7
Lattice gauge theory

In this chapter we introduce Wilson's (1974) elegant formulation of gauge fields on a space-time lattice. The idea is heavily motivated by the concept of a gauge field as a path-dependent phase factor. The basic degrees of freedom are group elements associated with bonds or straight-line paths connecting nearest neighbor pairs of lattice sites. The group element associated with an arbitrary path connecting a sequence of neighboring sites is the group product of these fundamental variables. This particular formulation is also remarkable in that the gauge freedom remains as an exact local symmetry.

Considering a general gauge group G, we associate an independent element of G with each nearest-neighbor pair of lattice sites (i,j)

$$U_{ij} \in G. \tag{7.1}$$

The indices i and j label the lattice sites at the ends of the bond on which U_{ij} is defined. We suppress those indices associated with the fact that U_{ij} is itself a matrix in the gauge group. On traversing a link in the opposite direction, one should obtain the inverse element

$$U_{ji} = (U_{ij})^{-1}. \tag{7.2}$$

We can define a vector potential by writing

$$U_{ij} = \mathrm{e}^{\mathrm{i}g_0 A_\mu a}. \tag{7.3}$$

Here a is the lattice spacing and the Lorentz index μ is the direction of the given bond. We use the matrix notation for A_μ, which is an element of the Lie algebra of the gauge group. The spatial coordinate x_μ associated with A_μ should be in the vicinity of the link in question; for convenience we take it to lie half way along the bond

$$x_\mu = \tfrac{1}{2}a(i_\mu + j_\mu). \tag{7.4}$$

In the continuum limit, this choice and the fact that U_{ij} should be path-ordered along the bond become irrelevant conventions.

We need an action to determine the dynamics of these field variables. The Lagrangian should reduce in the continuum limit to the classical Yang–Mills theory of the last chapter. The field strength is a generalized

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curl of the potential. This suggests using integrals of A_μ around small closed contours. Thus motivated, Wilson proposed that the action should be a sum over all elementary squares of the lattice

$$S = \sum_{\square} S_{\square}. \tag{7.5}$$

The action on each of these squares or 'plaquettes' is the trace of the product of the group elements surrounding the plaquette

$$S_{\square} = \beta[1-(1/n)\operatorname{Re}\operatorname{Tr}(U_{ij}\,U_{jk}\,U_{kl}\,U_{li})]. \tag{7.6}$$

Here the sites i, j, k and l circulate about the square in question. The factor $-1/n$ establishes normalization and sign conventions; n is the dimension of the group matrices. The normalization factor β will be defined later. The additive constant in eq. (7.6) is chosen to make the action vanish whenever the group elements approach the identity. The trace in this equation can be in any representation; for now we only consider the fundamental one.

The demonstration that this action reduces to the usual Yang–Mills theory begins with eq. (7.3) for the U_{ij} in terms of the vector potential. Consider, for example, a plaquette with center at x_μ and oriented in the $(\mu\,\nu) = (1,2)$ plane. Writing out eq. (7.6) gives

$$\begin{aligned} S_{\square} = \beta(1-(1/n)\operatorname{Re}\operatorname{Tr}(&\exp \mathrm{i}g_0\, A_1(x_\mu - \tfrac{1}{2}a\delta_{\mu 2}) \\ &\times \exp \mathrm{i}g_0\, A_2(x_\mu + \tfrac{1}{2}a\delta_{\mu 1}) \\ &\times \exp -\mathrm{i}g_0\, A_1(x_\mu + \tfrac{1}{2}a\delta_{\mu 2}) \\ &\times \exp -\mathrm{i}g_0\, A_2(x_\mu - \tfrac{1}{2}a\delta_{\mu 1}))). \end{aligned} \tag{7.7}$$

We now consider vector potentials smooth enough that we can Taylor expand about x. A little suppressed algebra, which the reader should carry out for himself, yields

$$S_{\square} = \beta(1-(1/n)\operatorname{Re}\operatorname{Tr}\exp(\mathrm{i}g_0\, a^2 F_{12} + O(a^4))). \tag{7.8}$$

Here F_{12} is the field strength tensor including the non-linear terms in A arising from manipulation of the orderings of the exponentials in eq. (7.7). Expanding the exponential, we find

$$S_{\square} = (\beta g_0^2/(2n))\, a^4 \operatorname{Tr}(F_{12}^2) + O(a^6). \tag{7.9}$$

The term of order a^2 vanishes because for unitary groups, the only type considered here, the group generators are Hermitian. We now approximate the sum over all plaquettes with a space-time integral to obtain

$$S = (\beta g_0^2/(2n)) \int \tfrac{1}{2} \operatorname{Tr}(F_{\mu\nu}\, F_{\mu\nu})\, \mathrm{d}^4x + O(a^6). \tag{7.10}$$

The factor of one-half under the integral comes from the symmetry under

$\mu\nu$ interchange. Thus we obtain the usual gauge theory action if we identify

$$\beta = 2n/g_0^2. \tag{7.11}$$

The terms with higher powers of the cutoff in eq. (7.10) vanish in the classical continuum limit. Because of divergences in the quantum theory, they can give rise to a finite renormalization of the coupling constant.

We now have our variables and Lagrangian. To proceed to the quantum theory, we insert the action into a path integral

$$Z = \int (\mathrm{d}U)\,\mathrm{e}^{-S(U)}. \tag{7.12}$$

Here we integrate over all possible values for the gauge variables. As they are elements of a compact group, it is natural to use the invariant group measure for this integration. The next chapter discusses this measure in some detail.

Eq. (7.12) defines the partition function for the statistical system motivating this book. Correlation functions are expectation values as discussed in earlier chapters. If H is some function of the field variables U, then its expectation is defined

$$\langle H \rangle = Z^{-1} \int (\mathrm{d}U)\, H(U)\,\mathrm{e}^{-S(U)}. \tag{7.13}$$

In the quantum mechanical Hilbert space, this is the vacuum expectation value of the corresponding time-ordered operator.

Note that we have not included any gauge fixing terms in the path integral. In usual continuum formulations, such terms eliminate a divergence from integrating over all gauges. Here, however, the variables are elements of a compact group. As a consequence, the gauge orbits are themselves compact. For gauge-invariant observables, it is harmless to include an integral over all gauges. We will, however, need to introduce the concept of gauge fixing in order to formulate perturbation theory or to use the transfer matrix to find a Hamiltonian formalism. We will discuss these points later.

Up to this point we have been considering only pure gauge fields. Inclusion of quark degrees of freedom simply involves taking the fermionic action from chapter 3 and inserting a factor of U_{ij} on the fermi field whenever a quark hops from site i to site j. The quark fields have an additional supressed internal symmetry index upon which this matrix acts. Adopting Wilson's projection operator technique for dealing with species doubling, we take eq. (4.55) and write the action for the full interacting gauge theory of quarks and gluons on a lattice

$$
\begin{aligned}
S = \beta \sum_{\square} (1-(1/n)\,\mathrm{Re}\,\mathrm{Tr}\, U_{\square}) \\
+\tfrac{1}{2} i a^3 \sum_{\{i,j\}} \bar{\psi}_i (1+\gamma_\mu e_\mu)\, U_{ij}\, \psi_j \\
+(a^4 m_0 + 4a^3) \sum_i \bar{\psi}_i \psi_i .
\end{aligned}
\tag{7.14}
$$

Here we have used several shorthand notations to keep the expression manageable. First, $U_{\square}$ represents the product of group elements around the plaquette in question. Second, the sum over $\{i, j\}$ is over all nearest-neighbor pairs and includes one term for each ordering of i and j. Finally, e_μ represents a unit vector in the direction from site i to site j. We have placed the subscript 0 on the mass m_0 to emphasize that this is the bare mass and will need to be renormalized for a continuum limit of this interacting theory. It is straightforward to introduce other matter fields, such as scalars. As these do not seem to play any role in strong interaction physics, we will only briefly mention them in chapter 9, where we point out some peculiarities of the Higgs mechanism for generating gauge meson masses.

On a kinematic level, the lattice theory has by construction the appropriate classical continuum limit as the Yang–Mills theory. Before such a limit, however, the model still keeps many other aspects of a gauge theory. For one, we work directly with a theory of phases. Furthermore, a local gauge symmetry remains exact. If we associate an arbitrary group element g_i with each lattice site, then the action is invariant under the change

$$
\begin{aligned}
U_{ij} &\rightarrow g_i\, U_{ij}\, g_j^{-1} \\
\psi_i &\rightarrow g_i\, \psi_i \\
\bar{\psi}_i &\rightarrow \bar{\psi}_i\, g_i^{-1}.
\end{aligned}
\tag{7.15}
$$

Only the definition of a gauge theory in terms of the Lorentz properties of the fields appears to be irrelevant to the lattice formulation, which rather severely mutilates space-time symmetries.

Faithfulness to an exact gauge symmetry should not be a requirement of a cutoff scheme. Indeed, the physics of a renormalizable theory should not depend on the details of the regulator. Nevertheless, this elegant formalism introduced by Wilson greatly simplifies strong coupling treatments of confinement and has been nearly universally adopted in lattice treatments.

Problems

1. Explicitly carry out the steps between eqs (7.7) and (7.10).

2. Consider taking the trace in eq. (7.8) in the adjoint rather than the fundamental representation of the group. What happens to eq. (7.11)?

3. Show that the fermionic terms in eq. (7.14) have the correct classical continuum limit.

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