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FOURIER SERIES WITH SMALL GAPS

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Abstract

A trigonometric series has "small gaps" if the difference of the orders of successive terms is bounded below by a number exceeding one. Wiener, Ingham and others have shown that if a function represented by such a series exhibits a certain behavior on a large enough subinterval I, this will have consequences for the behavior of the function on the whole circle group. Here we show that the assumption that f is in any one of various classes of functions of generalized bounded variation on I implies that the appropriate order condition holds for the magnitude of the Fourier coefficients. A generalized bounded variation condition coupled with a Zygmund-type condition on the modulus of continuity of the restriction of the function to I implies absolute convergence of the Fourier series.

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1. Introduction

A trigonmetric series $\sum a_n \cos nx + b_n \sin nx = \sum A_n(x)$ is said to be *lacunary* if it exhibits large gaps, that is, $a_n = b_n = 0$ except for $n \in \{n_k\}$ where $n_{k+1}/n_k \geq q > 1$, $k = 1, 2, \ldots$. Such series have been extensively studied (Zygmund [10], Chapter V, Sections 6–8; Bary [1], Chapter XI) with results of the following character: some particular local property of the series implies that it has a certain global property.

We say that a trigonometric series $\sum A_{n_k}(x)$ has "small gaps" if $n_{k+1} - n_k \geq q > 1$. Although this case is briefly treated in the standard sources ([10], Chapter V, §9; [1], Chapter XI, §13), the literature is not well known and we will review it briefly.

These problems seem first to have been considered by Wiener [9], who showed that if a series with small gaps behaves well in a large enough interval, then some aspects of this behavior hold for $[0, 2\pi]$. His primary interest was the generalization of the Hadamard gap theorem.

Wiener's result was improved by Ingham [3] who employed it to establish a generalized form of the Hardy-Littlewood "high-indices" theorem. Kennedy [4] established results on nonharmonic trigonometrical series which specialize to the harmonic case as follows: If $n_{k+1} - n_k \to \infty$ and $\sum A_{n_k}(x)$ is the Fourier series of an integrable function f, then

(i) f of bounded variation (BV) on an interval I implies $a_n, b_n = O(n^{-1})$.

(ii) $f \in \Lambda_\alpha$ on I, $0 < \alpha < 1$, implies $a_n, b_n = O(n^{-\alpha})$.

(iii) $f \in \Lambda_\alpha$ on I, $\frac{1}{2} < \alpha < 1$, implies $\sum |a_n| + |b_n| < \infty$.

(iv) $f \in \Lambda_\alpha$ on I, $0 < \alpha < 1$, and $f \in BV$ implies $\sum |a_n| + |b_n| < \infty$.

Here Λ_α denotes the Lipschitz class of order α. Actually he showed these results hold if $\underline{\lim}(n_{k+1} - n_k) > 32\pi/|I|$ and, therefore hold regardless of the length of I if $n_{k+1} - n_k \to \infty$.

Noble [5] had established the above conclusions under the assumption $(n_{k+1} - n_k)/\log n_k \to \infty$. Bary [1], Chapter XI, §13, reports on the work of Noble and slight generalizations of it by Ul'yanov. Her comment to the effect that Kennedy required $f \in L^2([0, 2\pi])$ in his results is misleading. Patadia [6] has shown that a similar extension of Stečkin's theorem on absolute convergence ([1, page 196]) can be made by assuming the conditions of that theorem to hold on an arbitrary interval and requiring further $n_{k+1} - n_k \to \infty$.

2. Definitions and background

Let f be a real function defined on the circle group $T([0, 2\pi))$. $\{I_n\}$ will denote a collection of nonoverlapping intervals in T. If $I = [a, b]$, then $f(I) = f(b) - f(a)$.

If $\Lambda = \{\lambda_n\}$ is a nondecreasing sequence of positive real numbers such that $\sum 1/\lambda_n = \infty$, we say that f is of Λ-bounded variation (ΛBV) if $\sum_1^\infty |f(I_n)|/\lambda_n < \infty$ for every $\{I_n\}$. This is known to imply that the collection of sums $\sum |f(I_n)|/\lambda_n$ is bounded [8].

Let $\varphi(x)$ be a nonnegative convex function defined on $[0, \infty)$ such that $\varphi(x)/x \to 0$ as $x \to 0$. We say that f is of φ-bounded variation (φBV) if for some $c > 0$, $\sup\{\sum \varphi(c|f(I_n)|) | \{I_n\}\} = V_c(f) < \infty$.

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If $h(n)$ is a nondecreasing concave-downward function on the positive integers, we say that $f \in V[h]$ if there is a constant C such that $\sum_1^n |f(I_k)| \leq Ch(n)$, $n = 1, 2, \ldots$, for every collection $\{I_n\}$.

We say that f is in one of the classes on $I \subset T$ if, in the definition, we restrict $\{I_n\}$ by $I_n \subset I$.

We concern ourselves here with the known estimates of the order of magnitude of the Fourier coefficients for functions in these classes [7] and with a condition for the absolute convergence of Fourier series of $V[n^\alpha]$ [2], showing that these results hold if the conditions are satisfied on a (large enough) small interval $I \subset T$.

3. Statement of results

We suppose throughout that f is a real function in $L^1(T)$ with Fourier series $\sum c_{n_k} e^{in_k x}$, $n_{-k} = -n_k$, satisfying $n_{k+1} - n_k \geq q > 1$, $k = 0, 1, 2, \ldots$. Let $I \subset T$ be a closed interval with length $|I| = (1+\delta)2\pi/q$, $\delta > 0$.

We have the following results.

THEOREM 1. *With f and I as above,*
(i) *$f \in V[h]$ on I implies $c_n = O(h(|n|)/|n|)$.*
(ii) *$f \in \Lambda BV$ on I implies $c_n = O(1/\sum_1^{|n|} 1/\lambda_i)$.*
(iii) *$f \in \varphi BV$ on I implies $c_n = O(\varphi^{-1}(1/|n|))$.*

THEOREM 2. *Let f and I be as above. Let $\omega_I(f,t)$ be the modulus of continuity of f restricted to I. If $f \in V[n^\alpha]$ on I, $0 \leq \alpha < \frac{1}{2}$, and*

$$\sum_{n=1}^{\infty} \frac{1}{n} \omega_I^{(1-2\alpha)/2(1-\alpha)}\left(f, \frac{1}{n}\right) < \infty,$$

then the Fourier series of f converges absolutely.

It is clear that if we make the assumption $n_{k+1} - n_k \to \infty$, then the conclusions hold for any nondegenerate interval I.

4. Preliminaries

The proofs of our theorems rest on two other results:

LEMMA 1. *Let $f \in L^1(T) \cap L^2(I)$, I a closed subinterval of T. The sequence of partial sums of the Fourier series of f converges to f in $L^2(I')$ for any closed interval I' in the interior of I.*

PROOF. With I and I' as above, let $g = f$ on I and $g = 0$ on $T \backslash I$. Let $S_n(\cdot)$ be the nth partial sum of the Fourier series of a function. Since $g \in L^2(T)$ we have $\|S_n(g) - g\|_{L^2(T)} \to 0$ as $n \to \infty$. Therefore,

$$\begin{aligned}\|S_n(f) - f\|_{L^2(I')} &\leq \|S_n(f) - S_n(g)\|_{L^2(I')} + \|S_n(g) - f\|_{L^2(I')} \\ &\leq \|S_n(f) - S_n(g)\|_{L^2(I')} + \|S_n(g) - g\|_{L^2(T)} \\ &= o(1),\end{aligned}$$

since the localization principle implies that $S_n(f) - S_n(g) \to 0$ uniformly on I'.

Lemma 1 enables us to extend a result due to Ingham [3] as follows.

LEMMA 2. *Let $\sum c_{n_k} e^{in_k x}$ be the Fourier series of a function $f \in L^1(T) \cap L^2(I)$ where $-n_k = n_{-k}$, $n_{k+1} - n_k \geq q > 1$, $k = 0, 1, 2, \ldots$, and $|I| = (1+\delta)2\pi/q$ for some $\delta > 0$. Then*

$$\sum_{-\infty}^{\infty} |c_{n_k}|^2 \leq (A_\delta/|I|) \int_I |f(x)|^2 \, dx$$

where $A_\delta = 2\pi(1+\delta)^2/4\delta(2+\delta)$.

PROOF. Let I_η be a closed interval of length $(1+\eta)2\pi/q$ concentric with I, $\eta < \delta$, and let $S_N(x) = \sum_{k=-N}^{N} c_{n_k} e^{in_k x}$. Then by [10], Chapter V, Theorem 9.1

$$\sum_{k=-N}^{N} |c_{n_k}|^2 \leq (A_\eta/|I_\eta|) \int_{I_\eta} |S_N(x)|^2 \, dx.$$

By Lemma 1, $S_n \to f$ in $L^2(I_\eta)$ as $N \to \infty$ and therefore $\int_{I_\eta} |S_N|^2 \to \int_{I_\eta} |f|^2$, implying

$$\sum_{-\infty}^{\infty} |c_{n_k}|^2 \leq A_\eta/|I_\eta| \int_{I_\eta} |f(x)|^2 \, dx.$$

Letting η increase to δ we have the desired result.

5. Proof of Theorem 1

Let $|I| = 2\pi/q + 2\varepsilon$ and let I' be the concentric interval of length $2\pi/q + \varepsilon$. Consider $|k|$ so large that $0 < 2\pi/|n_k| < \varepsilon/2$ and let $N = N_k = [(|n_k|\varepsilon/4\pi) - 1]$.

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For $j = 1, \pm 1, \pm 2, \ldots, \pm N$, let

$$g_j(x) = f(x + 2\pi j/|n_k| + \pi/2|n_k|) - f(x + 2\pi j/|n_k| - \pi/2|n_k|).$$

Then, for every s,

$$\hat{g}_j(n_s) = 2ic_{n_s} e^{i2\pi j n_s/|n_k|} \sin n_s\pi/2|n_k|.$$

(i) If $f \in V[h]$ on I, then g_j is bounded on I' and, therefore $\sum_{-N}^{N} g_j \in L^2(I')$. By Lemma 2.

$$\left|\sum_{j=-N}^{N} \hat{g}_j(n_k)\right|^2 \leq \sum_{S=-\infty}^{\infty} \left|\sum_{j=-N}^{N} \hat{g}_j(n_s)\right|^2 \leq C \int_{I'} \left|\sum_{-N}^{N} g_j(x)\right|^2 dx.$$

(The letter C will denote various constants independent of k.) Since $\hat{g}_j(n_k) = 2ic_{n_k} \operatorname{sgn} n_k$ and $0 < C \leq (2N+1)/|n_k| \leq C'$ for some constants C and C' (that is, $2N + 1 \sim |n_k|$), it follows that

$$(*) \qquad \begin{aligned} |c_{n_k}|^2 &\leq \frac{C}{|n_k|^2} \int_{I'} \left|\sum_{-N}^{N} g_j(x)\right|^2 dx \\ &\leq \frac{C}{|n_k|^2} \int_{I'} \left(\sum_{-N}^{N} |g_j(x)|\right)^2 dx \\ &\leq \frac{C}{|n_k|^2} (h(2N+1))^2 \\ &\leq \frac{C}{|n_k|^2} (h(|n_k|))^2 \end{aligned}$$

which establishes (i).

(ii) If $f \in \Lambda BV$ on I we observe that

$$\left(\sum_{j=-N}^{N} |g_j(x)|\right) \left(\sum_{i=1}^{2N+1} 1/\lambda_j\right) \leq C(2N+1).$$

Thus from $(*)$ we have

$$|c_{n_k}| \leq \frac{C}{|n_k|} (2N+1) / \sum_{j=1}^{2N+1} 1/\lambda_j \leq C / \sum_{1}^{|n_k|} 1/\lambda_j.$$

This establishes (ii).

(iii) We observe first that for small enough $\alpha > 0$ (independent of k),

$$\varphi\left(\frac{1}{2N+1} \sum_{-N}^{N} \alpha c |g_j(x)|\right) \leq \frac{\alpha}{2N+1} V_c(f) < 1/|n_k|$$

implying

$$\sum_{-N}^{N} |g_j(x)| < C(2N+1)\varphi^{-1}(1/|n_k|).$$

Thus, from (*) we have

$$|c_{n_k}| \leq \frac{C}{|n_k|}(2N+1)\varphi^{-1}(1/|n_k|)$$

which yields (iii).

6. Proof of Theorem 2

Let $f \in V[n^\alpha]$ on I and let g_j and I' be as in the proof of Theorem 1. From Lemma 2 we have

$$\sum_{S=-\infty}^{\infty} |\hat{g}_j(n_s)|^2 = \sum_{S=-\infty}^{\infty} 4|c_{n_s}|^2 \sin^2 n_s\pi/2|n_k| \leq C\int_{I'} |g_j(x)|^2\,dx$$

for $j = 0, \pm 1, \ldots, \pm N$. For s such that $|n_k|/2 < |n_s| \leq |n_k|$, we have

$$\sin^2 n_s\pi/2|n_k| \geq \tfrac{1}{2};$$

letting $\sum^*$ indicate summation over *these* values of s,

$$\sum\nolimits^* |c_{n_s}|^2 \leq C\int_{I'} |g_j(x)|^2\,dx.$$

Summing over j and noting that $2N+1 \sim |n_k|$ we have

$$\sum\nolimits^* |c_{n_s}|^2 \leq \frac{C}{|n_k|}\int_{I'} \sum_{-N}^{N} |g_j(x)|^2\,dx.$$

Fix $x \in I'$ and for each integer $m \geq 0$, let

$$E_m = \{j | 2^{-(m+1)}\omega_I(f, \pi/|n_k|) < g_j(x) \leq 2^{-m}\omega_I(f, \pi/|n_k|)\}.$$

Since $|g_j(x)| \leq \omega_I(f, \pi/|n_k|)$, each j belongs to one and only one E_m. Let σ_m be the cardinality of E_m. Since $f \in V[n^\alpha]$ on I,

$$\sigma_m 2^{-(m+1)}\omega_I(f, \pi/|n_k|) \leq \sum_{j \in E_m} |g_j(x)| \leq C\sigma_m^\alpha,$$

implying

$$\sigma_m \leq C 2^{m/(1-\alpha)}\omega_I^{-1/(1-\alpha)}(f, \pi/|n_k|).$$

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Therefore

$$\sum_{j=-N}^{N} |g_j(x)|^2 = \sum_{m} \sum_{j \in E_m} |g_j(x)|^2 \le \sum_{m} \sigma_m 4^{-m} \omega_I^2(f, \pi/|n_k|)$$
$$\le C \sum_{m} 2^{-m(1-2\alpha)/(1-\alpha)} \omega_I^{(1-2\alpha)/(1-\alpha)}(f, \pi/|n_k|)$$
$$\le C \omega_I^{(1-2\alpha)/(1-\alpha)}(f, \pi/|n_k|),$$

implying

$$\sum\nolimits^{*} |c_{n_s}|^2 \le \frac{C}{|n_k|} \omega_I^{(1-2\alpha)/(1-\alpha)}(f, \pi/|n_k|).$$

For large m, if there is n_s such that $2^m < n_s \le 2^{m+1}$, let $k = \max\{s | n_s \le 2^{m+1}\}$. For such an m,

$$\sum_{2^m < |n_s| \le 2^{m+1}} |c_{n_s}|^2 \le \sum\nolimits^{*} |c_{n_s}|^2 \le \frac{C}{|n_k|} \omega_I^{(1-2\alpha)/(1-\alpha)}(f, \pi/|n_k|)$$
$$\le C 2^{-m} \omega_I^{(1-2\alpha)/(1-\alpha)}(f, \pi/2^m).$$

Hence,

$$\sum_{2^m < |n_s| \le 2^{m+1}} |c_{n_s}| \le C \omega_I^{(1-2\alpha)/2(1-\alpha)}(f, \pi/2^m).$$

The theorem follows then from the fact that the convergence of $\sum_{-\infty}^{\infty} |c_{n_s}|$ is implied by the convergence of

$$\sum_{0}^{\infty} \omega_I^{(1-2\alpha)/2(1-\alpha)}(f, \pi/2^m),$$

and the convergence of this last series is equivalent to that of

$$\sum_{n=1}^{\infty} \frac{1}{n} \omega_I^{(1-2\alpha)/2(1-\alpha)}\left(f, \frac{1}{n}\right).$$

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