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INTEGRAL GRADIENT ESTIMATES ON A CLOSED SURFACE

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ABSTRACT. Let (Σ, g) be a closed Riemann surface, and let u be a weak solution to the equation

$$-\Delta_g u = \mu,$$

where μ is a signed Radon measure. We aim to establish L^p estimates for the gradient of u that are independent of the choice of the metric g. This is particularly relevant when the complex structure approaches the boundary of the moduli space. To this end, we consider the metric $g' = e^{2u}g$ as a metric of bounded integral curvature. This metric satisfies a so-called quadratic area bound condition, which allows us to derive gradient estimates for g' in local conformal coordinates. From these estimates, we obtain the desired estimates for the gradient of u.

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1. Introduction

Let f_k be a sequence of conformal immersions from the 2-dimensional disk D into $\mathbb{R}^n$ satisfying $\int_D |A_k|^2 < 4\pi - \tau$ for some $\tau > 0$, where A denotes the second fundamental form. Define the induced metric by $g_k = f_k^*(g_{\mathbb{R}^n}) = e^{2u_k} g_{euc}$. The celebrated Hélein convergence theorem asserts that if the areas of $f_k(D)$ are bounded, then $\|u_k\|_{L^\infty(D_{\frac{1}{2}})}$ remains bounded and f_k converges to a $W^{2,2}$-conformal immersion, or $u_k \to -\infty$ and f_k collapses to a single point. Hélein's theorem plays a crucial role in variational problems involving the Willmore functional. However, we can not directly get a meaningful limit in the collapsing case. When collapse occurs, one typically finds suitable translations y_k and scalings λ_k and studies the convergence of $\lambda_k(f_k - y_k)$. However, the areas potentially tend to infinity, so we need a gradient estimate to ensure that the areas remain locally bounded.

In [7], the authors employed the gradient estimate $|\nabla_x G(x,y)| \le \frac{C}{d(x,y)}$ to derive gradient bounds for solutions to equations of the form $-\Delta_g u = f$ on closed surfaces. However, the constant C depends on the metric g, rendering this approach unsuitable for sequences whose induced conformal classes diverge in moduli space.

In [8], the first author utilized the conformal invariance of the Willmore energy to establish gradient estimates under the assumption that $f_k(D)$ extends to a closed immersed surface satisfying $\|A\|_{L^2} < C$. Several years later, in [10], the authors showed that such gradient estimates also hold under the density condition

$$\frac{\text{Area}(f_k(D) \cap B_r^n(y))}{\pi r^2} < C$$

for sufficiently small r and all $y \in f_k(D)$. By Simon's monotonicity inequality([15]), the extension assumption implies this density condition.

In [9],the authors proved an intrinsic version of these estimates. In this paper, we apply the method in [9] to provide gradient estimates for solutions to the equation

$$-\Delta_g u = \mu, \tag{1.1}$$

where μ is a signed Radon measure.

Here, we say that u is a solution to (1.1) if $u \in L^1(\Sigma, g)$ and

$$-\int_\Sigma u\Delta_g \varphi dV_g = \int_\Sigma \varphi d\mu \tag{1.2}$$

holds for any $\varphi \in \mathcal{D}(\Sigma)$, where $\mathcal{D}(\Sigma)$ denotes the set of compactly supported smooth functions on Σ. It follows easily from Brezis-Merle's result (see Theorem 2.1 in section 2) and Weyl's lemma (cf. [11, Theorem 2.3.1]) that $u \in W^{1,p}_{loc}$ for any $p \in [1, 2)$. Since $\int_\Sigma u\Delta_g \varphi dV_g$ is independent of g when the conformal class is fixed, actually the definition of equation (1.1) depends only on the conformal class of g.

Equations of this type arise naturally and frequently. For instance, if $f \in L^1(\Sigma, g)$, then fV_g can be regarded as a signed Radon measure. More importantly, the Green's function may be viewed as a solution to such an equation.

In fact, (1.2) is equivalent to

$$-2\sqrt{-1}\int_\Sigma u\partial\bar{\partial}\varphi = \int_\Sigma \varphi d\mu,$$

i.e. the current $-2\sqrt{-1}\partial\bar{\partial}u$ is represented by the Radon measure μ. From this perspective, it is natural to expect an estimate of u that is independent of g.

To state our estimates, we introduce the following notation. We set $\mathfrak{C}$ to be the set of conformal embeddings from D into Σ and define

$$E_p(u) = \sup\{\|\nabla u \circ \varphi\|_{L^p(D_{\frac{1}{2}})} : \varphi \in \mathfrak{C}\},$$

where ∇ denotes the standard gradient operator in the Euclidean space.

When the complex structure approaches the boundary of the moduli space, it becomes necessary to consider estimates on collars. For $0 \leq a < \frac{1}{\pi}\log 2$, we let $\mathfrak{C}_a$ be the set of conformal embeddings from $D \setminus \overline{D_a}$ into Σ and define

$$E_{a,p}(u) = \sup\{\|\nabla u \circ \varphi\|_{L^p(D_{\frac{1}{2}} \setminus \overline{D_{2a}})} : \varphi \in \mathfrak{C}_a\}.$$

By convention, we set $D_0 = \{0\}$ and define $E_{a,p}(u) = 0$ when $\mathfrak{C}_a = \emptyset$.

Our main result is the following:

Theorem 1.1. *Let Σ be a closed Riemann surface of genus $\mathfrak{g}$ and u be a solution to* (1.1). *Then for any $p \in [1, 2)$ and $0 \leq a < \frac{1}{\pi}\log 2$,*

$$E_p(u) \leq C(p, \mathfrak{g})|\mu|(\Sigma), \quad E_{a,p} \leq C(a, p, \mathfrak{g})|\mu|(\Sigma).$$

Remark 1.2. *Such a result does not hold locally. For example, on a disk, for the harmonic functions $u_k = kx^1$ we have $\|\nabla u_k\| \to +\infty$ uniformly.*

Remark 1.3. *Applying the Collar Theorem, from Theorem 1.1 we obtain a new proof of Theorem 0.1 in* [16].

The proof of Theorem 1.1 is geometric in nature. Choose a smooth metric g with $\int_\Sigma |K_g|dV_g < C$ and set $g' = e^{2u}g$. If we assume $|\mu|(\Sigma) = 1$, then g' is a metric with bounded integral curvature in the sense of Alexandrov (we refer to Section 2 for the definition). By a result in [5], g' has quadratic area bound which, together with a blow-up argument, yields the desired estimate.

As a corollary, we obtain an estimate on a constant curvature surface involving the metric.

Corollary 1.4. *Let (Σ, g) be a closed surface with $K_g = 0$ or -1, and u be a solution of* (1.1). *Then there exists a constant C depending only on the genus, such that*

$$\int_{B^g_r(x_0)} |\nabla_g u|dV_g < Cr|\mu|(\Sigma). \tag{1.3}$$

For $p \in (1,2)$, our method does not provide a uniform estimate—the constant C depends on certain geometric data: when $K_g = 0$,

$$r^{\frac{p-2}{p}} \|\nabla_g u\|_{L^p(B_r^g(x_0))} < C(p) \left(\frac{\pi r^2}{\mathrm{Area}(B_r^g(x_0))} \right)^{\frac{p-1}{p}} |\mu|(\Sigma); \tag{1.4}$$

and when $K_g = -1$,

$$\|\nabla_g u\|_{L^p(\Sigma,g)} < C(p) \frac{|\chi(\Sigma)|^{\frac{1}{p}}}{\mathrm{Inj}(\Sigma, g)} |\mu|(\Sigma), \tag{1.5}$$

where Inj is the injectivity radius.

2. Preliminaries

In this section, we list some geometric and analytic results that will be used later.

2.1. **Brezis-Merle's estimates.**

The following theorem, established by Brezis and Merle [3], plays a crucial role in this paper.

Proposition 2.1 (Brezis-Merle). *Given a signed Radon measure μ supported in $D \subset \mathbb{R}^2$ with $0 < |\mu|(\mathbb{R}^2) < +\infty$, let*

$$I_\mu(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| d\mu(y).$$

Then $I_\mu \in W^{1,q}_{loc}(\mathbb{R}^2)$ for any $q \in [1,2)$ and weakly solves the equation:

$$-\Delta I_\mu = \mu. \tag{2.1}$$

Moreover, we have

$$r^{q-2} \int_{D_r(x)} |\nabla I_\mu|^q dx \le C(q) |\mu|(\mathbb{R}^2)^q, \quad \forall x, r, \tag{2.2}$$

and

$$\int_{D_R} e^{\frac{(4\pi-\epsilon)|I_\mu|}{|\mu|(\mathbb{R}^2)|}} dx \le C R^{\frac{\epsilon}{2\pi}}, \quad \forall R > 0 \ \ \textit{and} \ \ \epsilon \in (0, 4\pi). \tag{2.3}$$

Corollary 2.2. *Let μ be a signed Radon measure on D with $|\mu|(D) < \tau$. Suppose that u solves (1.1) weakly on D and $\|u\|_{L^1(D)} < \gamma$. Then for any $p < \frac{4\pi}{\tau}$ there exists $\beta = \beta(\tau, p, \gamma)$ such that*

$$\int_{D_{\frac{1}{2}}} e^{p|u|} dx \le \beta.$$

The proof of this corollary can be found in [5].

2.2. **BIC metrics.**

We begin with the following definition.

Definition 2.3. *Let (Σ, g_0) be a smooth Riemann surface without boundary. Let $\mathcal{M}(\Sigma, g_0)$ denote the set of measurable tensor $g = e^{2u} g_0$ with $u \in L^1_{loc}(\Sigma)$ such that there exists a signed Radon measure μ satisfying*

$$\int_\Sigma \varphi \, d\mu = \int_\Sigma (\varphi \, K_{g_0} - u \Delta_{g_0} \varphi) \, dV_{g_0}, \quad \forall \varphi \in \mathcal{D}(\Sigma).$$

We call μ the Gauss curvature measure of g, and denote it by $\mathbb{K}_g$.

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Given $g \in \mathcal{M}(\Sigma, g_0)$, we define

$$d_{g,\Sigma}(x,y) = \inf\left\{\int_\gamma e^u ds_{g_0} : \gamma \text{ is a piecewise smooth curve from } x \text{ to } y \text{ in } \Sigma\right\} \tag{2.4}$$

where γ is parametrized by its g_0-arclength. As mentioned in the introduction, it follows from the equation in the above definition that $u \in W^{1,p}_{loc}(\Sigma)$ for any $p \in [1,2)$.Therefore, $d_{g,\Sigma}$ is a well-defined distance function whenever $d_g(x,y)$ is finite for any x and y. In this case, we say g is a metric of *bounded integral curvature* in the sense of Alexandrov. For other equivalent definitions, see [17].

If $|\mathbb{K}_g|(\{x\}) < 2\pi$ for any $x \in \Sigma$, d_g is finite (see [5, 12, 13, 14] for more details).

The following estimate proved in [5] is essential for the results developed in the next section.

Theorem 2.4. *Let (Σ, g_0) be a closed surface and $g = e^{2u}g_0 \in \mathcal{M}(\Sigma, g_0)$ with $|\mathbb{K}_g|(\Sigma) < +\infty$. Assume $d_{g,\Sigma}$ is finite in $\Sigma \times \Sigma$. Then*

$$\frac{\mathrm{Area}(B^g_R(x))}{\pi R^2} \le 1 + \frac{1}{2\pi}\mathbb{K}^-_g(\Sigma). \tag{2.5}$$

3. The gradient estimate

We begin by proving a gradient estimate on the two-dimensional disk. As mentioned in the introduction, the examples $u_k = kx^1$ show that no gradient estimates can hold when k is sufficiently large. Therefore, such examples must be excluded.

Definition 3.1. *We say $g = e^{2u}g_{euc} \in \mathcal{M}(D)$ has quadratic area bound with constant Λ, if for any x and r , we have*

$$\frac{\mathrm{Area}(B^D_r(x,d_g))}{\pi r^2} < \Lambda.$$

Here, by $B^D_r(x,d_g)$ we mean the disk is defined with respect to the metric d^D_g,which is d_g restricted to D (see the remark below).

Remark 3.2. *Suppose $\Omega_1 \subset\subset \Omega_2$ and $g \in \mathcal{M}(\Omega_2)$. Then we also have $g \in \mathcal{M}(\Omega_1)$. Let d_1 and d_2 denote the distances induced by g on Ω_1 and Ω_2, respectively. Note that $d_2(x,y)$ may be strictly less than $d_1(x,y)$. However, for any $x,y \in \Omega_1$ with $d_1(x,y) < r$, we have*

$$d_2(x,y) \le d_1(x,y) < r,$$

which implies $y \in B_r(x,d_2)$, and thus

$$B_r(x,d_1) \subset B_r(x,d_2).$$

Therefore, Theorem 2.4 implies that any topological disk domain of Σ has quadratic area bound.

Having the above in mind, in the rest of this section we will simply write $B_r(x,d_g)$ in place of $B^D_r(x,d_g)$ when there is no risk of confusions.

First, we show Theorem 1.3 in [9] still holds if we only assume that $g \in \mathcal{M}(D)$.

Lemma 3.3. *Let $g = e^{2u}g_{euc} \in \mathcal{M}(D)$ has quadratic area bound with constant Λ. Then for any $p \in [1,2)$, there exist positive constants C and ϵ_0, such that if $|\mathbb{K}_g(D)| < \epsilon_0$, then*

$$\|\nabla u\|_{L^p(D_{\frac{1}{2}})} < C.$$

Proof. Assume the result does not hold, then there exists a sequence $g_k = e^{2u_k}g_{euc} \in \mathcal{M}(D)$ such that $\|\mathbb{K}_{g_k}\|(D) \to 0$, while $\|\nabla u_k\|_{L^p(D_{\frac{1}{2}})} \to +\infty$.

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Let $v_k = I_{\mathbb{K}_{g_k}}$ be as in Proposition 2.1. By Proposition 2.1,

$$\sup_{D_r(x)\subset D} r^{p-2}\int_{D_r(x)} |\nabla v_k|^p \to 0.$$

Set $w_k = u_k - v_k$, which is harmonic on D. Then $\|\nabla w_k\|_{C^0(D_{\frac{1}{2}})} \to +\infty$.

Define

$$\rho_k = \sup_{x\in D_{\frac{2}{3}}} (\frac{2}{3} - |x|)|\nabla w_k|(x).$$

There exists $x_k \in D_{\frac{2}{3}}$, such that $\rho_k = (\frac{2}{3} - |x_k|)|\nabla w_k|(x_k)$. Set $r_k = 1/|\nabla w_k(x_k)|$.

Since

$$\rho_k \geq (\frac{2}{3} - \frac{1}{2})\|\nabla w_k\|_{C^0(D_{\frac{1}{2}})} \to +\infty,$$

we have

$$\frac{r_k}{\frac{2}{3} - |x_k|} = \frac{1}{\rho_k} \to 0,$$

which implies that $D_{Rr_k}(x_k) \subset D_{\frac{2}{3}}$ for any fixed $R > 0$, when k is sufficiently large.

Define $w'_k(x) = w_k(x_k + r_k x) - w_k(x_k)$. We claim that for large k

$$\sup_{x\in D_R} |\nabla w_k|(x) \leq 2.$$

Indeed, for large k and any $x \in D_R$, $x_k + r_k x \in D_{\frac{2}{3}}$,

$$(\frac{2}{3} - |x_k|)|\nabla w_k|(x_k) \geq (\frac{2}{3} - |x_k + r_k x|)|\nabla w_k|(x_k + r_k x),$$

which gives

$$|\nabla w'_k|(x) \quad = \quad r_k|\nabla w_k(x_k + r_k x)| = \frac{|\nabla w_k(x_k + r_k x)|}{|\nabla w_k(x_k)|} \leq \frac{\frac{2}{3} - |x_k|}{\frac{2}{3} - |x_k + r_k x|} \leq \frac{\frac{2}{3} - |x_k|}{\frac{2}{3} - |x_k| - r_k R} \leq 2,$$

for k large enough.

By the standard elliptic estimate, we may pass to some subsequence and assume that w'_k smoothly converges to a harmonic function w on every bounded domain of $\mathbb{R}^2$, such that $w(0) = 0$, $|\nabla w|(0) = 1$, $|\nabla w| \leq 2$. By Liouville's theorem for harmonic functions,∇w is constant, therefore $w = ax^1 + bx^2$ with $a^2 + b^2 = 1$. By rotation, we may assume $w = x^1$.

Next, we define

$$v'_k(x) = v_k(r_k x + x_k) - c_k, \quad u'_k(x) = v'_k(x) + w'_k(x),$$

where c_k is chosen such that $\int_D v'_k = 0$. Then for any $x \in D_R$, we have

$$\int_{D(x)} |\nabla v'_k|^p = r_k^{p-2}\int_{D_r(x_k + r_k x)} |\nabla v_k|^p \to 0.$$

By the Poincaré inequality (cf. [1, Theorem 5.4.3]) and Proposition 2.1, we may assume v'_k weakly converges to 0 in $W^{1,p}_{loc}(\mathbb{R}^2)$. By Corollary 2.2 and the Lebesgue-Vitali Convergence Theorem(cf. [2, Theorem 4.5.4]), for any $q \in [1, +\infty)$ and $R > 0$,

$$\int_{D_R} |e^{u'_k} - e^{x^1}|^q \to 0, \quad \int_{D_R} |e^{2u'_k} - e^{2x^1}| \to 0. \tag{3.1}$$

Next, we compute the area bound of the metric $g_0 := e^{2x^1} g_{euc}$. Let $T(\theta)$ be the constant such that

$$Length((\cos\theta, \sin\theta)t|_{t\in[0,T(\theta)]}, g_0) = R, \quad i.e. \quad \int_0^{T(\theta)} e^{r\cos\theta} dr = R.$$

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It is easy to verify that

$$T(\theta) = \frac{\log(1 + R\cos\theta)}{\cos\theta}, \quad i.e. \ \ e^{T(\theta)\cos\theta} = R\cos\theta + 1$$

Let $a > 0$ be sufficiently small and define

$$\Omega(R) = \{(r,\theta) : r \in (0, T(\theta)), \quad \theta \in (-\frac{\pi}{2} + a, \frac{\pi}{2} - a)\}.$$

Then

$$\begin{aligned}
\text{Area}(\Omega(R), g_0) &= \int_{-\frac{\pi}{2}+a}^{\frac{\pi}{2}-a} \int_0^{T(\theta)} e^{2r\cos\theta} r dr d\theta \\
&= \int_{-\frac{\pi}{2}+a}^{\frac{\pi}{2}-a} \left(\frac{e^{2r\cos\theta} r}{2\cos\theta} - \frac{e^{2r\cos\theta}}{4\cos^2\theta} \right)\Bigg|_0^{T(\theta)} d\theta \\
&= \int_{-\frac{\pi}{2}+a}^{\frac{\pi}{2}-a} \left(\frac{(R\cos\theta + 1)^2 T(\theta)}{2\cos\theta} - \frac{(R\cos\theta+1)^2 - 1}{4\cos^2\theta} \right) d\theta \\
&= R^2 \int_{-\frac{\pi}{2}+a}^{\frac{\pi}{2}-a} \left(\frac{1}{2}\log(1 + R\cos\theta) - \frac{1}{4} \right) d\theta + \int_{-\frac{\pi}{2}+a}^{\frac{\pi}{2}-a} RT(\theta) d\theta \\
&\quad + \int_{-\frac{\pi}{2}+a}^{\frac{\pi}{2}-a} \frac{T(\theta) - R}{2\cos\theta} d\theta \\
&= R^2 \int_{-\frac{\pi}{2}+a}^{\frac{\pi}{2}-a} \left(\frac{1}{2}\log(1 + R\cos\theta) - \frac{1}{4} \right) d\theta + O(R\log R).
\end{aligned}$$

Therefore,

$$\lim_{R\to+\infty} \frac{\text{Area}(\Omega(R), g_0)}{\pi R^2} = +\infty.$$

Since $\Omega(R) \subset B_R(0, d_{g_0})$, it follows that

$$\lim_{R\to+\infty} \frac{\text{Area}(B_R(0, d_{g_0}, g_0))}{\pi R^2} = +\infty.$$

Let $T_1 = \max_{|\theta| \le \frac{\pi}{2} - a} T(\theta)$. Fix $q > 2$. By the first part of (3.1) and Hölder's inequality,

$$\int_0^{2\pi} \int_0^{T_1} |e^{u'_k(r,\theta)} - e^{w(r,\theta)}| dr d\theta = \int_0^{2\pi} \int_0^{T_1} |e^{u'_k(r,\theta)} - e^{w(r,\theta)}| r^{\frac{1}{q}} r^{-\frac{1}{q}} dr d\theta \le C \|e^{u'_k} - e^w\|_{L^q(D_{T_1})} \to 0.$$

Passing to a subsequence, we have $\int_0^{T_1} |e^{u'_k(r,\theta)} - e^{w(r,\theta)}| dr$ converges to 0 for a.e. $\theta \in S^1$. For small $\epsilon > 0$, by Egorov's theorem there exists $A \subset S^1$ with Lebesgue measure less than ϵ such that $\int_0^{T_1} |e^{u'_k} - e^w| dr \to 0$ uniformly on $S^1 \setminus A$. Set $g'_k = e^{2u'_k} g_{euc}$ and

$$\Omega(R, A) = \Omega(R) \setminus \{(r,\theta) : \theta \in A\}.$$

By the Trace Embedding Theorem, for large k,

$$Length((\cos\theta, \sin\theta)t|_{t\in[0,T(\theta)]}, g'_k) < R + 1, \quad \forall \theta \in S^1 \setminus A,$$

so

$$\Omega(R, A) \subset B_{R+1}^{g'_k}(0).$$

By the second part of (3.1), for small $\epsilon > 0$ and large k,

$$\text{Area}(\Omega(R, A), g'_k) \ge \frac{1}{2}\text{Area}(\Omega(R), g_0).$$

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Therefore, for suitable $R > 0$,

$$\frac{\mathrm{Area}(B^{g_k'}_{R+1}(0), g_k')}{\pi(R+1)^2} = \frac{\mathrm{Area}(B^{g_k}_{(R+1)r_k}(x_k), g_k)}{\pi((R+1)r_k)^2} > \Lambda,$$

which contradicts the quadratic area bound condition for g_k. $\square$

Next, we remove the assumption that $|\mathbb{K}_g|(D) < \epsilon_0$.

Theorem 3.4. *Let $g = e^{2u}g_{euc} \in \mathcal{M}(D)$. Assume g has quadratic area bound with constant Λ and $|\mathbb{K}_g|(D) < \Lambda'$. Then for any $p \in [1, 2)$, there exists a constant $C = C(\Lambda, \Lambda', p)$ such that*

$$\|\nabla u\|_{L^p(D_{\frac{1}{2}})} < C.$$

Proof. Suppose the conclusion fails, then there exists a sequence $g_k = e^{2u_k}g_{euc}$ with $|\mathbb{K}_{g_k}|(D) < \Lambda'$ and quadratic area bound with constant Λ such that

$$\|\nabla u_k\|_{L^p(D_{\frac{1}{2}})} \to +\infty.$$

Let $v_k = I_{\mathbb{K}_{g_k}}$ and $w_k = u_k - v_k$. By Proposition 2.1,

$$r^{p-2}\int_{D_r(x)} |\nabla v_k|^p < C, \quad \forall D_r(x) \subset D.$$

Thus,

$$\sup_{D_{\frac{1}{2}}} |\nabla w_k| \to +\infty.$$

Passing to a subsequence if necessary, we may assume that $|\mathbb{K}_{g_k}|$ converges weakly to a Radon measure μ. Let $\epsilon_0 > 0$ be as in 3.3 and set

$$\mathcal{S} = \{x \in D : \mu(\{x\}) > \frac{\epsilon_0}{2}\}.$$

Clearly, $\mathcal{S}$ is a finite set. For any $x \notin \mathcal{S}$, we can find $r > 0$, such that $|\mathbb{K}_{g_k}|(D_r(x)) < \epsilon_0$ for sufficiently large k. Then $\|\nabla u_k\|_{L^p(D_{r/2}(x))} < C(r)$, which implies that $\|\nabla w_k\|_{L^p(D_{r/2}(x))} < C(r)$. By the standard elliptic estimate, after passing to a subsequence we may assume that there exist constants c_k(for instance, fix a point $p_0 \notin \mathcal{S}$ and let $c_k = w_k(p_0)$) and a harmonic function w, such that $w_k - c_k \to w$ smoothly on Ω for any $\Omega \subset\subset D \setminus \mathcal{S}$.

Choose $r_0 > 0$ such that for any distinct $p, p' \in \mathcal{S}$, $D_{r_0}(p) \cap D_{r_0}(p') = \emptyset$. For any $x \in D_{r_0/4}(p)$, $\partial D_{r_0/2}(x) \subset D_{r_0}(p) \setminus D_{r_0/4}(p)$. Applying the mean value property of harmonic functions on $\partial D_{r_0/2}(x)$, it turns out that $w_k - c_k$ are uniformly bounded on $D_{r_0/4}(p)$. Therefore, $|\nabla w_k|$ are uniformly bounded on $D_{r_0/4}(p)$, hence $|\nabla w_k|$ are uniformly bounded on $D_{\frac{1}{2}}$, which leads to a contradiction. $\square$

Proof of Theorem 1.1: Without loss of generality, we assume $|\mu|(\Sigma) = 1$. Choose a smooth metric g with $\int_\Sigma |K_g| dV_g < C(\mathfrak{g})$, where $C(\mathfrak{g})$ depends only on the genus $\mathfrak{g}$ of Σ. Set $g' = e^{2u}g \in \mathcal{M}(\Sigma, g)$. Then

$$\mathbb{K}_{g'} = \mu - K_g dV_g.$$

Moreover, we have $|\mathbb{K}|_{g'}(\{x\}) < 2\pi$ for any x, so $d_{g'}$ is finite. By Theorem 2.4, both g' and g have quadratic area bound with some constant $\Lambda > 0$.

For any $\varphi \in \mathfrak{C}$, φ defines a local isothermal coordinate chart. In this chart, we write

$$\varphi^*(g) = e^{2v}g_{\mathrm{euc}}, \quad \varphi^*(g') = e^{2(u+v)}g_{\mathrm{euc}}.$$

By Theorem 3.4,

$$\|\nabla v\|_{L^p(D_{\frac{1}{2}})} < C, \quad \|\nabla(u+v)\|_{L^p(D_{\frac{1}{2}})} < C$$

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which implies the first part of Theorem 1.1.

Next, we consider $\varphi \in \mathfrak{C}_a$ and define v as above. For simplicity, let $a = 2^{-m}$ with $m > 2$ being a positive integer. Note that for any $x \in D_{2^{2-j}} \setminus \overline{D_{2^{1-j}}}$(where $2 < j \leq m$), we have $D_{2^{-j}}(x) \subset D \setminus \overline{D_a}$. By a rescaling argument,

$$\int_{D_{2^{-j}}(x)} |\nabla u|^p \leq C(2^{-j})^{2-p}.$$

By a covering argument,

$$\int_{D_{2^{-i}} \setminus D_{2^{-i-1}}} |\nabla u|^p \leq C(2^{-i})^{2-p},$$

so

$$\int_{D_{2^{-1}} \setminus D_{2^{-m-1}}} |\nabla u|^p dx \leq C \sum_{i=1}^{m} (2^{-i})^{2-p}.$$

For the case $a = 0$, letting $m \to \infty$ we get

$$\int_{D_{2^{-1}} \setminus \{0\}} |\nabla u|^p \leq C \sum_{i=1}^{\infty} (2^{-i})^{2-p}.$$

□

4. Estimates on a surface of constant curvature

In this section, we assume that $K_g = 0$ or -1 on Σ and that u is a solution of (1.1). The main aim is to prove Corollary 1.4, as well as (1.4) and (1.5). For simplicity, we also assume $|\mu|(\Sigma) = 1$ throughout the section.

4.1. **The flat case.** We first consider the case $K_g = 0$. Note that (1.3) and (1.4) hold for g if and only if they hold for λg for any $\lambda > 0$. Since the lattices $\{1, z_1\}$ and $\{1, z_2\}$ induce the same complex structure if $z_2 = \frac{a_{11}z_1 + a_{12}}{a_{21}z_1 + a_{22}}$, where $(a_{ij}) \in SL(2, \mathbb{Z})$(see the subsection IV.7.3 of [6]), there is no loss of generality in assuming that (Σ, h) is induced by the lattice $\{1, a + b\sqrt{-1}\}$ in $\mathbb{C}$, where $|a| \leq \frac{1}{2}$, $b > 0$, and $a^2 + b^2 \geq 1$. Set

$$\rho = \sqrt{a^2 + b^2}, \quad \theta = \arccos \frac{a}{\rho}, \quad v = (\rho, 0). \quad w = (\cos\theta, \sin\theta).$$

Then (Σ, h) is also induced by the lattice $\{v, w\}$, where it necessarily holds that $\rho \geq 1$ and $\frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3}$.

Let $\Pi : \mathbb{C} \to \Sigma$ be the covering map. Let u' denote the lift of u, i.e. $u' = u \circ \Pi$. For any $x \in \mathbb{C}$, and $r < \frac{\sqrt{3}}{4}$, $D_r(x)$ can be viewed as an isothermal coordinate chart of Σ around $\Pi(x)$ (Indeed, the inscribed circle of a fundamental parallelogram has radius at least $\frac{\sqrt{3}}{4}$). By Theorem 1.1,

$$\int_{D_r(x)} |\nabla u'|^p dx \leq Cr^{2-p}.$$

By setting $r = \frac{\sqrt{3}}{8}$ and using a covering argument, we have

$$\int_{[i,i+1] \times [-\frac{\sin\theta}{2}, \frac{\sin\theta}{2}]} |\nabla u'|^p < C.$$

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Without loss of generality, we assume 0 is a lift of x_0. When $r < 3$, note that $r/8 < \frac{\sqrt{3}}{4}$. Covering $D_r(0)$ with finitely many $D_{\frac{r}{8}}(x)$ (having a universal upper bound on the number of disks) gives

$$\int_{B_r^g(x_0)} |\nabla_g u|^p dV_g \le \int_{D_r(0)} |\nabla u'|^p dx \le Cr^{2-p}.$$

When $p = 1$, this yields (1.3). When $p \in (1,2)$, since $0 < C \le \frac{\mathrm{Area}(B_r^g(x_0))}{\pi r^2} \le 1$, we obtain (1.4).

When $r > 3$, let m be the smallest integer such that $r \le m$. It is easy to check that if $x \in \Pi([-\frac{m}{2}, \frac{m}{2}] \times [-\frac{\sin\theta}{2}, \frac{\sin\theta}{2}])$, then

$$d_g(x, x_0) \le \sqrt{(\frac{m}{2})^2 + (\frac{\sin\theta}{2})^2} \le m - 1 \le r,$$

so

$$\Pi([-\frac{m}{2}, \frac{m}{2}] \times [-\frac{\sin\theta}{2}, \frac{\sin\theta}{2}]) \subset B_r^g(x_0) \subset \Pi([-m, m] \times [-\frac{\sin\theta}{2}, \frac{\sin\theta}{2}]), \tag{4.1}$$

Then

$$\int_{B_r(p)} |\nabla_g u|^p dV_g \le \int_{[-m,m]\times[-\frac{\sin\theta}{2},\frac{\sin\theta}{2}]} |\nabla u'|^p = \sum_{i=-m}^{m-1} \int_{[-m,m]\times[-\frac{\sin\theta}{2},\frac{\sin\theta}{2}]} |\nabla u'|^p \le Cm.$$

When $p = 1$, using $m - 1 < r$, we get (1.3). When $p \in (1,2)$, by (4.1),

$$\mathrm{Area}(B_r^g(x_0)) \le Cm < 2Cr,$$

so

$$\frac{r^p}{\mathrm{Area}^{p-1}(B_r^g(x_0))} \ge Cr^p/r^{p-1} = Cr,$$

which yields (1.4).

4.2. **On a hyperbolic surface.** Now, we assume $K_g = -1$.

Let r_x denote the injectivity radius of g at x. Since

$$\mathrm{Area}(B_{r_x}^g(x)) = 2\pi(\cosh r_x - 1) \le \mathrm{Area}(\Sigma) = 2\pi|\chi(\Sigma)|,$$

it follows that r_x is bounded above by a constant depending only on the genus of Σ.

It is well-known that for any $r < r_x$ $(B_r^g(x), g)$ is biholomorphic to (D, g_r), where

$$g_r = \left(\frac{2\sinh(r/2)}{1 - \sinh^2(r/2)|x|^2}\right)^2 g_{euc}.$$

By applying Theorem 1.1, one readily verifies that when $r \le r_x/2$,

$$\int_{B_r^g(x)} |\nabla_g u|^p dV_g \le Cr^{2-p}.$$

Let $a > 0$ be fixed and Σ_a be the set of points where $r_x \ge a$. By Vitali's 5-times covering theorem, there exist pairwise disjoint closed balls $\{\overline{B_{a/10}^g(x_i)}\}$ with $x_i \in \Sigma_a$ such that $\{\overline{B_{a/2}(x_i)}\}$ cover Σ_a. Then

$$\#\{x_i\} \le \frac{\mathrm{Area}(\Sigma)}{V_{a/10}},$$

where $V_{a/10}$ is the area of a disk of radius $a/10$ in the hyperbolic plane. Therefore

$$\int_{\Sigma_a} |\nabla_g u|^p < C\frac{|\chi(\Sigma)|}{a^2}\left(\frac{a}{2}\right)^{2-p} < C\frac{|\chi(\Sigma)|}{a^p},$$

which gives (1.5) by letting $a = \frac{1}{2}\mathrm{Inj}(\Sigma, \mathrm{g})$.

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Next, we consider the collar regions, which are components of $\Sigma \setminus \Sigma_{\text{arcsinh}\,1}$. For more details on collars, we refer the reader to [4]. In particular, we note that the total number of collar regions is bounded by $3\mathfrak{g} - 3$, a constant depending only on the genus.

Let γ be a geodesic loop of length $\ell(\gamma) < 2\,\text{arcsinh}\,1$, γ defines a collar $U \subset \Sigma$, isometric to $S^1 \times (-T, T)$ with the metric

$$\left(\frac{\lambda}{\cos \lambda t}\right)^2 (dt^2 + d\theta^2).$$

See the appendix for the definitions of T and λ.

We claim that for small $\lambda > 0$ and any t_1, t_2 satisfying $|t_1| < T - 1$,$|t_2| < T - 1$, $|t_2 - t_1| < 2$, there exists a universal constant $C_0 > 1$ such that

$$C_0^{-1} < \frac{\cos \lambda t_2}{\cos \lambda t_1} < C_0 \tag{4.2}$$

In fact,(4.2) holds with $C_0 = e^2$. Indeed, Lagrange's mean value theorem (together with the fact $\lambda(T-1) < \frac{\pi}{2} - \lambda$) gives that

$$|\log \cos \lambda t_2 - \log \cos \lambda t_1| \le \lambda |t_2 - t_1| \sup_{|t| < \frac{\pi}{2} - \lambda} |\tan t| < \frac{2\lambda}{\tan \lambda} < 2$$

From which the claim follows.

Let k, m be integers.We claim that

$$\int_{[k,m]\times S^1} |\nabla_g u| dV_g \le C d_{k,m}$$

for any $-T + 2 < k < m < T - 2$, where we define

$$d_{t,t'} = d_g(\{t\} \times S^1, \{t'\} \times S^1). \tag{4.3}$$

We first prove the case $k \ge 0$. Let $Q_i = [i, i+1] \times S^1$. By Theorem 1.1,

$$\int_{Q_i} |\nabla u| dt d\theta < C. \tag{4.4}$$

Since $t \mapsto \frac{\lambda}{\cos \lambda t}$ is increasing on $[k, m]$, (4.4) implies that

$$\int_{[k,m]\times S^1} |\nabla_g u| dV_g \le C \sum_{i=k}^{m-1} \frac{\lambda}{\cos \lambda (i+1)} \le C \int_{k+1}^{m+1} \frac{\lambda}{\cos \lambda s} ds.$$

Note that, by (4.2) and the monotonicity of $\cos t$ on $[0, \frac{\pi}{2}]$,

$$\int_{i-1}^{i} \frac{\lambda ds}{\cos \lambda s} \le \int_{i}^{i+1} \frac{\lambda ds}{\cos \lambda s} \le C \int_{i-1}^{i} \frac{\lambda ds}{\cos \lambda s}. \tag{4.5}$$

Hence

$$\int_{k+1}^{m+1} \frac{\lambda ds}{\cos \lambda s} \le C \int_{k+1}^{m-1} \frac{\lambda ds}{\cos \lambda s} \le C d_{k,m}.$$

Thus

$$\int_{[k,m]\times S^1} |\nabla_g u| dV_g \le C d_{k,m}.$$

When $k < 0$, we have

$$\int_{[k,m]\times S^1} |\nabla_g u| dV_g = \int_{[k,0]\times S^1} |\nabla_g u| dV_g + \int_{[0,m]\times S^1} |\nabla_g u| dV_g \le C(d_{k,0} + d_{0,m}) = C d_{k,m}.$$

This proves our claim.

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Next, we estimate $\int_{(B_r^g(x_0)\setminus\Sigma_a)\cap U}|\nabla u|$. Choose $a>0$ sufficiently small and assume $(B_r^g(x_0)\setminus\Sigma_a)\cap U\neq\emptyset$, then $\frac{1}{2}\ell(\gamma)=\min_{x\in U}r_x<a$. We can choose a such that (see the calculations in the appendix)

$$\inf_{t\in(-T,-T+10)\cup(T-10,T)} r_{(t,\theta)}>a, \quad \textit{and} \quad d_{T-10,T}>2a.$$

We have two cases:

Case 1. $B_r^g(x_0)\subset[k-1,k+1]\times S^1\subset(-T+5,T-5)\times S^1$. Without loss of generality, we assume $x_0=(t_0,0)$.

It suffices to consider the case where $r>r_{x_0}/2$. Then there exists a unit-speed geodesic $\gamma:[0,2r_{x_0}]\to B_{2r}^g(x_0)$ with $\gamma(0)=\gamma(2r_{x_0})$. In particular, γ must pass through a point $x_0'=(t_0',\pi)$. By (4.2), we have $0<\cos\lambda t\le C\cos\lambda t_0$, which implies

$$\frac{\lambda}{\cos\lambda t_0}\le Cd(x_0',x_0)\le 2Cr.$$

Consequently,

$$\int_{B_r^g(x_0)}|\nabla_g u|dV_g\le\frac{\lambda}{\cos\lambda t_0}\int_{[t_0-1,t_0+1]\times S^1}|\nabla u|dtd\theta\le C\frac{\lambda}{\cos\lambda t_0}\le Cr.$$

Case 2. Case 1 does not hold. Let t_1 and t_2 be the smallest and largest $t\in[-T+5,T-5]$ respectively, such that $\{t\}\times S^1\cap\overline{B_r^g(x_0)}\neq\emptyset$. Then

$$d_g(\{t_1\}\times S^1,\{t_2\}\times S^1)\le 2r.$$

Let $m=[t_2]+1$, $k=[t_1]-1$. By (4.5),

$$d_{k,m}\le Cd_{t_1,t_2}\le Cr.$$

Thus

$$\int_{B_r(x_0)\setminus\Sigma_a\cap U}|\nabla_g u|dV_g\le Cd_{k,m}\le Cr.$$

We are now in a position to prove (1.3). When $r<a$, we have already proved it for x_0 in either Σ_a or $\Sigma\setminus\Sigma_a$. When $r\ge a$, since Σ has at most $3\mathfrak{g}-3$ collars, we have

$$\int_{B_r^g(x_0)}|\nabla_g u|dV_g=\int_{B_r^g(x_0)\cap\Sigma_a}|\nabla_g u|dV_g+\int_{B_r^g(x_0)\setminus\Sigma_a}|\nabla_g u|dV_g\le C+Cr\le C'r.$$

□

5. Appendix

In this appendix, we list some facts about collars.

Let γ be a geodesic loop with length $\ell(\gamma)<2\operatorname{arcsinh}1$, and

$$w=\operatorname{arcsinh}\frac{1}{\sinh(\frac{1}{2}\ell(\gamma))}.$$

By [4, Theorem 4.1.6], γ defines a collar $U\subset\Sigma$, isometric to $S^1\times(-w,w)$ with the metric

$$g=d\rho^2+\ell^2(\gamma)\cosh^2\rho ds^2=\left(\frac{\ell(\gamma)\cosh\rho}{2\pi}\right)^2\left(\left(\frac{2\pi d\rho}{\ell(\gamma)\cosh\rho}\right)^2+d\theta^2\right),$$

where $s=\frac{\theta}{2\pi}$.

Let

$$(t,\theta)=\phi(\rho,\theta)=\left(\frac{4\pi\arctan e^\rho}{\ell(\gamma)},\theta\right).$$

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Then ϕ is a diffeomorphism from $(-w,w)\times S^1$ to

$$\left(\frac{4\pi\arctan e^{-w}}{\ell(\gamma)},\frac{4\pi\arctan e^{w}}{\ell(\gamma)}\right)\times S^1$$

with

$$\phi*(g)=\ell^2(\gamma)\cosh^2\rho(dt^2+d\theta^2)=\left(\frac{\ell(\gamma)}{2\pi\sin\frac{\ell(\gamma)t}{2\pi}}\right)^2(dt^2+d\theta^2).$$

Set

$$T=\frac{4\pi\arctan e^{w}}{\ell(\gamma)}-\frac{\pi^2}{\ell(\gamma)},\quad \lambda=\frac{\ell(\gamma)}{2\pi}.$$

Then the collar is also isometric to

$$\left((-T,T)\times S^1,\left(\frac{\lambda}{\cos\lambda t}\right)^2(dt^2+d\theta^2)\right).$$

For $-T\le t<t'\le T$, recall that we have defined (see(4.3))

$$d_{t,t'}=d_g(\{t\}\times S^1,\{t'\}\times S^1).$$

We compute $d_{T-t,T}$ and $r_{(T-t,\theta)}$ for fixed t and small $\ell:=\ell(\gamma)$. Recall the asymptotics as $x\to+\infty$

$$\arctan x=\frac{\pi}{2}-\frac{1}{x}+O(\frac{1}{x^2}),\quad \operatorname{arcsinh}x=\log(2x)+O(\frac{1}{x^2}).$$

We have

$$w=\operatorname{arcsinh}\frac{1}{\ell/2+O(\ell^3)}=\operatorname{arcsinh}(\frac{2}{\ell}+O(\ell))=\log\frac{4}{\ell}+O(\ell),$$

Then it follows from Lagrange's mean value theorem that $e^{-w}=\frac{\ell}{4}+O(\ell^2)$, hence

$$\arctan e^{w}=\frac{\pi}{2}-\frac{\ell}{4}+O(\ell^2),\quad T=\frac{\pi^2}{\ell}-\pi+O(\ell).$$

Thus

$$\begin{aligned}d(t):=d_{T-t,T}&=\int_{T-t}^{T}\frac{\lambda}{\cos\lambda s}ds=\int_{\lambda(T-t)}^{\lambda T}\frac{ds}{\cos s}=\log\frac{1+\sin\lambda T}{1+\sin\lambda(T-t)}-\log\frac{\cos\lambda T}{\cos\lambda(T-t)}\\&=-\log\frac{\pi}{\pi+t}+O(\ell),\end{aligned}$$

by [4, Theorem 4.1.6] again

$$\sinh r_{(T-t,\theta)}=\cosh\frac{\ell}{2}\cosh d(t)-\sinh d(t)=e^{-d(t)}+O(\ell)=\frac{\pi}{\pi+t}+O(\ell),$$

so

$$r_{(T-t,\theta)}=\operatorname{arcsinh}\frac{\pi}{\pi+t}+O(\ell).$$

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