Canad. Math. Bull. Vol. **23** (2), 1980

A CONSTRUCTION OF APPROXIMATELY FINITE-DIMENSIONAL NON-ITPFI FACTORS

BY
ALAIN CONNES AND E. J. WOODS

A von Neumann algebra is said to be approximately finite-dimensional if it is of the form

$$M = \left(\bigcup_{n=1}^{\infty} M_n\right)''$$

where $M_n \subseteq M_{n+1}$ for each n and each M_n is a finite-dimensional matrix algebra. A factor is said to be ITPFI if it is of the form

$$M = \bigotimes_{n=1}^{\infty} (M_n, \omega_n)$$

where each M_n is a type I factor (and ω_n is a state on M_n) [2]. The existence of factors which are approximately finite-dimensional but not ITPFI is an interesting problem. The first construction of such factors was given by Krieger [8]. However in [8] it is only proved that the factors are not "weakly equivalent" to any ITPFI factor. The first proof that these factors are not ITPFI was given by Connes [3]. Alternatively one could now use Krieger's theorem [9] that unitary equivalence implies "weak equivalence" to complete the argument. However Krieger's construction is rather involved, and the arguments of both Krieger [8] and Connes [3] were quite delicate. We give here a new construction for which, in the context of the flow of weights, the argument is rather elementary.

Section 1 reviews the relevant aspects of the flow of weights [4], and gives some terminology. Section 2 contains the technical lemmas. In Section 3 we discuss the examples.

1. **Preliminary material.** Let M be a factor, $\operatorname{Aut} M$ the group of all automorphisms of M with the topology of pointwise norm convergence in the predual, and $\operatorname{Int} M$ the subgroup of inner automorphisms of M. The flow of weights of M is an ergodic action of R_+^* on some measure space (X_M, μ_M). The construction of [4] gives not the measure space, but the measure algebra whose elements are equivalence classes $[\phi]$ of integrable weights ϕ of infinite type. The flow is then defined by $F_M(\lambda)[\phi] = [\lambda\phi]$. Let $\alpha \in \operatorname{Aut} M$. The equation $\operatorname{Mod} \alpha[\phi] = [\phi \circ \alpha]$ defines a Borel (and hence continuous) homomorphism

Received by the editors November 27, 1978 and, in revised form, January 19, 1979.
AMS (MOS) subject classifications (1970). Primary 46L10. Key words and phrases, approximately finite-dimensional factors, ITPFI factors.

https://doi.org/10.4153/CMB-1980-030-5 Published online by Cambridge University Press

from the polish group Aut M into the polish group of automorphisms of the measure space (X_M, μ_M). Clearly $\alpha \in \text{Int}\, M$ implies that $\text{Mod}\,\alpha = 1$. If M is a factor of type III_0 then the flow of weights $F_{M\otimes M}(\lambda)$ for $M\otimes M$ is given by the action of $F_M(\lambda)\otimes 1$ on the measure algebra of the $F_M(\lambda)\otimes F_M(\lambda^{-1})$ invariant sets on $X_M \times X_M$.

All Borel spaces considered in this paper are standard (i.e. Borel isomorphic to a Borel subset of the unit interval). A transformation S on a measure space (X, μ) is called non-singular if it is invertible and both S and S^{-1} are μ-measurable. Given a non-singular S, the orbit of x under S is the set

$$O_S(x) = \{S^j x : j \in Z\}.$$

The full group of S is the set $[S]$ of all non-singular transformations T such that for a.e. x, $Tx \in O_S(x)$. A set $W \subset X$ such that $\mu(S^j W \cap S^k W) = 0$ for all $j \neq k$ is called a wandering set for S. S is said to be dissipative if there is a wandering set W such that $X = \bigcup_{j=-\infty}^{\infty} S^j W$.

2. **The technical lemmas.** Let M be a von Neumann algebra, $x, y \in M$. The automorphism σ of $M\otimes M$ defined by the equation $\sigma(x\otimes y) = y\otimes x$ is called the Sakai flip.

LEMMA 2.1. *Let M be an ITPFI factor, σ the Sakai flip on $M\otimes M$. Then $\sigma \in \overline{\text{Int}}\,(M\otimes M)$*

Proof. Let $M = \otimes_{n=1}^{\infty}(M_n, \omega_n)$ be given on $\otimes(H_n, \Omega_n)$ where $\omega_n(x) = (x\Omega_n, \Omega_n)$. Then $M\otimes M = \otimes_n (M_n\otimes M_n, \omega_n\otimes\omega_n)$ acts on $K = \otimes_n (H_n\otimes H_n, \Omega_n\otimes\Omega_n)$. Let $\psi \in (M\otimes M)_*$, $\varepsilon > O$. We can assume that $\otimes_n (\Omega_n\otimes\Omega_n)$ is a separating vector for $M\otimes M$ (see Lemma 3.15 of [2]). Hence there is a vector $\Psi \in K$ such that $\psi(x) = (x\Psi, \Psi)$. By Lemma 3.1 of [1] there exists $m < \infty$ and $\Psi(m) \in \otimes_{n=1}^{m}(H_n\otimes H_n)$, $\|\Psi(m)\| = 1$, such that $\|\Psi - \Psi_\varepsilon\| < \varepsilon$ where

$$\Psi_\varepsilon = \Psi(m)\otimes\left(\bigotimes_{n=m+1}^{\infty}(\Omega_n\otimes\Omega_n)\right).$$

Let ψ_ε be the state defined by Ψ_ε, and let σ_m be the Sakai flip on $\otimes_{n=1}^{m}(M_n\otimes M_n)$. Then $\psi_\varepsilon \circ \sigma = \psi_\varepsilon \circ (\sigma_m \otimes 1)$. Hence

$$\|\psi\circ\sigma - \psi\circ(\sigma_m\otimes 1)\| < 2\varepsilon$$

Since σ_m is inner, it follows that $\sigma = \lim_{m\to\infty} \sigma_m \otimes 1 \in \overline{\text{Int}}\,(M\otimes M)$. QED.

LEMMA 2.2. *Let R, S be non-singular transformations on the standard measure space (X, μ). If S is dissipative and R leaves invariant (modulo μ) all S-invariant measurable sets, then $R \in [S]$.*

Proof. We first note that if (E, ν) is a countably separated measure space and $f: E \to E$ satisfies $f(B) = B$ (modulo ν) for all measurable $B \subset E$, then

https://doi.org/10.4153/CMB-1980-030-5 Published online by Cambridge University Press

$f(x)=x$ (a.e. ν). Namely let $(B_n)_{n\in N}$ separate points in E. Then

$$\{x : f(x)\neq x\}\subset \bigcup_n \{B_n \setminus f(B_n)\}$$

which is a set of measure zero.

Now let W be a wandering set for S such that $X=\cup_{k=-\infty}^{\infty} S^k W$. Let P_k be the projection of X onto $S^k W$ defined by $P_k x = y$ if $x = S^j y$ for some j such that $y\in S^k W$. Let A be any measurable subset of $S^k W$. Then $\cup_{p=-\infty}^{\infty} S^p A$ is S-invariant and it follows that $(P_k R P_k)A = A$ (modulo μ). Now clearly $R\in[S]$ if and only if $P_k R P_k(x)=x$ (a.e.) for all k. QED.

The following theorem uses the base and ceiling function construction of a flow. For this purpose it is more convenient to have the flow as an action of R rather than R_+^*. Hence we shall use $\mathfrak{F}_M(\lambda)=F_M(e^\lambda)$.

THEOREM 2.3. *Let M be a factor of type* III_0 *whose flow of weights can be built under a constant ceiling function with a base transformation T such that* $T\times T^{-1}$ *is dissipative. Then the Sakai flip* $\sigma\notin\overline{\mathrm{Int}}\,(M\otimes M)$ *and hence M is not ITPFI.*

Proof. Clearly $\mathrm{Mod}\,\sigma$ acts on $X_M\times X_M=(B\times I)\times(B\times I)$ by $\sigma(x,s,y,t)=(y,t,x,s)$. Let E be any $T\times T^{-1}$ invariant set in $B\times B$, σ_B the flip on $B\times B$. Then $E\times I\times I$ is an $\mathfrak{F}_M(\lambda)\otimes\mathfrak{F}_M(-\lambda)$ invariant set in $X_M\times X_M$. Now assume $\mathrm{Mod}\,\sigma=1$. Then σ_B must preserve E, hence $\sigma_B\in[T\times T^{-1}]$ by the preceding lemma. But this implies that for a.e. $(x,y)\in B\times B$ there exists an integer $n(x,y)$ such that

$$\sigma_B(x,y)=(y,x)=(T^{n(x,y)}x,\ T^{-n(x,y)}y),$$

i.e. $y\in O_T(x)$. But $O_T(x)$ is countable. QED.

3. **The examples.** It remains to demonstrate the existence of approximately finite-dimensional factors of type III_0 satisfying the conditions of Theorem 2.3. For this we first need the existence of ergodic transformations T such that $T\times T^{-1}$ is dissipative. It is a classical result in ergodic theory that such transformations exist [6]. As a specific example, one can use the Markov shift obtained from a two-dimensional random walk. (These transformations preserve an infinite measure.) The existence now follows from the fact that any flow arises as the flow of weights of some approximately finite-dimensional factor [4, 9]. (The proof of this in the general case is not so easy. However for measure preserving flows the argument is not difficult (see for example [7]).)

We remark that $\sigma\in\overline{\mathrm{Int}}\,(M\otimes M)$ is not a sufficient condition for M to be ITPFI. Namely let M be an approximately finite-dimensional factor whose flow can be built under a constant ceiling function with a base transformation T which preserves a finite measure. If T is a Bernoulli shift then M is not ITPFI [5]. But then $T\times T^{-1}$ is ergodic, and it follows easily that $\mathrm{Mod}\,\sigma=1$. Hence

https://doi.org/10.4153/CMB-1980-030-5 Published online by Cambridge University Press

$\sigma \in \overline{\text{Int}}\,(M \otimes M)$ [4]. In fact if T is any ergodic transformation preserving a finite measure, it follows from the proof of part (2) of lemma 1 of [7] that $\text{Mod}\,\sigma = 1$ (see also [10]).

References

1. H. Araki and E. J. Woods, *Complete Boolean Algebras of Type I Factors*, Publ. Res. Inst. Math. Sci Ser. A **2** (1966), 157–242.
2. H. Araki and E. J. Woods, *A classification of factors*, Publ. Res. Inst. Math. Sci. Ser. A **4** (1968), 51–130.
3. A. Connes, *Une classification des facteurs de type III*, Ann. Sci. E.N.S. 4 eme Serie **6** (1973), 133–25.
4. A. Connes and M. Takesaki, *Flow of weights on type III factors*, Tohoku Math J. **29** (1977), 473–575.
5. A. Connes and E. J. Woods, to appear.
6. T. E. Harris and H. Robbins, *Ergodic theory of Markov chains admitting an infinite invariant measure*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 860–864.
7. T. Hamachi, Y. Oka and M. Osikawa, *Flows Associated with Ergodic Non-Singular Transformation Groups*, Publ. Res. Inst. Math. Sci. **11** (1975), 31–50.
8. W. Krieger, *On the infinite product construction of nonsingular transformations of a measure space*. Invent. Math. **15** (1972), 144–163.
9. W. Krieger, *On ergodic flows and the isomorphism of factors*, Math. Ann. **223** (1976), 19–70.
10. T. Hamachi and M. Osikawa, *Fundamental Homomorphism of Normalizer Group of Ergodic Transformation*, preprint.

Université de Paris VI

Department of Mathematics and Statistics
Jeffery Hall
Queen's University
Kingston, Ontario K7L 3N6

https://doi.org/10.4153/CMB-1980-030-5 Published online by Cambridge University Press