, and briefly discuss their relationship to content R-algebras. In Section 2 we give some applications of these notions to divisibility properties of domains. The last section contains a result on when a weak content R-algebra is R-flat.

1. **General properties of R-algebras which are content modules.** Let M be an R-module and let $x \in M$. The *content* $c(x)$ of x is defined as the intersection

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of those ideals A of R such that $x \in AM$. If $x \in c(x)M$ for every $x \in M$, then M is called a *content R-module*. These modules have been studied in [7] and [21]. It is immediate that M being a content R-module is equivalent to $(\bigcap_{i \in I} A_i)M = \bigcap_{i \in I} A_iM$ for any family of ideals $\{A_i\}_{i \in I}$ of R. Further if $x = \sum_{i=1}^n a_ix_i$ with $a_i \in c(x)$ and $x_i \in M$, then $c(x) = \sum_{i=1}^n Ra_i$. Thus if M is a content R-module and $x \in M$, then $c(x)$ is finitely generated. In the following proposition we summarize some facts needed about R-algebras which are content R-modules. Parts (ii) and (iii) are given in [24] but are included here for the reader's convenience.

1.1. Proposition. *Let $\varphi: R \to B$ be an R-algebra which is a content R-module. The following properties hold.*

(i) *$c(fg) \subseteq c(f)c(g)$ for every $f, g \in B$.*

(ii) *$c(B) = c(1_B)$ is generated by an idempotent.*

(iii) *$\{p \in \operatorname{Spec}(R) \mid pB = B\} = V(c(B))$ and thus is open and closed.*

(iv) *$c(B) = R$ iff $pB \neq B$ for all $p \in \operatorname{Spec}(R)$.*

(v) $\operatorname{Ann}_R(c(b)) \subseteq Ann_R(b)$ *for $b \in B$ (see Lemma* 3.1.)

(vi) *If φ is injective then $c(B) = R$. The converse holds if B is flat.*

Proof. (i) $f \in c(f)B \Rightarrow f = \sum a_ix_i$, $x_i \in B$, $a_i \in R$, and $c(f) = (a_1, \ldots, a_n)$. Similarly $g = \sum b_iy_i$, $y_i \in B$ and $(b_1, \ldots, b_m) = c(g)$. Then $fg = \sum a_ib_jx_iy_j \subseteq c(f)c(g)B \Rightarrow c(fg) \subseteq c(f)c(g)$. The proofs of the other statements are also straightforward.

Definition. An R-algebra $\varphi: R \to B$ is called a *weak content R-algebra* if B is a content R-module and $c(x)c(y) \subset \sqrt{c(xy)}$ for every $x, y \in B$. (Here $\sqrt{A}$ denotes the radical of A.)

1.2. Theorem. *Let $\varphi: R \to B$ be an R-algebra such that B is a content R-module. The following are equivalent.*

(i) *B is a weak content R-algebra.*

(ii) *For each $p \in \operatorname{Spec}(R)$, either pB is a prime ideal of B or $pB = B$.*

Proof. (i) $\Rightarrow$ (ii). Let $p \in \operatorname{Spec}(R)$ with $pB \neq B$ and let $x, y \in B$. If $xy \in pB$ then $c(xy) \subseteq p \Rightarrow c(x)c(y) \subseteq p \Rightarrow c(x) \subseteq p$ or $c(y) \subseteq p \Rightarrow x \in pB$ or $y \in pB$.

(ii) $\Rightarrow$ (i). Let $x, y \in B$ and suppose $c(xy) \subseteq p$. Then $xy \in pB$ and (ii) implies $x \in pB$ or $y \in pB$. Thus $c(x)c(y) \subseteq p$. q.e.d.

In [21] an R-algebra B was called a *content R-algebra* if B is a faithfully flat content R-module such that for every $f, g \in B$ there exists an integer $n \geq 1$ such that $c(f)^{n+1}c(g) = c(f)^nc(fg)$. Some examples of content R-algebras are:

(1) Semigroup algebras $R[G]$ where G is a cancellative, torsion-free, abelian semigroup [19].

(2) Pure subalgebras of content R-algebras [21, Theorem 1.3]. (This includes the symmetric algebras $S(M)$ where M is a pure submodule of a free

R-module [17, Proposition 4.4], as well as the invertible (= stably polynomial) R-algebras that arise in connection with the cancellation problem for commutative rings [3, 6, 27].)

(3) If B is a content R-algebra, T is a multiplicative subset of B, and $S=T\cap R$, then $T^{-1}B$ is a content $S^{-1}R$-algebra provided $c(t)\cap S\neq\phi$ for every $t\in T$ [21, Theorem 6.2]. This includes for example the case that R is the complement of a prime, as well as any set T consisting of elements of content R. In particular, the R-algebra $R(X)$ used by Nagata [18, p. 18], as well as the R-algebra, also denoted by $R(X)$, used in Quillen's recent solution to the Serre conjecture [8, 23], are content R-algebras. Some further examples of content R-algebras are given in [21].

It is immediate that content R-algebras are weak content R-algebras. To see that the converse does not hold, observe that $R[[X_1,\ldots,X_n]]$ is a flat weak content R-algebra if R is Noetherian. (It is a content module by [7, Theorem 2.6] and primes extend to primes by [1].) That these algebras are not necessarily content R-algebras is seen as follows. Let $R=k[u,v]$, k a field and u, v indeterminates. Let $F=u+vX+vX^2+\cdots$, and $g=v+X\in R[[X]]$. Then $c(g)=1$ but $c(fg)=(uv,u+v^2,v+v^2)\neq(u,v)=c(f)$. Thus for each $n\geq 1$, $c(g)^{n+1}c(f)\neq c(g)^n c(fg)$. (It does hold that $c(f)^{n+1}c(g)=c(f)^n c(fg)$ for some n, however, [12, Theorem 3.6].)

For R a Prüfer domain it does hold that every flat weak content R-algebra B is a content R-algebra. This follows since in this case $c(f)$ is locally principal for every $f\in B$.

Since content R-algebras are weak content algebras, then prime ideals of R extend to prime ideals in a content R-algebra. However, it is easily seen that content R-algebras B have the additional property that if q is a primary ideal of R, then qB is also primary. In fact, primary decompositions in R extend to primary decompositions in B. Further, if R and B are Noetherian, then the primary ideals q and qB have the same length.

2. **Divisibility.** In this section R is an integral domain with quotient field K. If I is an ideal of R, let $I^{-1}=\{x\in K \mid xI\subseteq R\}$. If B is a content R-algebra and $b\in B$, we say b is *primitive* if $c(b)$ is not contained in a proper principal ideal of R, and say b is *super-primitive* if $c(b)^{-1}=R$. It is immediate that super-primitive $\Rightarrow$ primitive and that these are equivalent for GCD-domains.

The following useful lemma is perhaps of interest even for polynomial rings.

2.1. Lemma. *Let B be a content R-algebra, $S=R-\{0\}$, and $b\in B$. Then $bS^{-1}B\cap B=bB$ if and only if $c(b)^{-1}=R$.*

Proof. ($\Rightarrow$) Let $t\in c(b)^{-1}$. Then $tb\in bS^{-1}B\cap B=bB\Rightarrow tb=bh$ some $h\in B$. Thus $t=h\in B\cap K$ and $B\cap K=R$ since B is faithfully flat over R.

($\Leftarrow$) Let $g\in bS^{-1}B\cap B$. Then $g=bh(1/s)$, $s\in R$, $s\neq 0$, and $h\in B$. For some $n>0$, $c(b)^{n+1}c(h)=c(b)^n c(bh)=c(b)^n c(sg)=sc(b)^n c(g)\Rightarrow c(h)c(b)^n/s\subseteq c(b)^{-1}=R\Rightarrow c(h)c(b)^{n-1}/s\subseteq c(b)^{-1}=R\cdots\Rightarrow c(h)/s\subseteq c(b)^{-1}=R$. q.e.d.

The above lemma clearly implies [25, Theorem A], and many of the other results of [25] carry over to content R-algebras by arguments similar to the one above. Also, it is immediate from Lemma 2.1, that if $b\in B$ is super-primitive, then $bB\cap sB=sbB$ for every $s\in S$. In the terminology of [13], this says that super-primitive elements of B are *LCM prime* to S. It is now easy to extend Gilmer and Parker's result on group rings [13, Theorem 4.4] to the following result. See also [17, Section 4].

2.2 Theorem. *Let R be a GCD-domain, B a content R-algebra, and $S=R-\{0\}$. Then B is a GCD-domain if and only if $S^{-1}B$ is a GCD-domain.*

Note 1. Similar considerations also show that if T is the set of primitive elements of B, then B is a GCD-domain if and only if $T^{-1}B$ is a GCD-domain and each element of T has a least common multiple in B.

Note 2. The analogous result to 2.2 for *UFD's* also holds.

We conclude this section with some results similar to 2.2 for weak content algebras. We first extend slightly our notion of content. Let R be an integral domain, $S=R-\{0\}$, and M a flat content R-module. If $x\in S^{-1}M$, then $x=y/s$, $y\in M$, $s\in S$ and we define $c(x)$ as the fractional iseal $(1/s)\cdot c(y)$. It is immediate that this is well defined and $x\in M$ if and only if $c(x)\subseteq R$.

2.3 Theorem. *Let $R\to B$ be a weak content R-algebra with R a Krull domain. If B_p is a Krull domain for every prime p of R, of height ≤ 1, then B is Krull. ($B_p=S^{-1}B$ where $S=R\setminus P$.)*

Proof. First we note that $B={}_{ht}\bigcap_{p=1} B_p$. To see this observe that if $x\in B_p$ for every height one prime p of R, then $c(x)\subseteq R_p$ for every height one prime of R. Thus $c(x)\subseteq {}_{ht}\bigcap_{p=1} R_p=R$ since R is Krull.

Next we show that if p is a height one prime of R, then $B_p=B_{pB}\cap S^{-1}B$ where $S=R\setminus\{0\}$. Let $f/g=h/s$, f, g, $h\in B$, $s\in S$, $g\notin pB$, p a height prime of R. Then $sf=gh$ and localizing at p we get $(s/1)c(f)R_p=c(gh)R_p=c(h)R_p$ since $c(g)R_p=R_p$. Thus $h=(s/1)h'$ some $h'\in B_p\Rightarrow f/g=h/s=sh'/s=h'\in B_p$. Now we have $B={}_{ht}\bigcap_{p=1} B_p={}_{ht}\bigcap_{p=1}(B_{pB}\cap S^{-1}B)=({}_{ht}\bigcap_{p=1} B_{pB})\cap S^{-1}B$. Further B_{pB} is Krull since it is a localization of the Krull domain B_p. Thus it suffices to show that the above intersection has finite character. Let $b\in B$. Then if b is a non-unit of B_{pB}, then $b\subseteq pB\Rightarrow c(b)\subseteq p$. But since R is Krull, this can hold for only finitely many height one primes p. q.e.d.

Note. This generalizes [7, Theorem 2.29] since content module + locally polynomial $\Rightarrow$ weak content and locally Krull.

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2.4. Corollary. *If $R \subseteq B$ are domains with R a Dedekind domain such that primes of R extend to primes of B, and B_p is Krull for every prime p of R, then B is Krull if and only if B is a weak content R-algebra (if and only if B is a content R-algebra).*

Proof. ($\Rightarrow$) Since R is Dedekind, B is R-flat. Thus for each ideal A of R, AB is divisorial. Thus for any infinite set $\{A_i\}_{i\in I}$ of ideals of R, $\bigcap_{i\in I}(A_iB)=0$. But for any finite set $A_1,\ldots,A_n$ of ideals $\bigcap A_iB=(\cap A_i)B$ by flatness. Thus B is a content R-module and thus a weak content R-algebra.

2.5. Corollary. *If R is a Krull domain and B is a finitely presented locally polynomial R-algebra, then B is a Krull domain.*

Proof. The hypotheses imply that B is a symmetric algebra $S(P)$ where P is a projective R-module [3, Theorem 3]. Thus B is a content R-algebra and hence Krull by Theorem 2.3.

Note. If $R \subseteq B$ is a weak content R-algebra, then R has the property that finitely generated flat modules are projective if and only if B does. The same holds for the property that pure ideals are generated by idempotents [24].

3. **Flatness.** In this section we give criteria for weak content R-algebras to be R-flat.

3.1. Lemma. *Let M be a content R-module. Then M is R-flat if and only if* $\mathrm{Ann}_R(m)=\mathrm{Ann}_R(c(m))$ *for every $m\in M$.*

Proof. ($\Rightarrow$) If M is R-flat then $rc(m)=c(rm)$ for every $r\in R$, $m\in M$ by [21, Theorem 1.5]. The result follows.
($\Leftarrow$) By [21, Theorem 1.5 and Corollary 1.6] it suffices to show $(o:r)_M\subseteq (o:r)_RM$ for every $r\in R$. Let $m\in(o:r)_M$. Then $rm=0\Rightarrow rc(m)=0$. Thus $c(m)\subseteq(o:r)_R$ and $m\in c(m)M\subseteq(o:r)_RM$.

3.2. Theorem. *Let $\varphi: R\to B$ be a weak content R-algebra with R reduced. The following are equivalent.*

(i) *φ is injective.*
(ii) *B is R-flat.*
(iii) *B is faithfully flat over R.*

Proof. (i) $\Rightarrow$ (ii). By the above lemma it suffices to show $\mathrm{Ann}_R(b)\subseteq \mathrm{Ann}_R(c(b))$ for every $b\in B$. If $rb=0$, $r\in R$, then $c(r)c(b)\subset\sqrt{c(rb)}=\sqrt{0}=0$ and since $r\in c(r)B$, then $r\cdot c(b)=0$. Thus $r\in\mathrm{Ann}_R(c(b))$.
(ii) $\Rightarrow$ (iii). This follows from Parts (iv) and (vi) of Proposition 1.1.
(iii) $\Rightarrow$ (i). Obvious. q.e.d.

It follows from the above theorem that if $\varphi: R\to B$ is a flat weak content R-algebra, then $\mathrm{Spec}(R)$ decomposes into the disjoint clopen sets $\{p\in \mathrm{Spec}(R)\mid pB=B\}=V(c(B))$ and $\{p\in\mathrm{Spec}(R)\mid pB\cap R=p\}$.

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