DIRECT THEOREMS ON METHODS OF SUMMABILITY II

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1. Introduction

1.1. This paper is a continuation of the papers of the author [14], [15]. We begin by recapitulating the main definitions. If $\{n_\nu\}$ is an increasing sequence of positive integers, the value of the *characteristic or the counting function* $\omega(n)$ of $\{n_\nu\}$ is, for any $n \geqslant 0$, the number of n_ν satisfying the inequality $n_\nu \leqslant n$. Suppose that A is a linear method of summation corresponding to the transformation

$$\sigma_m = \sum_{n=0}^{\infty} a_{mn} s_n \qquad (m = 0, 1, \ldots). \tag{1.1(1)}$$

In what follows, $\Omega(n)$ is always a non-decreasing positive function defined for all real $n \geqslant 0$ and tending to $+\infty$ with n. A function $\Omega(n)$ is a *summability function of the first kind of a method* A if all real bounded sequences s_n such that $s_n = 0$ except for a sequence $\{n_\nu\}$ of values of n whose counting function $\omega(n) \leqslant \Omega(n)$, $n \geqslant 0$, are A-summable. $\Omega(n)$ is a *summability function of the second kind of a method* A if $S_n = s_0 + s_1 + \ldots + s_n = O(\Omega(n))$ implies that s_n is A-summable.

In [15] we have given necessary and sufficient conditions for summability functions of an arbitrary method A and have found all summability functions of some special methods. Here in §2 and §3 we solve the last problem for the Riesz and Abel methods $\mathrm{R}(\lambda_n, \kappa)$, $\kappa > 0$ and $\mathrm{A}(\lambda_n)$ (for the properties of these methods compare Hardy and Riesz [6], Hardy [5]). We have had to make some hypotheses on the regularity of the sequence λ_n (which are in most cases very modest). In §4 we discuss summability functions for absolute summability. Theorem 6 gives necessary and sufficient conditions for absolute summability functions, Theorem 7 describes methods which possess such functions. We also determine all absolute summability functions for some special methods. Thus for the Cesàro methods C_α, $\alpha > 0$ they are given by the condition $\sum n^{-1-\beta}\Omega(n) < +\infty$ ($\beta = \alpha$ for $\alpha \leqslant 1$, $\beta = 1$ for $\alpha \geqslant 1$) in contrast to the condition $\Omega(n) = o(n)$ which describes ordinary summability functions of C_α. Finally, in §5 we give applications of theorems of this and the previous papers. Of these we note Theorem 10, whose application is a good way to show that certain Tauberian conditions are the best possible of their kind.

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2. Summability functions of Riesz and Abel methods. Case when $\Delta\lambda_n$ is increasing

2.1. Let $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots, \lambda_n \to \infty$ be a given sequence and $\kappa > 0$. A series $\sum u_n$, or the sequence s_n of its partial sums, is $R(\lambda_n, \kappa)$ summable to s if

2.1(1)
$$v^{-\kappa} \sum_{\lambda_n \leqslant v} (v - \lambda_n)^\kappa u_n$$

converges to s for $v \to \infty$. And $\sum u_n$ is $A(\lambda_n)$ -summable to s, if

2.1(2)
$$\sigma(x) = \sum_{n=0}^{\infty} e^{-\lambda_n x} u_n \to s, \qquad x \to 0+.$$

We shall find it convenient to extend the definition of λ_n also to non-integral values of n and to consider a monotone continuous function $\lambda(\omega)$, $\omega \geqslant 0$ such that $\lambda(n) = \lambda_n$. Then we can write 2.1(1) in the form

2.1(3)
$$\begin{aligned}\sigma(\omega) &= \lambda(\omega)^{-\kappa} \sum_{n \leqslant \omega} (\lambda(\omega) - \lambda_n)^\kappa u_n \\ &= \lambda(\omega)^{-\kappa} \sum_{n \leqslant n_0 - 1} \left\{(\lambda(\omega) - \lambda_n)^\kappa - (\lambda(\omega) - \lambda_{n+1})^\kappa\right\} s_n + \lambda(\omega)^{-\kappa} (\lambda(\omega) - \lambda_{n_0})^\kappa s_{n_0},\end{aligned}$$

where $n_0 = [\omega]$. On the other hand, the expression 2.1(2) is equivalent to

2.1(4)
$$\sigma(x) = \sum_{n=0}^{\infty} (e^{-\lambda_n x} - e^{-\lambda_{n+1} x}) s_n$$

for any $A(\lambda_n)$ -summable sequence s_n (see for instance [13, Theorem 10]).

In the sequel we seek to find all summability functions of the methods $R(\lambda_n, \kappa)$, $A(\lambda_n)$ in a simpler form than that given by general theorems [15, §2]. We first make the following remark. *Any of the methods* $R(\lambda_n, \kappa)$, $\kappa > 0$, $A(\lambda_n)$ *possesses summability functions if and only if*

2.1(5)
$$\Delta\lambda_n / \lambda_n \to 0 \text{ or } \lambda_{n+1}/\lambda_n \to 1 \qquad (\Delta\lambda_n = \lambda_{n+1} - \lambda_n).$$

In fact, if the method $R(\lambda_n, \kappa)$ has summability functions, the coefficients of the transformation 2.1(3) must converge uniformly to zero for $\omega \to \infty$ by [14, Theorem 8*]. In particular the last coefficient converges to zero, and this gives 2.1(5). And if 2.1(5) is true, the coefficients in 2.1(4) converge uniformly to 0:

$$e^{-\lambda_n x}(1 - e^{-\Delta\lambda_n x}) \leqslant C_1 e^{-\lambda_n x} \Delta\lambda_n x \leqslant C_2 \Delta\lambda_n / \lambda_n \to 0,$$

since $e^{-u}u$ is bounded for $u \geqslant 0$. Since $R(\lambda_n, \kappa) \subset A(\lambda_n)$ for $\kappa > 0$ [6, p. 39], the proof is complete.

2.2. To obtain further results we suppose some regularity of the sequence λ_n. In this section we shall suppose that $\Delta\lambda_n$ is increasing. A first consequence of this hypothesis together with 2.1(5) is that $\lambda_n / \Delta\lambda_n = O(n)$. For

$$\Delta\left(\frac{\lambda_n}{\Delta\lambda_n} - n\right) = \left(\frac{\Delta\lambda_n}{\Delta\lambda_{n+1}} - 1\right) - \frac{\lambda_n \Delta^2\lambda_n}{\Delta\lambda_n \Delta\lambda_{n+1}} \leqslant 0,$$

and thus $\lambda_n/\Delta\lambda_n - n$ is decreasing. Therefore, $\lambda_n/\Delta\lambda_n \leqslant n + C$ for some constant C. Theorems 1 and 2 below give full information about the summability functions of the first and the second kind. In Theorem 1 we suppose that $\Delta\lambda_n/\lambda_n \to 0$ (which is no restriction because of 2.1(5)), in Theorem 2 slightly more, namely that $\Delta\lambda_n/\lambda_n$ decreases to 0.

THEOREM 1. *If $\Delta\lambda_n/\lambda_n$ converges to zero and $\Delta\lambda_n$ increases, all summability functions (of the first kind) of the methods* $\mathrm{R}(\lambda_n, \kappa)$, $\kappa > 0$ *and* $\mathrm{A}(\lambda_n)$, *and only these functions, are given by*

2.2(1) $$\Omega(n) = o(\lambda_n/\Delta\lambda_n).$$

Proof. (a) *Every function $\Omega(n)$ satisfying* 2.2(1) *is a summability function of the method* $\mathrm{R}(\lambda_n, \kappa)$, $0 < \kappa \leqslant 1$. We have to show that 2.2(1) implies that $A(\omega, \Omega) \to 0$ for $\omega \to \infty$ [15, 2.3]. We recall that for a method of summation defined by $s = \lim_{\omega\to\infty} \sum_{n=1}^{\infty} a_n(\omega)s_n$ and a function $\Omega(n)$, $A(\omega, \Omega)$ is the least upper bound of $\sum_{\nu=1}^{\infty} |a_{n_\nu}(\omega)|$ for all sequences n_ν with the counting function $\leqslant \Omega(n)$. Because of 2.1(5) we may disregard the last coefficient in 2.1(3). For $n \leqslant n_0 - 1$ the coefficient

$$a_n(\omega) = -\lambda(\omega)^{-\kappa}\Delta(\lambda(\omega) - \lambda_n)^{\kappa} = \kappa\lambda(\omega)^{-\kappa}(\lambda(\omega) - \lambda'_n)^{\kappa-1}\Delta\lambda_n$$

(λ'_n is between λ_n and λ_{n+1}) is increasing with n. Therefore,

$$A(\omega, \Omega) \leqslant \sum_{n_0 - \Omega(\omega) \leqslant n \leqslant n_0 - 1} a_n(\omega) \leqslant \lambda(\omega)^{-\kappa}[\lambda(\omega) - \lambda(n_0 - \Omega(\omega))]^{\kappa}$$

$$\leqslant C\left[\frac{\Delta\lambda_{n_0+1}}{\lambda_{n_0+1}}(\Omega(\omega) + 2)\right]^{\kappa} \to 0$$

by 2.1(5) and 2.2(1). This proves (a).

(b) *Any summability function of the method* $\mathrm{A}(\lambda_n)$ *satisfies* 2.2(1). Suppose that 2.2(1) does not hold, then for some $\delta > 0$ and an infinity of n, $\Omega(n) \geqslant \delta\lambda_n/\Delta\lambda_n$. For these n define the integer n_1 by

2.2(2) $$\lambda_{n_1} \leqslant (1 + \delta)\lambda_n < \lambda_{n_1+1}.$$

For a fixed n of the above kind, we denote by $\Omega_1(\nu)$ the counting function of the set of integers ν defined by $n \leqslant \nu < n_1$. We have

$$n_1 - n \leqslant (\lambda_{n_1} - \lambda_n)/\Delta\lambda_n \leqslant \delta\lambda_n/\Delta\lambda_n,$$

and therefore $\Omega_1(n_1) \leqslant \Omega(n)$. Thus $\Omega_1(u) \leqslant \Omega(u)$ in $n \leqslant u < n_1$, and since Ω_1 is constant outside of this interval, the same inequality holds for all u. Therefore for the function $A(x, \Omega)$ of the method $\mathrm{A}(\lambda_n)$ we have

$$A(x, \Omega) \geqslant \sum_{n \leqslant \nu < n_1} (e^{-\lambda_\nu x} - e^{-\lambda_{\nu+1} x}) = e^{-\lambda_n x} - e^{-\lambda_{n_1} x}$$

$$= e^{-\lambda'_n x} x(\lambda_{n_1} - \lambda_n)$$

for some λ'_n between λ_n and λ_{n_1}. Here

$$\lambda_{n_1}-\lambda_n = \lambda_{n_1+1}-\lambda_n + o(\lambda_n) \geqslant \delta\lambda_n + o(1) \geqslant \tfrac{1}{2}\delta\lambda_n$$

for large n. Choosing $x_n = \lambda_n^{-1}$, we obtain $\lambda'_n x_n \leqslant 1 + \delta$ and therefore

$$A(x_n, \Omega) \geqslant \tfrac{1}{2}\delta e^{-(1+\delta)} = \text{const.} > 0,$$

so that $A(x,\Omega)$ does not tend to zero for $x \to \infty$, which proves (b) by [15, 2.3].

From (a) and (b) the theorem follows in virtue of the inclusions $\mathrm{R}(\lambda_n, \kappa) \subset \mathrm{R}(\lambda_n, \kappa') \subset \mathrm{A}(\lambda_n)$, $0 < \kappa < \kappa'$.

2.3. We now treat summability functions of the second kind.

THEOREM 2. *If* $\Delta\lambda_n/\lambda_n$ *decreases to* 0 *and* $\Delta\lambda_n$ *increases,* (i) *all summability functions of the second kind of the methods* $\mathrm{R}(\lambda_n, \kappa)$, $\kappa \geqslant 1$ *and* $\mathrm{A}(\lambda_n)$ *and only these are given by*

2.3(1) $$\Omega(n) = o(\lambda_n/\Delta\lambda_n).$$

(ii) *For* $\mathrm{R}(\lambda_n, \kappa)$, $0 < \kappa < 1$ *the condition is*

2.3(2) $$\Omega(n) = o(\lambda_n/\Delta\lambda_n)^\kappa.$$

Proof. (a) *If* 2.3(1) *holds, then* $\Omega(n)$ *is a summability function of the second kind of* $\mathrm{R}(\lambda_n, 1)$. From this (i) will follow by Theorem 1. By [15, 2.3] we have to show that if 2.3(1) holds, and $a_n(\omega)$ is the coefficient of s_n in the transformation 2.1(3) for $\kappa = 1$, then

2.3(3) $$\Delta(\omega, \Omega) = \sum_{\nu=0}^{\infty} \Omega(\nu)\left|\Delta a_\nu(\omega)\right| \to 0.$$

We have $a_\nu(\omega) = \Delta\lambda_\nu/\lambda(\omega)$ for $\nu \leqslant n_0-1$, $a_{n_0}(\omega) = (\lambda(\omega)-\lambda_{n_0})/\lambda(\omega)$ and $a_\nu(\omega) = 0$ for $\nu > n_0$. The last non-vanishing term of the sum 2.3(3) with $\nu = n_0$ converges to 0 because $\Delta\lambda_n/\lambda_n \to 0$. Therefore, 2.3(3) is equivalent to

2.3(4) $$\lambda(n)^{-1} \sum_{\nu=0}^{n} \Omega(\nu)\, \Delta^2\lambda_\nu \to 0 \qquad \text{for } n \to \infty.$$

With $\Delta\lambda_\nu/\lambda_\nu$ also $\lambda_{\nu+1}/\lambda_\nu$ is decreasing, and so $\lambda_\nu\lambda_{\nu+2} \leqslant \lambda_{\nu+1}^2$ and

2.3(5) $$\lambda_\nu\Delta^2\lambda_\nu = \lambda_\nu\lambda_{\nu+2} - 2\lambda_\nu\lambda_{\nu+1} + \lambda_\nu^2 \leqslant (\lambda_{\nu+1}-\lambda_\nu)^2 = (\Delta\lambda_\nu)^2,$$

$$\lambda_n^{-1}\sum_{\nu=0}^{n} \lambda_\nu\Delta^2\lambda_\nu/\Delta\lambda_\nu \leqslant \lambda_n^{-1}\sum_{\nu=0}^{n} \Delta\lambda_\nu = 1.$$

By a variant of the theorem of Silverman-Toeplitz we now see that

$$\lambda_n^{-1}\sum_{0}^{n} \Omega(\nu)\,\Delta^2\lambda_\nu = \lambda_n^{-1}\sum_{\nu=0}^{n} \frac{\lambda_\nu\Delta^2\lambda_\nu}{\Delta\lambda_\nu}\,\Omega(\nu)\frac{\Delta\lambda_\nu}{\lambda_\nu} \to 0,$$

if $\Omega(\nu)\,\Delta\lambda_\nu/\lambda_\nu \to 0$.

(b) *We prove* (ii). For $0 < \kappa < 1$ the necessary and sufficient condition is again 2.3(3), where $a_\nu(\omega)$ is defined by the transformation 2.1(3). Con-

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sidering the last non-vanishing term we see that $\Omega(n)(\Delta\lambda_n/\lambda_n)^\kappa \to 0$, that is 2.3(2) is necessary. Let 2.3(2) be true. Then 2.3(3) is equivalent to

$$2.3(4) \qquad S = \lambda(\omega)^{-\kappa} \sum_{n=0}^{\omega_1} \Omega(n) \left| \Delta^2(\lambda(\omega)-\lambda_n)^\kappa \right| \to 0,$$

where ω_1 is some integer of the form $\omega_1 = \omega - p$, and p is constant. It will be sufficient to take $p \geqslant 5$.

We have, if $0 \leqslant c < b < a$ and $a - 2b + c \leqslant 0$,

$$\begin{aligned} a^\kappa - 2b^\kappa + c^\kappa &= c^\kappa - (2b-a)^\kappa + a^\kappa - 2b^\kappa + (2b-a)^\kappa \\ &= \kappa(a-2b+c)\xi^{\kappa-1} + \kappa(\kappa-1)(b-a)^2\eta^{\kappa-2}, \end{aligned}$$

where $c < \xi < 2b - a < b$, $c < \eta < a$. Applying this to S, we obtain

$$S \leqslant C_1 S_1 + C_2 S_2,$$

$$2.3(6) \qquad \begin{cases} S_1 = \lambda(\omega)^{-\kappa} \sum_{n=0}^{\omega_1} \Omega(n)\, \Delta^2\lambda_n \left|\lambda(\omega) - \lambda'_n\right|^{\kappa-1}, \\ S_2 = \lambda(\omega)^{-\kappa} \sum_{n=0}^{\omega_1} \Omega(n)\, (\Delta\lambda_n)^2 \left|\lambda(\omega) - \lambda''_n\right|^{\kappa-2}, \end{cases}$$

where λ'_n and λ''_n are between λ_n and λ_{n+2}. If μ_n is such that $-\Delta(\lambda(\omega)-\lambda_n)^\kappa = \kappa(\lambda(\omega)-\mu_n)^{\kappa-1}\Delta\lambda_n$, $\lambda_n < \mu_n < \lambda_{n+1}$, we have

$$\frac{\lambda(\omega) - \lambda'_n}{\lambda(\omega) - \mu_n} = 1 - \frac{\lambda'_n - \mu_n}{\lambda(\omega) - \mu_n} \geqslant 1 - \frac{\lambda_{n+2} - \lambda_n}{\lambda_{n+5} - \lambda_{n+1}} \geqslant \tfrac{1}{2},$$

and therefore, using again 2.3(5),

$$\begin{aligned} S_1 &\leqslant C_3 \lambda(\omega)^{-\kappa} \sum_{n=0}^{\omega_1} \Omega(n) \frac{\Delta^2\lambda_n}{\Delta\lambda_n} [(\lambda(\omega)-\lambda_n)^\kappa - (\lambda(\omega)-\lambda_{n+1})^\kappa] \\ &\leqslant C_3 \lambda(\omega)^{-\kappa} \sum_{n=0}^{\omega_1} \Omega(n) \frac{\Delta\lambda_n}{\lambda_n} [(\lambda(\omega)-\lambda_n)^\kappa - (\lambda(\omega)-\lambda_{n+1})^\kappa]. \end{aligned}$$

We may regard this as a transformation of the sequence $\Omega(n)\Delta\lambda_n/\lambda_n$ and obtain as before $S_1 \to 0$ for $\omega \to \infty$.

To deal with S_2, 2.3(1) will not be enough and we need 2.3(2) in full. We have

$$\begin{aligned} S_2 &= o(1)\lambda(\omega)^{-\kappa} \sum_{n=0}^{\omega_1} \lambda_n^\kappa (\Delta\lambda_n)^{1-\kappa} (\lambda(\omega)-\lambda_n)^{\kappa-2} \Delta\lambda_n \\ &\leqslant o(1)\,(\Delta\lambda_{n_0})^{1-\kappa} \sum_{n=0}^{\omega_1} (\lambda(\omega)-\lambda_n)^{\kappa-2}\Delta\lambda_n. \end{aligned}$$

As before, it is easy to see that $(\lambda(\omega)-\lambda_n)^{\kappa-2}\Delta\lambda_n = O(\Delta(\lambda(\omega)-\lambda_n)^{\kappa-1})$, and therefore

$$S_2 = o(1)\,(\Delta\lambda_{n_0})^{1-\kappa}[(\lambda(\omega)-\lambda(\omega_1+1))^{\kappa-1} - \lambda(\omega)^{\kappa-1}].$$

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Since $\Delta\lambda_n/\lambda_n$ is decreasing, $1 \leqslant \Delta\lambda_{n+1}/\Delta\lambda_n \leqslant \lambda_{n+1}/\lambda_n \to 1$ and so $\Delta\lambda_{n+1}/\Delta\lambda_n \to 1$ for $n \to \infty$. But this implies $[\lambda(\omega) - \lambda(\omega_1 + 1)]/\Delta\lambda_{n_0} = O(1)$ and $S_2 = o(1)$. Therefore $S \to 0$ and the proof of the theorem is complete.

3. Riesz and Abel methods. Case when $\Delta\lambda_n$ is decreasing

3.1. If $\Delta\lambda_n$ is decreasing, the condition 2.1(5) is automatically fulfilled. By the argument used in §**2.2** it is seen that we even have $\lambda_n/\Delta\lambda_n \geqslant Cn$ for some constant $C > 0$.

THEOREM 3. *If $\Delta\lambda_n$ is decreasing, all functions $\Omega(n) = o(n)$ are summability functions of the methods* $\mathrm{R}(\lambda_n, \kappa)$, $\kappa > 0$ *and* $\mathrm{A}(\lambda_n)$.

Proof. It is sufficient to consider $\mathrm{R}(\lambda_n, \kappa)$ for $0 < \kappa \leqslant 1$. We prove that $A(\omega, \Omega) \to 0$, if $\Omega(n) = o(n)$. Choose an $\epsilon > 0$ and break the matrix $A = (a_n(\omega))$ of $\mathrm{R}(\lambda_n, \mathrm{K})$ into the parts $A' = (a'_n(\omega))$, $A'' = (a''_n(\omega))$, where

$$a'_n(\omega) = \begin{cases} a_n(\omega) & \text{for } 0 \leqslant n \leqslant \omega_1 - 1, \\ 0 & \text{for } n > \omega_1 - 1; \end{cases}$$

$$a''_n(\omega) = \begin{cases} 0 & \text{for } 0 \leqslant n \leqslant \omega_1 - 1, \\ a_n(\omega) & \text{for } n > \omega_1 - 1, \end{cases}$$

and ω_1 is defined by $\lambda(\omega_1) = (1-\epsilon)\lambda(\omega)$. Clearly,

$$A(\omega, \Omega) \leqslant A'(\omega, \Omega) + A''(\omega, \Omega).$$

For $n \leqslant \omega_1 - 1$ we have, with some λ'_n between λ_n and λ_{n+1}

$$a'_n(\omega) = \kappa\lambda(\omega)^{-\kappa}(\lambda(\omega) - \lambda'_n)^{\kappa-1}\Delta\lambda_n$$

$$= \kappa\left(\frac{\lambda(\omega)}{\lambda(\omega)-\lambda'_n}\right)^{1-\kappa}\frac{\Delta\lambda_n}{\lambda(\omega)} \leqslant \frac{\kappa}{\epsilon^{1-\kappa}}\,\frac{\Delta\lambda_n}{\lambda(\omega)} = \alpha_n(\omega),$$

say. We put $\alpha_n(\omega) = 0$ for $n > \omega_1 - 1$. These $\alpha_n(\omega)$ are positive, decreasing and have uniformly bounded sums $\sum_n \alpha_n(\omega)$. Therefore, $A'(\omega, \Omega) \to 0$, by [15, Theorem 7]. On the other hand,

$$A''(\omega, \Omega) \leqslant \sum_{n=0}^{\infty} a''_n(\omega) \leqslant \lambda(\omega)^{-\kappa}(\lambda(\omega) - \lambda(\omega_1 - 1))^{\kappa}$$

$$= (1-(1-\epsilon) + o(1))^{\kappa} = (\epsilon + o(1))^{\kappa}.$$

Therefore $\overline{\lim}_{\omega\to\infty} A(\omega, \Omega) \leqslant \epsilon^{\kappa}$; and since $\epsilon > 0$ was arbitrary, $\lim A(\omega, \Omega) = 0$, q.e.d.

THEOREM 4. *If $\Delta\lambda_n$ decreases, all functions $\Omega(n) = o(n)$ are summability functions of the second kind of the methods* $\mathrm{A}(\lambda_n)$ *and* $\mathrm{R}(\lambda_n, \kappa)$, $\kappa \geqslant 1$.

Proof. It is sufficient to consider $\mathrm{R}(\lambda_n, 1)$. The assertion is then $\mathrm{R}(\lambda_n, 1) \supset \mathrm{C}_1$, and this is a theorem of Cesàro [5, p. 58].

Theorems 3 and 4 give only sufficient conditions, but it is clear that they

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may not be improved, since $\Omega(n) = n$ is not a summability function for any regular method. On the other hand, summability functions which do not satisfy $\Omega(n) = o(n)$ may exist. For instance the method R $(\log n, 1)$, which is equivalent to the method of logarithmic means, possesses summability functions $\Omega(n)$ such that $\Omega(n) \neq o(\varphi(n))$ provided $\varphi(n)$ has the property $\varphi(n) = o(n \log n)$.

3.2. Now we shall show that in case $0 < \kappa < 1$ the condition for a summability function $\Omega(n)$ of the second kind is again 2.3(2). But for this result we require a much greater amount of regularity of $\lambda(n)$ than up to now. However, any function $\lambda(n)$ which is a product of powers of n and iterated logarithms satisfies our conditions.

THEOREM 5. *If for all large real n*

(α) $\lambda(n + h) - \lambda(n)$ *is decreasing for any fixed* $h > 0$,

(β) $\lambda(\log n)/\lambda(n)$ *is decreasing,*

(γ) $\Delta\lambda_n/\Delta\lambda_{2n} \leqslant M$,

then the general form of a summability function $\Omega(n)$ *of the second kind of the method* R(λ_n, κ), $0 < \kappa < 1$ *is* 2.3(2).

For instance, if $\lambda_n = \log \log n$, the conditions are satisfied and we obtain $\Omega(n) = o(n \log n \log \log n)^\kappa$.

Proof. We first observe that (γ) implies

3.2(1) $$1 \leqslant \Delta\lambda_n/\Delta\lambda_{n+1} \leqslant M.$$

As in Theorem 2(b) we see that the condition 2.3(2) is necessary, further that to prove it sufficient it is enough to derive from it that the sums S_1 and S_2 in 2.3(6) converge to 0 as $\omega \to \infty$. We shall first deduce $S_2 \to 0$ from $S_1 \to 0$. Using the inequality $\Delta(a_n b_n) \geqslant b_{n+1}\Delta a_n$ if $a_n \geqslant 0$, b_n increases, we see that with the λ''_n of 2.3(6),

$$\begin{aligned}\Delta[(\lambda(\omega) - \lambda_{n+2})^{\kappa-1}\Omega(n)] &\geqslant \Omega(n+1)\,\Delta(\lambda(\omega) - \lambda_{n+2})^{\kappa-1} \\ &\geqslant (1-\kappa)\Omega(n)\,(\lambda(\omega) - \lambda_{n+2})^{\kappa-2}\Delta\lambda_{n+2} \\ &\geqslant C\Omega(n)(\lambda(\omega) - \lambda''_n)^{\kappa-2}\Delta\lambda_n.\end{aligned}$$

Therefore, using the formula of partial summation, 2.3(2) and 3.2(1),

$$\begin{aligned}S_2 &\leqslant C_1\lambda(\omega)^{-\kappa}\sum_{n=0}^{\omega_1}\Delta\lambda_{n+3}\Delta[(\lambda(\omega) - \lambda_{n+2})^{\kappa-1}\Omega(n)] \\ &= C_1\lambda(\omega)^{-\kappa}\Big\{-\Delta\lambda_3(\lambda(\omega) - \lambda_2)^{\kappa-1}\Omega(0) + \Delta\lambda_{\omega_1+3}(\lambda(\omega) - \lambda_{\omega_1+3})^{\kappa-1}\Omega(\omega_1+1) \\ &\qquad - \sum_{n=1}^{\omega_1}(\lambda(\omega) - \lambda_{n+2})^{\kappa-1}\Omega(n)\,\Delta^2\lambda_{n+2}\Big\} \\ &\leqslant o(1) + C_1\lambda(\omega)^{-\kappa}\sum_{n=0}^{\omega_1+2}(\lambda(\omega) - \lambda_n)^{\kappa-1}\Omega(n)|\Delta^2\lambda_n|.\end{aligned}$$

But the second term is $-S_1$ with ω_1 replaced by $\omega_1 + 2$. Thus we have only to show that $S_1 \to 0$ if $\omega_1 < \omega - 1$ or that

$$S' = \sum_{n \leqslant \omega - 1} (\Delta\lambda_n)^{-\kappa} |\Delta^2\lambda_n| (\lambda(\omega) - \lambda_n)^{\kappa-1}$$

is bounded. We break up S' into three parts $\sum_1, \sum_2, \sum_3$ according to the inequalities $n \leqslant \log \omega$, $\log \omega < n \leqslant \frac{1}{2}\omega$, $\frac{1}{2}\omega < n \leqslant \omega - 1$. For $\sum_1$ we have $\lambda(\omega) - \lambda_n \geqslant \lambda(\omega) - \lambda(\log \omega) \to +\infty$ by (β), and therefore

$$\sum\nolimits_1 = o(1) \sum_{n \leqslant \log \omega} (\Delta\lambda_n)^{-\kappa} |\Delta^2\lambda_n| = o(1) \left| \sum_{n=0}^{\infty} \Delta(\Delta\lambda_n)^{1-\kappa} \right|$$
$$= o(1)\,(\Delta\lambda_0)^{1-\kappa} = o(1).$$

On the other hand, since $\Delta(\lambda(\log n)/\lambda(n)) \leqslant 0$,

3.2(2) $$\frac{\lambda(\log(n+1)) - \lambda(\log n)}{\Delta\lambda_n} \leqslant \frac{\lambda(\log n)}{\lambda(n)} \leqslant 1.$$

Using (α), 3.2(2) and (β) we see that

$$\begin{aligned}\Delta\lambda(\log n) &= \lambda(\log n + 1) - \lambda(\log n) \\ &\leqslant [\lambda(\log 4n) - \lambda(\log(4n-1))] + \ldots + [\lambda(\log(n+1)) - \lambda(\log n)] \\ &\leqslant 3n[\lambda(\log(n+1)) - \lambda(\log n)] \leqslant C_2 n\,\Delta\lambda_n\end{aligned}$$

and therefore

3.2(3) $$\Delta\lambda(\log n)/(n\,\Delta\lambda_n) \leqslant C_2$$

for some constant C_2. We have further

$$\lambda(\omega) - \lambda_n \geqslant (\omega - n)\,\Delta\lambda(\omega) \geqslant (\omega/2)\,\Delta\lambda(\omega)$$

if $0 \leqslant n \leqslant \frac{1}{2}\omega$. Therefore

$$\begin{aligned}\sum\nolimits_2 &\leqslant C_3(\omega\,\Delta\lambda(\omega))^{\kappa-1} \sum_{\log \omega \leqslant n \leqslant \omega} (\Delta\lambda_n)^{-\kappa} |\Delta^2\lambda_n| \\ &\leqslant C_4 \left(\frac{\Delta\lambda(\log \omega)}{\omega\,\Delta\lambda(\omega)} \right)^{1-\kappa} = O(1),\end{aligned}$$

by 3.2(3). Finally,

$$\sum\nolimits_3 \leqslant (\Delta\lambda(\omega))^{\kappa-1} \sum_{\frac{1}{2}\omega < n \leqslant \omega - 1} (\Delta\lambda_n)^{-\kappa} |\Delta^2\lambda_n| \leqslant C_5 \left(\frac{\Delta\lambda(\frac{1}{2}\omega)}{\Delta\lambda(\omega)} \right)^{1-\kappa} = O(1)$$

by (γ). This completes the proof.

4. Absolute summability functions

4.1. Let $\Omega(n)$ be, as before, a non-decreasing positive function which tends to $+\infty$ with n. In analogy with our former definitions we shall say that $\Omega(n)$ is an *absolute summability function of a method of summation* A (given by 1.1(1)), if any bounded sequence s_n for which $s_n = 0$ except for a subsequence $\{n_\nu\}$ with the counting function $\omega(n) \leqslant \Omega(n)$, is absolutely A-summable, that is if $\sum |\sigma_m - \sigma_{m-1}| < +\infty$ for any such sequence.

The following Lemma will be useful. (With another proof, the Lemma has been communicated to the author by Dr. K. Zeller, Tübingen).

LEMMA 1. *The transformation*

$$v_m = \sum_{\nu=0}^{\infty} b_{m\nu} s_\nu \qquad (m = 0, 1, \ldots) \tag{4.1(1)}$$

maps any bounded sequence $s = \{s_\nu\}$ *into a sequence* $v = \{v_m\}$ *with* $\sum |v_m| < +\infty$ *if and only if one of the following three conditions is fulfilled*:

$$\Big| \sum_{m\epsilon e_1} \sum_{\nu\epsilon e} b_{m\nu}\Big| \leqslant M, \tag{4.1(2)}$$

$$\sum_{m=0}^{\infty} \Big| \sum_{\nu\epsilon e} b_{m\nu}\Big| \leqslant M, \tag{4.1(3)}$$

$$\sum_{m=0}^{\infty} \Big| \sum_{\nu\epsilon E} b_{m\nu}\Big| \leqslant M. \tag{4.1(4)}$$

Here E is an arbitrary subset and e, e_1 arbitrary finite subsets of the set of all positive integers, and the M independent of e, e_1, E.

Proof. The conditions are equivalent. It is clear, that 4.1(4) implies 4.1(3) and this imples 4.1(2), and we leave to the reader the elementary proof that 4.1(2) implies 4.1(4). Further, $\sum_{\nu=0}^{\infty} |b_{m\nu}| < +\infty$, $m = 0, 1, \ldots$ is necessary and is also a consequence of any of our conditions.

Let S and V be Banach spaces of bounded sequences $s = \{s_\nu\}$ and of sequences $v = \{v_m\}$ with $\sum |v_m| < +\infty$, respectively. Suppose that $v = B(s)$, defined by 4.1(1), maps S into V. For a fixed m, $\sum_\nu b_{m\nu} s_\nu$ is a linear functional in S. Therefore the transformation $v = B_m(s)$, defined by $v_\mu = \sum_{\nu=0}^{\infty} b_{\mu\nu} s_\nu$ for $0 \leqslant \mu \leqslant m$, $v_\mu = 0$ for $\mu > m$, is a linear operation mapping S into V. But clearly $B_m(s) \to B(s)$ for $s \in S$ in the norm of the space V. Therefore $v = B(s)$ is also a linear operation and there is an M such that $||v|| \leqslant M\,||s||$. But this is identical with 4.1(4), if we put $s_\nu = 1$ for $\nu \in E$, $s_\nu = 0$ for $\nu \overline{\in} E$.

It remains to show that if 4.1(4) is true, then $v = B(s)$ maps S into V. The function $F(s) = \sum_{m=0}^{\infty} \big| \sum_{\nu=0}^{\infty} b_{m\nu} s_\nu \big| \leqslant +\infty$ is clearly lower semi-continuous in S. If the sequence $s = \{s_\nu\}$ is positive, takes only a finite number of values and if $||s|| \leqslant 1$, then $s = a^{(1)} s^{(1)} + \ldots + a^{(p)} s^{(p)}$, where the $s^{(i)}$ are sequences of 0's and 1's, and $a^{(i)} \geqslant 0$, $\sum a^{(i)} \leqslant 1$. Using 4.1(4) we obtain $F(s) \leqslant \sum a^{(i)} F(s^i) \leqslant M$. Without the condition of positiveness of s we have $F(s) \leqslant 2M$. But these new s are dense in the unit sphere of S. Therefore $F(s) \leqslant 2M$ for any s with $||s|| \leqslant 1$, and $F(s) < +\infty$ everywhere. This completes the proof of the Lemma.

4.2. From Lemma 1 we obtain

THEOREM 6. *In order that* $\Omega(n)$ *be an absolute summability function of the method* 1.1(1) *for which* $\sum |a_{0n}| < +\infty$, *it is necessary and sufficient that for for any finite or infinite sequence* $n_1 < n_2 < \ldots$ *with the counting function* $\omega(n) \leqslant \Omega(n)$ *there is an M such that*

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4.2(1) $$\operatorname*{var}_{m} \sum_{\nu=1}^{\infty} a_{mp_\nu} \leqslant M$$

for any subsequence p_ν of the sequence n_ν.

Proof. We apply Lemma 1 to the transformation 4.1(1), where $b_{m\nu}$ is $a_{mn_\nu} - a_{m-1,\, n_\nu}$ and $a_{-1,\, n} = 0$. Then 4.2(1) is equivalent to 4.1(4).

There are of course two other forms of the condition which are obtained from 4.1(2) or 4.1(3). More useful is the following *sufficient* condition:

4.2(2) $$\sum_{\nu=1}^{\infty} \operatorname*{var}_{m} a_{mn_\nu} < +\infty$$

for any sequence $n_1 < n_2 < \ldots$ whose counting function does not exceed $\Omega(n)$.

THEOREM 7. *The method of summation A generated by the matrix (a_{mn}) for which $\sum |a_{0n}| < +\infty$ has absolute summability functions if and only if the variation of the n-th column $V_n = \operatorname*{var}_{m} a_{mn}$ converges to 0 for $m \to \infty$.*

Proof. (a) *The condition is sufficient.* Suppose that $V_n \to 0$ for $n \to \infty$. Put $W_n = \max_{p \leqslant n} V_p$, take a sequence n_ν such that $\sum W_{n_\nu} < +\infty$ and denote by $\Omega(n)$ the counting function of $\{n_\nu\}$. If n'_ν is an increasing sequence of integers with the counting function $\omega(n) \leqslant \Omega(n)$, then $n'_\nu \geqslant n_\nu$ for all ν [15, 2.1]. But this implies $\sum V_{n'_\nu} < +\infty$. Applying the sufficient condition 4.2(2) we see that the matrix $A' = (a_{mn'_\nu})$ sums absolutely every bounded sequence, and the matrix A every bounded sequence s_n such that $s_n = 0$ if $n \neq n'_\nu (\nu = 1, 2, \ldots)$. Therefore, $\Omega(n)$ is an absolute summability function for A.

(b) *The condition is necessary.* Suppose that V_n does not tend to 0 and that $\Omega(n)$ is an absolute summability function for the method A. We shall show that there is a sequence n_ν with the counting function $\omega(n) \leqslant \Omega(n)$ such that

4.2(3) $$\operatorname*{var}_{m} \sum_{\nu=1} a_{mn_\nu} = +\infty.$$

This contradiction with Theorem 6 will show that no absolute summability function $\Omega(n)$ can exist.

If the integer p is sufficiently large, the sequence consisting of p alone has certainly the counting function $\leqslant \Omega(n)$; therefore 4.2(1) shows that almost all V_n are finite. We write $b_{mn} = a_{mn} - a_{m-1,\, n}$ $(a_{-1,\, n} = 0)$. Then for any sequence n_ν with the counting function $\leqslant \Omega(n)$ all series $\sum_{\nu=1}^{\infty} b_{mn_\nu} s_{n_\nu}$, $m = 0, 1, \ldots$ must converge for all bounded s_{n_ν}. It follows that all series $\sum_\nu |b_{mn_\nu}|$ converge. It is now clear that there is a monotone sequence of integers p_r whose counting function is $\leqslant \Omega(n)$, such that all series $\sum_m |b_{mp_r}|$ and $\sum_r |b_{mp_r}|$ are convergent and that

4.2(4) $$\sum_{m} |b_{mp_r}| \geqslant \epsilon \qquad (r = 1, 2, \ldots)$$

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for some constant $\epsilon > 0$. For simplicity we write c_{mr} instead of b_{mp_r}. Inductively we choose two increasing sequences of integers r_ν, M_ν. If all numbers with indices less than ν are defined, we choose first an $M_\nu > M_{\nu-1}$ which satisfies

$$4.2(5) \qquad A_\nu = \sum_{m>M_\nu} \sum_{\mu=1}^{\nu-1} |c_{mr_\mu}| < \epsilon/5,$$

then $r_\nu > r_{\nu-1}$ such that

$$4.2(6) \qquad B_\nu = \sum_{m\leqslant M_\nu} \sum_{r\geq r_\nu} |c_{mr}| < \epsilon/5.$$

We have then

$$\sum_{M_\nu<m\leqslant M_{\nu+1}} \left|\sum_{\mu=1}^{\infty} c_{mr_\mu}\right| \geqslant \sum_{M_\nu<m\leqslant M_{\nu+1}} |c_{mr_\nu}| - A_\nu - B_{\nu+1}$$

$$\geqslant \sum_{m=0}^{\infty} |c_{mr_\nu}| - \sum_{m\leqslant M_\nu} |c_{mr_\nu}| - \sum_{m>M_{\nu+1}} |c_{mr_\nu}| - 2\epsilon/5$$

$$\geqslant \epsilon - 4\epsilon/5 = \epsilon/5$$

by 4.2(5), 4.2(6), and 4.2(4). It follows that $\sum_m |\sum_\nu c_{nr_\nu}| = +\infty$, and this proves 4.2(3). The proof is complete.

4.3. As an example of application of Theorem 7 we consider Abel, Riesz and Hausdorff methods.

(i) The method $A(\lambda_n)$ has absolute summability functions if it has summability functions, that is if and only if $\Delta\lambda_n/\lambda_n \to 0$ (compare §**2.1**).

In fact, the coefficient $a_n(x) = e^{-\lambda_n x} - e^{-\lambda_{n+1} x}$ of the $A(\lambda_n)$ transformation 2.1(4) has its maximum for some value x_n of x between λ_n^{-1} and λ_{n+1}^{-1}, and is monotone in $0 \leqslant x \leqslant x_n$ and $x \geqslant x_n$. Therefore,

$$V_n = \operatorname*{var}_{0\leqslant x<+\infty} a_n(x) = 2a_n(x_n) \to 0, \qquad n \to \infty,$$

if $A(\lambda_n)$ has summability functions of the first kind. This proof applies also to $R(\lambda_n, \kappa)$, $\kappa > 0$ and gives the same result (in fact, to any regular method A for which a_{mn} has one single maximum in every column).

(ii) A regular Hausdorff method H_g with the generating function $g(t)$ of bounded variation has absolute summability functions whenever H_g has summability functions, that is if and only if $g(t)$ is continuous at $t = 1$ [14, Theorem 13].

For the method H_g,

$$a_{mn} = \int_0^1 p_{nm}(t)dg(t), \quad p_{nm}(t) = \binom{m}{n} t^n (1-t)^{m-n}, \qquad 0 \leqslant n \leqslant m,$$

and $a_{mn} = 0$ for $n > m$. Therefore, if H_g has summability functions,

$$4.3(1) \qquad V_n = \operatorname*{var}_m a_{mn} \leqslant |a_{nn}| + \int_0^1 \sum_{m=n}^{\infty} |p_{nm}(t) - p_{n,\,m+1}(t)|\,|dg(t)|$$

$$= o(1) + \int_0^1 P(t)|dg(t)|,$$

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say. But for fixed n and t, $p_{nm}(t)$ is first increasing with m and then decreasing, the maximal value being $O(n^{-\frac{1}{2}}) = o(1)$ for $n \to \infty$ uniformly in any interval $\delta \leqslant t \leqslant 1 - \delta$, $\delta > 0$. Moreover $P_n(t) \leqslant 2$ for all n and t. Since $g(t)$ is continuous at $t = 0$ (by the regularity of H_g) and at $t = 1$, 4.3(1) implies $V_n \to 0$, which proves our result.

4.4. In this and the next section we use conditions 4.2(1) and 4.2(2) to find all absolute summability functions of the Cesàro, Euler-Knopp and Borel methods.

THEOREM 8. *A function $\Omega(n)$ is an absolute summability function of the method C_α if and only if*

4.4(1) $$\sum_{n=1}^{\infty} n^{-1-\alpha}\,\Omega(n) < +\infty, \qquad 0 < \alpha < 1,$$

or

4.4(2) $$\sum_{n=1}^{\infty} n^{-2}\,\Omega(n) < +\infty, \qquad \alpha \geqslant 1.$$

We shall need two lemmas.

LEMMA 2. *For a sequence of integers $0 < n_1 < n_2 < \ldots$ with the counting function $\omega(n)$ the two following conditions are equivalent* ($\alpha > 0$):

4.4(3) $$\sum_{n=1}^{\infty} n^{-1-\alpha}\,\omega(n) < +\infty$$

4.4(4) $$\sum_{\nu=1}^{\infty} n_\nu^{-\alpha} < +\infty.$$

In fact,

$$\sum n^{-1-\alpha}\,\omega(n) = \sum_{n=1}^{\infty} \sum_{n_\nu \leqslant n} n^{-1-\alpha} = \sum_{\nu=1}^{\infty} \sum_{n \geqq n_\nu} n^{-1-\alpha}$$
$$= \theta \sum_{\nu=1}^{\infty} n_\nu^{-\alpha},$$

where θ is some number, contained in a fixed interval (a, b), $0 < a < b < \infty$.

LEMMA 3. *Let $\sum n^{-1-\alpha}\Omega(n) = +\infty$, $\alpha > 0$ and let $a > 1$ be an integer. Set $p_\nu = a^\nu$. Then*

4.4(5) $$\sum_{\nu=1}^{\infty} p_\nu^{-\alpha}[\Omega(p_\nu) - \Omega(p_{\nu-1})] = +\infty.$$

For we have, with positive constants C_1, C_2,

$$\sum_{\nu=1}^{N} p_\nu^{-\alpha}[\Omega(p_\nu) - \Omega(p_{\nu-1})]$$
$$= -\Omega(p_0)p_1^{-\alpha} + \sum_{\nu=1}^{N-1} \Omega(p_\nu)\,(p_\nu^{-\alpha} - p_{\nu+1}^{-\alpha}) + p_N^{-\alpha}\Omega(p_N)$$

$$\geqslant O(1) + C_1 \sum_{\nu=1}^{N} \Omega(p_\nu) p_\nu^{-\alpha}$$

$$\geqslant O(1) + C_2 \sum_{\nu=1}^{N-1} \Omega(p_\nu) \sum_{n=p_{\nu-1}}^{p_\nu - 1} n^{-1-\alpha}$$

$$\geqslant O(1) + C_2 \sum_{n=1}^{p_{N-1}-1} n^{-1-\alpha} \Omega(n).$$

Proof of Theorem 8. (a) *The conditions are sufficient*. Suppose that 4.4(1) holds with some α, $0 < \alpha \leqslant 1$, and let $n_1 < n_2 < \ldots$ have a counting function $\omega(n) \leqslant \Omega(n)$. Then $\sum n_\nu^{-\alpha} < +\infty$, by Lemma 2. It will be sufficient to show that 4.2(2) holds. But for the method C_α, $a_{mn} = 0$ for $m < n$,

4.4(6) $$a_{mn} = (A_m^\alpha)^{-1} A_{m-n}^{\alpha-1} \text{ for } m \geqslant n,\ A_n^\alpha = \binom{n+\alpha}{n} \cong n^\alpha / \Gamma(\alpha+1),$$

and a_{mn} is a decreasing function of m for $m \geqslant n$. Therefore,

$$\operatorname*{var}_m a_{mn} = 2a_{nn} = 2(A_n^\alpha)^{-1} \leqslant Cn^{-\alpha},$$

and 4.2(2) follows. The rest follows from the inclusion $|C_\alpha| \subset |C_\beta|$ for $\alpha \leqslant \beta$.

(b) *The conditions are necessary*. First suppose $0 < \alpha < 1$. By [15, 5.1] we may assume that $\Omega(n) = o(n)$. Suppose that $\sum n^{-1-\alpha} \Omega(n) = +\infty$. We define $\omega_1(n)$ inductively by putting $\omega_1(1) = 0$ and, if $\omega_1(n)$ is known, $\omega_1(n+1) = \omega_1(n) + 1$ if this number is $\leqslant \Omega(n+1)$, and $\omega_1(n+1) = \omega_1(n)$ in the contrary case. Using $\Omega(n) = o(n)$ one proves easily that $\sum n^{-1-\alpha} \omega_1(n) = +\infty$. $\omega_1(n)$ is the counting function of some sequence. Omitting, if necessary, some terms of this sequence, we obtain another sequence of integers $n_1 < n_2 < \ldots$ such that (i) its counting function $\omega(n) \leqslant \Omega(n)$; (ii) $\sum n_\nu^{-\alpha} = +\infty$; (iii) for any ν, $n_\nu + 1$ does not belong to the sequence. We now observe that the coefficient a_{mn} given by 4.4(6) is decreasing for $m \geqslant n$ and that

$$a_{nn} - a_{n+1,\,n} = (A_n^\alpha)^{-1} - (A_{n+1}^\alpha)^{-1} A_1^{\alpha-1}$$

$$= (A_n^\alpha)^{-1} \frac{(1-\alpha)n + 1}{n+1+\alpha} \geqslant Cn^{-\alpha}$$

with some constant $C > 0$. Using (iii) and (ii) we obtain

$$\operatorname{var} \sum_{\nu=1}^{\infty} a_{mn_\nu} \geqslant \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} (a_{n_\mu n_\nu} - a_{n_\mu+1,\, n_\nu})$$

$$\geqslant \sum_{\mu=1}^{\infty} (a_{n_\mu n_\mu} - a_{n_\mu+1,\, n_\mu}) \geqslant C \sum n_\mu^{-\alpha} = +\infty,$$

and the result follows by Theorem 6.

Next consider the case $\alpha \geqslant 1$. We may assume $\alpha > 1$. Without restriction of generality we may also suppose that $\Omega(n) = o(n)$ and takes only integral values. We choose $k > e\alpha$ and then an integer $a > k\alpha$. If 4.4(2) is not fulfilled, we must have $\sum_{\nu=1}^{\infty} p_\nu^{-1} q_\nu = +\infty$, $q_\nu = \Omega(p_\nu) - \Omega(p_{\nu-1})$, by Lemma 3. Consider the sequence consisting of all groups of integers n, $p_\nu \leqslant n < p_\nu + q_\nu$ $(\nu = 1, 2, \ldots)$. The counting function of the sequence is $\leqslant \Omega(n)$. Put $f(m) = \sum_{\nu=1}^{\infty} f_\nu(m)$, $f_\nu(m) = \sum_{p_\nu \leqslant n < p_\nu + q_\nu} a_{mn}$. If we can show that

$$4.4(7) \qquad \operatorname*{var}_m f(m) = +\infty$$

our result will follow by Theorem 6. Since

$$a_{m+1,n}\, a_{mn}^{-1} - 1 = \frac{\alpha n - m - 1}{(m - n + 1)(m + \alpha + 1)}, \qquad m \geqslant n,$$

the coefficient a_{mn} is surely decreasing as a function of m for $m > \alpha\, n$. Therefore, $f_\nu(m)$ decreases if $m > \alpha(p_\nu + q_\nu)$. Let $m'_\nu = [\alpha p_\nu]$, $m''_\nu = [k\alpha p_\nu]$. Since $m''_\nu < p_{\nu+1}$, $f_\mu(m) = 0$ for $\mu > \nu$, $m \leqslant m''_\nu$. On the other hand, $f_\mu(m)$, $\mu < \nu$ are decreasing for $m \geqslant m'_\nu$. Therefore

$$4.4(8) \qquad f(m'_\nu) - f(m''_\nu) \geqslant f_\nu(m'_\nu) - f_\nu(m''_\nu).$$

Using 4.4(6) and $q_\nu = o(p_\nu)$ we have

$$4.4(9) \qquad f_\nu(m'_\nu) = \sum_{p_\nu \leqslant n < p_\nu + q_\nu} a_{m'_\nu n} \geqslant q_\nu a_{m'_\nu,\, p_\nu + q_\nu}$$
$$\cong C q_\nu (\alpha p_\nu)^{-\alpha} ((\alpha - 1) p_\nu)^{\alpha - 1} \geqslant C q_\nu (e \alpha p_\nu)^{-1},$$

where C denotes the constant $\Gamma(\alpha + 1)/\Gamma(\alpha)$. On the other hand

$$4.4(10) \qquad f_\nu(m''_\nu) \leqslant q_\nu a_{m''_\nu p_\nu} \cong C q_\nu (k\alpha p_\nu)^{-\alpha} ((k\alpha - 1) p_\nu)^{\alpha - 1}$$
$$\leqslant C q_\nu (k p_\nu)^{-1}.$$

Since $k > e\alpha$, from 4.4(8), 4.4(9), and 4.4(10) it follows that

$$f(m'_\nu) - f(m''_\nu) \geqslant C_1 q_\nu p_\nu^{-1}, \qquad C_1 > 0,$$

and we obtain 4.4(7).

We do not know whether the condition 4.4(2), which is clearly necessary, is also sufficient for the Abel method A. But there is a proof similar to the last case of Theorem 8 if $q_\nu p_\nu^{-1}$ is sufficiently smooth, if for instance $\Omega(n)$ is a quotient of n by iterated logarithms.

4.5. THEOREM 9. *A function* $\Omega(n)$ *is an absolute summability function of the Euler-Knopp method* E_t, $0 < t < 1$, *or of the Borel method* B *if and only if*

$$4.5(1) \qquad \sum_{n=1}^{\infty} n^{-3/2} \Omega(n) < +\infty.$$

Proof. In view of the inclusion $|\mathrm{E}_t| < |\mathrm{B}|$ (Knopp-Lorentz [11]) it will be sufficient to show that (i) 4.5(1) is sufficient for the method E_t; (ii) 4.5(1) is necessary for B.

Now the E_t transformation is

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$$\sigma_m = \sum_{n=0}^{m} p_{nm}(t)s_n \qquad (m = 0, 1, \ldots).$$

For fixed n and t, $p_{nm}(t)$ takes its maximal value at $m = m_0$, where m_0 is the least integer satisfying $m > nt^{-1} - 1$. This maximum is $\leqslant C(t)n^{-\frac{1}{2}}$. As $p_{nm}(t)$ is monotone in $n \leqslant m \leqslant m_0$ and $m \geqslant m_0$,

4.5(2) $$\operatorname*{var}_m p_{nm}(t) \leqslant 2C(t)n^{-\frac{1}{2}}.$$

Now if $\{n_\nu\}$ is a sequence with the counting function $\omega(n) \leqslant \Omega(n)$, we have $\sum n_\nu^{-\frac{1}{2}} < +\infty$ by Lemma 2, and from 4.5(2) we see that 4.2(2) holds. This proves (i).

Now suppose the series 4.5(1) be divergent. Taking $a = 4$ we apply Lemma 3 and obtain $\sum p_\nu^{-\frac{1}{2}}q_\nu = +\infty$ with $q_\nu = \Omega(p_\nu) - \Omega(p_{\nu-1})$. Again we may assume that $\Omega(n)$ takes only integral values and [15, 5.2] has the property $\Omega(n) = o(n^{\frac{1}{2}})$. Consider the sequence (with counting function $\leqslant \Omega(n)$) which consists of all integers n contained in the intervals $p_\nu \leqslant n < p_\nu + q_\nu$, $(\nu = 1, 2, \ldots)$. Let

$$f(x) = \sum_{\nu=1}^{\infty} f_\nu(x), \quad f_\nu(x) = \sum_{p_\nu \leqslant n < p_\nu + q_\nu} e^{-x}x^n/n!$$

To prove (ii) we have, by Theorem 6 (or rather its continuous analogue), to show that

4.5(3) $$\operatorname*{var}_{0 \leqslant x < +\infty} f(x) = +\infty.$$

But $a_n(x) = e^{-x}x^n/n!$ attains its maximum $\cong (2\pi n)^{-\frac{1}{2}}$ at $x = n$. Moreover, if $0 \leqslant r \leqslant Cn^{-\frac{1}{2}}$, then $a_{n+r}(n) \geqslant C_1 n^{-\frac{1}{2}}$. Since $q_\nu = o(p_\nu^{-\frac{1}{2}})$, we obtain

$$f(p_\nu) \geqslant f_\nu(p_\nu) \geqslant C_1 p_\nu^{-\frac{1}{2}} q_\nu.$$

On the other hand,

$$f(3p_\nu) = \sum_{\mu=1}^{\infty} f_\mu(3p_\nu) \leqslant \sum_{|n - 2p_\nu| \geqq p_\nu} a_n(3p_\nu) = O(e^{-\gamma p_\nu})$$

for some $\gamma > 0$ (see for instance [5, p.200]). We see that

$$\operatorname{var} f(x) \geqslant \sum_{\nu=1}^{\infty} \{f_\nu(p_\nu) + O(e^{-a p_\nu})\} \geqslant C_1 \sum p_\nu^{-\frac{1}{2}} q_\nu + O(1) = +\infty,$$

which proves 4.5(3).

5. Some further theorems, applications and remarks

5.1. In this section we wish to discuss some applications of the results in [14], [15] and this paper and their relation to known theorems. We begin with the following remark. The definition of a summability function of the second

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kind (see §**1.1**) may obviously be restated as follows: $\Omega(n)$ is a summability function of the second kind of a regular A if and only if $\sigma_n = (s_0+s_1+\ldots+s_n)/(n+1) = s + O(n^{-1}\Omega(n))$ implies the A-summability of s_n to s. Thus from [15, 5.2] follows the theorem of Knopp ([10], also [5, p. 213]): $\sigma_n = s + o(n^{-\frac{1}{2}})$ implies E_t -summability of s_n together with the result that this is the best possible theorem.

5.2. We observed in [15, 3.1] that summability functions may be used to show that Tauberian conditions of a certain kind may not be improved. Thus our results in §**2** and §**3** imply that under certain conditions $u_n = O(\Delta\lambda_n/\lambda_n)$ is the best possible Tauberian condition for $R(\lambda_n, \kappa)$ and $A(\lambda_n)$. This method however fails to give the full truth if $\Delta\lambda_n/\lambda_n$ is smaller than n^{-1}, since a regular method of summation cannot possess summability functions like $n \log n$. The following theorem, based on the sufficiency part of [14, Theorem 8], gives, as far as we know, a precise result for all practically interesting special methods of summation (compare also [12]).

THEOREM 10. (i) *Suppose that* $A = (a_{mn})$ *is a regular method of summation and* $n_1 < n_2 < \ldots$ *a sequence of integers for which*

$$5.2(1) \qquad \lim_{m\to\infty} \{\max_{\nu} \sum_{n=n_\nu}^{n_{\nu+1}-1} |a_{mn}|\} = 0.$$

Then $u_n = 0$ *for* $n \neq n_\nu$ *is not a Tauberian condition for* A.

(ii) *If, moreover,* $c_n \to 0$, $c_n \geqslant 0$ *and*

$$5.2(2) \qquad \sum_{n=n_\nu}^{n_{\nu+1}-1} c_n \geqslant \delta > 0 \qquad (\nu = 1, 2, \ldots),$$

then $u_n = O(c_n)$ *is not a Tauberian condition for* A.

Both statements are true even for bounded sequences $s_n = \sum_{p=0}^{n} u_p$.

Proof. Let $A' = (a'_{m\nu})$ and $a'_{m\nu} = \sum_{n=n_\nu}^{n_{\nu+1}-1} a_{mn}$. Then $\max_\nu a'_{m\nu} \to 0$ for $m \to \infty$, and by [14, Theorem 8 and 8*], there is a bounded divergent sequence which is A′ -summable. This implies (i).

To prove (ii) consider the method $A'' = (a''_{m\nu})$, where

$$5.2(3) \qquad a''_{m\nu} = \sum_{n=n_\nu}^{n_{\nu+1}-1} |a_{mn}|.$$

Since $\max_\nu a''_{m\nu} \to 0$ and $\sum_\nu |a''_{m\nu}| < +\infty$ for any m, by [14, Theorem 8], there is a divergent sequence of 0's and 1's A″ -summable to 0 (Theorem 8 is formulated for regular methods, but only the two properties of A″ stated above are used in the proof). In other words there is a subsequence $\nu(k)$ of the ν such that

5.2(4) $$\sum_{k=1}^{\infty} \sum_{n=n_{\nu(k)}}^{n_{\nu(k)+1}-1} |a_{mn}| \to 0 \qquad \text{for } m \to \infty.$$

Using 5.2(2) and $c_n \to 0$ we can choose, for all large k, an n'_k between $n_{\nu(k)}$ and $n_{\nu(k)+1}$ and u_n positive in $n_{\nu(k)} \leqslant n < n'_k$, negative in $n'_k \leqslant n < n_{\nu(k)+1}$ such that

$$u_n = O(c_n), \quad \sum_{n_{\nu(k)}}^{n'_k - 1} u_n = \delta/3, \quad \sum_{n_{\nu(k)}}^{n_{\nu(k)+1}-1} u_n = 0.$$

We put $u_n = 0$ for the remaining n. The sequence $s_n = \sum_{p=0}^{n} u_p$ is bounded, divergent, A -summable to 0 and has the property $u_n = O(c_n)$. This proves (ii).

It follows from the proof that Theorem 10 remains true if instead of 5.2(1) we assume only

5.2(5) $$\lim_{m \to \infty} \left\{\max_{r} a''_{m\nu_r}\right\} = 0, \qquad m \to \infty$$

for a subsequence ν_r of the ν.

5.3. From the possible applications of Theorem 10 we choose those to Riesz and Wiener methods.

THEOREM 11. *Suppose that $\lambda(n) = \lambda$ is a positive function increasing to $+\infty$ with n.*

(i) *If n_ν is a sequence of integers increasing to $+\infty$ and such that $\lim_{\nu \to \infty} [\lambda(n_{\nu+1})/\lambda(n_\nu)] = 1$, then $u_n = 0$, $n \neq n_\nu$ is not a Tauberian condition of the method* $R(\lambda_n, \kappa)$, $\kappa > 0$.

(ii) *If $c_n = \varphi(n)\Delta\lambda_n/\lambda_n \to 0$, where $\sum c_n = +\infty$ and $\varphi(n) \to +\infty$, then $u_n = O(c_n)$ is not a Tauberian condition for* $R(\lambda_n, \kappa)$.

Proof. We may assume $0 < \kappa < 1$. By 2.1(3) we have

5.3(1) $$a''_\nu(\omega) = \sum_{n_\nu \leqslant n < n_{\nu+1}} a_n(\omega)$$
$$= \begin{cases} \lambda(\omega)^{-\kappa}\{[\lambda(\omega) - \lambda(n_\nu)]^\kappa - [\lambda(\omega) - \lambda(n_{\nu+1})]^\kappa\} & \text{if } \omega \geqslant n_{\nu+1} \\ \lambda(\omega)^{-\kappa}[\lambda(\omega) - \lambda(n_\nu)]^\kappa & \text{if } n_\nu \leqslant \omega < n_{\nu+1}, \\ 0 & \text{if } \omega < n_\nu. \end{cases}$$

Using the inequality $0 < \kappa < 1$ we see that for fixed ν, $a''_\nu(\omega)$ takes its maximum for $\omega = n_{\nu+1}$ which is equal to $\lambda(n_{\nu+1})^{-\kappa}[\lambda(n_{\nu+1}) - \lambda(n_\nu)]^\kappa$. Since the lower limit of this expression for $\nu \to \infty$ is 0, and since $a''_\nu(\omega) \to 0$ for fixed ν and $\omega \to \infty$, there is a subsequence ν_r such that 5.2(5) holds. Using the remark at the end of 5.2 we obtain (i).

In proving (ii) we may suppose that $c_n \leqslant 1$. We take n_1 arbitrary and define $n_{\nu+1}$, if n_ν is known, to be the first integer $> n_\nu$ such that $\sum_{n_\nu \leqslant n < n_{\nu+1}} c_n \geqslant 1$. Then

$$2 \geqslant \sum_{n_\nu \leqslant n < n_{\nu+1}} c_n \geqslant \varphi(n_\nu)\lambda(n_{\nu+1})^{-1}[\lambda(n_{\nu+1}) - \lambda(n_\nu)],$$

and therefore $\lambda(n_{\nu+1})/\lambda(n_\nu) \to 1$. As in the proof of (i) we see that this implies 5.2(1). The proof is completed by applying Theorem 10.

By a different and more difficult method, Theorem 11, (ii) had been proved by Ingham [8]. Instead of our hypothesis $c_n \to 0$ Ingham assumes that $\lambda_{n+1}/\lambda_n \to 1$. This difference is inessential, as in the latter case we may always replace $\varphi(n)$ by a smaller function tending to $+\infty$, for which $c_n \to 0$ holds.

Passing to Wiener's methods, we call a bounded function $f(x)$, $0 \leqslant x < +\infty$ summable to s by a Wiener method W_g, if $\int_0^{+\infty} |g(t)|dt < +\infty$ and

5.3(2) $$\frac{1}{x}\int_0^\infty g\left(\frac{t}{x}\right) f(t)dt \to s\int_0^\infty g(t)dt, \qquad x \to \infty.$$

The well known Tauberian theorem of Pitt [5, p. 296, Theorem 233] asserts that if $\int_0^\infty g(t)t^{ix}dt \neq 0$ for real x, then

5.3(3) $$f(x+\delta) - f(x) \to 0 \qquad \text{for } \delta > 0,\ \delta/x \to 0,\ x \to \infty$$

is a Tauberian condition for the method Wg. In particular, if $f(x)$ is absolutely continuous,

5.3(4) $$f'(x) = O(x^{-1}), \qquad x \to \infty$$

is a Tauberian condition. We use the analogue of Theorem 10 for integrals to show that these conditions cannot be improved.

THEOREM 12. *The conditions*

5.3(5) $$f(x+\delta) - f(x) \to 0 \qquad \text{for } \delta > 0,\ \delta\varphi(x)/x \to 0,\ x \to \infty$$

or

5.3(6) $$f'(x) = O(x^{-1}\varphi(x)), \qquad x \to \infty$$

where $\varphi(x)$ is bounded in any finite interval and $\varphi(x) \to \infty$ are not Tauberian conditions for any method W_g.

Proof. It will be sufficient to consider 5.3(6). We define t_ν, $(\nu = 1, 2, \ldots)$ inductively by $t_1 = 1$,

5.3(7) $$\int_{t_\nu}^{t_{\nu+1}} x^{-1}\varphi(x)dx = 1 \qquad (\nu = 1, 2, \ldots).$$

Then $t_{\nu+1}/t_\nu \to 1$. The expression corresponding to 5.2(3) is

$$a''_\nu(x) = \frac{1}{x}\int_{t_\nu}^{t_{\nu+1}} |g(t/x)|dt = \int_{x^{-1}t_\nu}^{x^{-1}t_{\nu+1}} |g(u)|du.$$

Taking $A > 0$ so large that $\int_A^\infty |g|du < \epsilon$, we observe that the maximal length of $(x^{-1}t_\nu, x^{-1}t_{\nu+1})$ for all ν with $x^{-1}t_\nu \leqslant A$ tends to 0 as $x \to \infty$. This implies that $a''_\nu(x) < \epsilon$ for all ν and all sufficiently large x. Thus we obtain 5.2(1) and 5.3(7) gives the condition 5.2(2) of Theorem 10. The proof is complete.

A theorem on absolute summability corresponding to Theorem 10, (i) may be obtained using Theorem 7, §**4.2** instead of [14, Theorem 8]. In this way we obtain that $u_n = 0$, $n \neq n_\nu$ $(\nu = 1, 2, \ldots)$ is not a Tauberian condition for absolute summability by the matrix $A = (a_{mn})$ if

5.3(8) $$\lim_{\nu\to\infty} \left\{ \operatorname*{var}_{m} \sum_{n_\nu \leqslant n < n_{\nu+1}} a_{mn} \right\} = 0.$$

(More precisely, if 5.3(8) holds, there are bounded divergent sequences with $u_n = 0$, $n \neq n_\nu$, which are absolutely A -summable.) As an example we have that the high indices theorem for absolute Abel summability of Zygmund [17] cannot be improved.

5.4. In [15, 6.2] it has been shown that $u_n = o(n^{-1})$ is a Tauberian condition for any regular Hausdorff method H_g. We show now that for an unspecified generating function $g(t)$ this condition cannot be improved. *There are regular methods* H_g *such that* $u_n = O(n^{-1})$ *is not a Tauberian condition, even for bounded sequences.*

Set
$$g(t) = \begin{cases} 0 & \text{in } [0, \frac{1}{3}), \\ \frac{1}{2} & \text{in } [\frac{1}{3}, \frac{2}{3}), \\ 1 & \text{in } [\frac{2}{3}, 1]. \end{cases}$$

The corresponding H_g transformation is given by

5.4(1) $$\sigma_n = \tfrac{1}{2} \sum_{\nu=0}^{n} \binom{n}{\nu} [t_1^\nu (1 - t_1)^{n-\nu} + t_2^\nu (1 - t_2)^{n-\nu}] s_\nu, \qquad t_1 = \tfrac{1}{3},\ t_2 = \tfrac{2}{3}.$$

Using the well known properties of the Newton probabilities $p_{n\nu}(t) = \binom{n}{\nu} t^\nu (1 - t)^{n-\nu}$ it is easy to prove that under the hypotheses $u_n = O(n^{-1})$, $s_n = O(1)$ the method 5.4(1) is equivalent to the method defined by

5.4(2) $$\sigma_n = \tfrac{1}{2}(s_{[n/3]} + s_{[2n/3]}).$$

Therefore it is sufficient to give a function $s(u)$ of the real argument $u \geqslant 1$ such that $s(u) = O(1)$, $s(u + 1) - s(u) = O(u^{-1})$ and $s(u) + s(2u) \to 0$. But a function of this kind is defined by

$$s(u) = \begin{cases} (-1)^\nu (\log_2 u - \nu) & \text{for } 2^\nu \leqslant u < 2^{\nu+\frac{1}{2}} \\ (-1)^\nu (\nu + 1 - \log_2 u) & \text{for } 2^{\nu+\frac{1}{2}} \leqslant u < 2^{\nu+1}, \end{cases} \qquad (\nu = 0, 1, \ldots).$$

Our proof in [15, 6.2] was based on a gap theorem of Agnew [2] for the methods H_g. It is perhaps worth while to remark that the following improvement of Agnew's result is true. *For any regular method* H_g *there is a constant* $\lambda = \lambda_g > 1$ *such that* $u_n = 0$ *for* $n \neq n_\nu (\nu = 1, 2, \ldots)$ *is a Tauberian condition for the method* H_g, *if*

5.4(3) $$n_{\nu+1}/n_\nu \geqslant \lambda.$$

(Agnew assumes $n_{\nu+1}/n_\nu \to \infty$ instead of this.) The proof is obtained by combining Agnew's argument with a well known elementary Mercerian theorem ([1], also [16]). It is not known whether we may take λ_g as near to 1 as we please.

5.5. In this section we make some minor remarks, and corrections to earlier papers.

We first observe, that almost convergence [14, 1] may be defined for sequences of elements x_n of a Banach space. We call x_n almost convergent to x, if

5.5(1) $$\left\| x - \frac{x_{n+1} + \ldots + x_{n+\nu}}{\nu} \right\| \to 0 \text{ for } \nu \to \infty \text{ uniformly in } n.$$

(This implies that the $\|x_n\|$ are bounded.) We have, for example, the following theorem. *Any weakly convergent sequence of elements of a uniformly convex Banach space contains a strongly almost convergent subsequence* (which is therefore strongly C_α -summable for any $\alpha > 0$). In fact, a modification of the argument used by Kakutani [9] shows that the subsequence x_n which he proves to be strongly C_1-summable, is even strongly almost convergent.

Dr. R. G. Cooke kindly points out that he has used our condition [15, 2.4(1)] for some other purpose in [3]. He also makes the following remark. The condition $\max_n |a_{mn}| \to 0$ is equivalent, for any method A with the property $\sum_n |a_{mn}| \leqslant M$, to the condition

5.5(2) $$\sum_{n=0}^{\infty} a_{mn}^2 \to 0 \qquad m \to \infty,$$

for

$$\max_n |a_{mn}|^2 \leqslant \sum_n a_{mn}^2 \leqslant M \max_n |a_{mn}|.$$

Now 5.5(2) is given by Hill [7] as a necessary condition for a method A to possess the Borel property. Hence, by [14, Theorem 8*] if a regular method A has the Borel property, then it possesses summability functions of the first kind.

We note that a theorem by Garabedian, Hille and Wall [4, Theorem 5.2] gives a set of necessary and sufficient conditions in order that all functions $\Omega(n) = o(n)$ be summability functions of the second kind of a Hausdorff method H_g.

We use this opportunity to rectify some mistakes in our previous papers.

In the proof of [14, Theorem 10] the sequence $n_1 < n_2 < \ldots$ depends upon m (it is erroneously stated there that it is the same for all m in question).

In the formulation of Theorem 5 in *Operations in linear metric spaces*, Duke Math. J., vol. 15 (1948) 755-761, replace "when" by "if and only if".

In a review of the above paper (Math. Reviews, vol. 10 (1949), 255) it is stated that the proof of the main Theorem 1 of this paper is incomplete. The slips are, however, of minor nature and are rectified as follows:

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(a). The (well known) definition of openness of a mapping is incorrectly formulated on p. 757, lines 1-3. To obtain a correct one, replace the first part of line 3 by: "for any $y \in U_\sigma(y_0)$ an element $x \in U_\epsilon(x_0)$ exists for which $y = Sx$". Only the correct definition is used in the proof.

(b). Lines 15-16 on p. 757 are not sufficient to insure that the set $B_{a,b} = [a < \Phi(y) < b]$ is analytical. But the argument in the text applies to the set $B_b = [\Phi(y) < b]$, and since the $B_{a,b}$ are unions of differences of the B_b, they, too, are analytical.

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