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LOCATION OF ZEROS OF CHROMATIC AND RELATED POLYNOMIALS OF GRAPHS

FRANCESCO BRENTI, GORDON F. ROYLE AND DAVID G. WAGNER

ABSTRACT. We consider the location of zeros of four related classes of polynomials, one of which is the class of chromatic polynomials of graphs. All of these polynomials are generating functions of combinatorial interest. Extensive calculations indicate that these polynomials often have only real zeros, and we give a variety of theoretical results which begin to explain this phenomenon. In the course of the investigation we prove a number of interesting combinatorial identities and also give some new sufficient conditions for a polynomial to have only real zeros.

0. **Introduction.** The chromatic polynomial $P(G;x)$ of a graph G has been studied extensively since its introduction by Birkhoff in 1912, as it encodes quite a lot of information about the graph. A good introduction to the literature can be gained from [6, 21, 23, 26] and so we will not go into the general theory in detail. An even more informative invariant of a graph is its Tutte polynomial, as developed in [3, 12, 31], for instance. Over the years, particular effort has gone into deriving conditions on a polynomial which are necessary if it is to be the chromatic polynomial of some graph. In this paper we approach this problem from a new direction: instead of working with the chromatic polynomial itself we consider three polynomials related to it by certain invertible linear transformations of $\mathbb{R}[x]$. These polynomials, denoted by $\sigma(G;x)$, $\tau(G;x)$, and $w(G;x)$, were introduced quite recently [9, 21] and seem to have a more regular behaviour with regard to location of zeros than does the chromatic polynomial.

In Section 1 we briefly review some of the theory of chromatic polynomials, with emphasis on results and conjectures concerning the location of their zeros and constraints on their coefficients. The only new result of the section (Theorem 1.3) provides a refined method for dealing with one of these conjectures. In Section 2 we define the polynomials $\sigma(G;x)$, $\tau(G;x)$, and $w(G;x)$ and describe some of their basic properties. By restricting the location of zeros or values of the coefficients of these polynomials we define several properties of graphs; in particular, it is not uncommon for these polynomials to have only real zeros. Many implications hold among the properties we define, as seen in [8]. Section 3 is an account of computations which produced a number of interesting examples and showed that a large majority of graphs on up to 9 vertices satisfy some of the most stringent of the properties of Section 2. Computations for regular graphs on up to 13 vertices, and 3-regular graphs on up to 16 vertices, were also made.

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Sections 4 and 5 consist of theoretical results which begin to explain the ubiquity of graphs satisfying the properties of Section 2. In Section 4 we concentrate on the σ-polynomial, giving several structural conditions on a graph G which are sufficient to imply that all the zeros of $\sigma(G;x)$ are real, *i.e.* that G is "σ-real". Our main results are a Complete Cutset Theorem for σ-reality (Corollary 4.6) and a theorem which indicates that σ-reality dominates σ-unreality in the presence of large chromatic number (Theorem 4.11). We also show that all cycles are σ-real, verifying part of Conjecture 7.3 of [9]. The results in Section 5 for the τ- and w-polynomials are not as strong. In particular, we do not yet have Complete Cutset Theorems for τ- or w-reality, although we have a number of partial results in this direction. For instance, Corollary 5.23 states that if G is w-real, H is chordal, and $G \cap H$ is complete then $G \cup H$ is also w-real, which generalizes Theorem 4.19 of [9].

We conclude in Section 6 with a variety of questions and conjectures which are suggested by both the theoretical and computational results of the preceding sections.

1. **Chromatic polynomials.** For a (finite, simple, undirected) graph $G = (V, E)$ and a positive integer m, a *proper m-colouring* of G is a function $f\colon V \longrightarrow \{1, 2, \ldots, m\}$ such that for all $i = 1, \ldots, m$, the set of vertices $f^{-1}(i)$ induces an edge-free subgraph of G. The smallest m for which G has a proper m-colouring is called the *chromatic number* of G and is denoted by $\chi(G)$. The *chromatic polynomial* of G is the unique polynomial $P(G;x)$ such that for every positive integer m, the value $P(G;m)$ is the number of proper m-colourings of G. That $P(G;m)$ truly is a polynomial function of m is evident from the elementary combinatorial expansion

$$P(G;x) = \sum_{j=0}^{n} a_j(G) x_{\langle j \rangle} \tag{1}$$

in which $x_{\langle j \rangle} = x(x-1)\cdots(x-j+1)$ is the j-th *falling factorial* polynomial, $n = \#V(G)$, and $a_j(G)$ is the number of partitions of $V(G)$ into exactly j blocks, each block inducing an edge-free subgraph of G (Theorem 2.1 of [27]).

Later, we will use the following fact, the Complete Cutset Theorem for chromatic polynomials. It appears with its simple proof as Theorem 2.5 of [27]. Notice that for the complete graph K_m we have $P(K_m;x) = x_{\langle m \rangle}$, and that Proposition 1.1 includes the case of disjoint union of graphs.

PROPOSITION 1.1. *Let G and H be graphs such that $G \cap H$ is a complete graph. Then*

$$P(G \cup H;x) = \frac{P(G;x)P(H;x)}{P(G \cap H;x)}.$$

There is another important property of chromatic polynomials, the Deletion-Contraction Algorithm (Theorem 2.6 of [27]), but we will not use it.

One of the best-known conjectures regarding chromatic polynomials is due to Read [24], who conjectured in 1968 that if the chromatic polynomial of any graph

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$P(G;x)$ is expanded in powers of x,

$$P(G;x) = \sum_{j=0}^{n}(-1)^{n-j}b_j(G)x^j, \tag{2}$$

then the sequence of coefficients $b_0, b_1, \ldots, b_n$ of G is *unimodal*: *i.e.* there is an index k such that $b_0 \leq \cdots \leq b_k \geq \cdots \geq b_n$. Read's conjecture was strengthened in 1974 by Hoggar [19] to claim that the sequence $b_0, b_1, \ldots, b_n$ is *strictly logarithmically concave* (SLC, for short): that is, that $b_j^2 > b_{j-1}b_{j+1}$ for all $j = 1, \ldots, n-1$. The advantage of this conjecture is that the SLC property of sequences is preserved by convolution, whereas mere unimodality need not be. Both conjectures are still open, but examples due to Björner show that they are likely to be false in general [7]. For a survey of known positive results on the Read-Hoggar conjecture, see [15].

As explained in Theorem 2.7 of [27] the Deletion-Contraction Algorithm implies that $b_j(G) \geq 0$ for all $j = 0, \ldots, n$. In fact, the coefficients $b_j(G)$ have a well-known combinatorial meaning, discovered by Whitney [39] in 1932, as the rank-sizes of a "broken circuit complex" associated with the graph G. It follows that $b_0(G), \ldots, b_n(G)$ has *no internal zeros*: *i.e.* that if $b_i(G) \neq 0$ and $b_j(G) \neq 0$ with $i < j$ then $b_k(G) \neq 0$ for all $i < k < j$. Note that if a sequence of nonnegative real numbers is SLC and has no internal zeros then it is unimodal. Wilf [40] explains Whitney's interpretation of the coefficients $b_j(G)$ in detail and uses it to derive some inequalities among the $b_j(G)$ *via* the Kruskal-Katona Theorem. Unfortunately, these inequalities do not imply unimodality of the sequence $b_j(G)$. More information about these coefficients of $P(G;x)$ can be found in [22, 27].

In relation to the Read-Hoggar conjecture the next proposition is interesting; see Proposition 2.9 of [30] or, for a more general result, Theorem 7.1 in Chapter 8 of [20].

PROPOSITION 1.2. *If all the zeros z of the polynomial $f(x) \in \mathbb{R}[x]$ are in the region $\{z \in \mathbb{C} : |\arg z| < \pi/3\}$ then the sequence of coefficients of $f(x)$ is* SLC *and alternates in sign.*

Proposition 1.2 can be extended slightly by also considering cubic factors of $f(x)$.

THEOREM 1.3. *Let $f(x) \in \mathbb{R}[x]$ factor over $\mathbb{C}$ as*

$$f(x) = C\prod_{i=1}^{k}(x - r_i)\prod_{j=1}^{m}(x - z_j)(x - \bar{z}_j)$$

where $C \in \mathbb{R}$, $r_i \in \mathbb{R}$ and $z_j \notin \mathbb{R}$ for all i, j. Suppose that $r_i \geq 0$ for all i, and that there exists a sequence of distinct positive integers $p_1, \ldots, p_m$ such that for all $j = 1, \ldots, m$ either $|\arg z_j| < \pi/3$, or $p_j \leq k$ and

$$1 - 2\cos\arg z_j < \min\left(\frac{|z_j|}{r_{p_j}}, \frac{r_{p_j}}{|z_j|}\right)^2.$$

Then the sequence of coefficients of $f(x)$ is SLC *and alternates in sign.*

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PROOF. Since convolution of sequences preserves sign-alternation and the SLC property it suffices to check that if $f(x) = (x-r)(x-z)(x-\bar{z})$ is a cubic polynomial with $r \geq 0$ and $z \notin \mathbb{R}$ which satisfies

$$1 - 2\cos\arg z < \min\left(\frac{|z|}{r}, \frac{r}{|z|}\right)^2 \tag{3}$$

then the sequence of coefficients of $f(x)$ is SLC and alternates in sign.

Note that (3) implies that $|\arg z| < \pi/2$. Letting $z = se^{i\theta}$ with $s > 0$ and $0 < \theta < \pi/2$ we have

$$f(x) = x^3 - (r + 2s\cos\theta)x^2 + (s^2 + 2rs\cos\theta)x - rs^2.$$

Thus the coefficients of $f(x)$ alternate in sign if and only if $r + 2s\cos\theta > 0$ and $s^2 + 2rs\cos\theta > 0$; that is, if and only if

$$\min\left(\frac{r}{s}, \frac{s}{r}\right) > -2\cos\theta$$

and this is implied by (3). Strict log-concavity of the coefficients of $f(x)$ is equivalent to

$$\frac{r^2}{s^2} + 2\frac{r}{s}\cos\theta > 1 - 4\cos^2\theta \tag{4}$$

and

$$\frac{s^2}{r^2} + 2\frac{s}{r}\cos\theta > 1 - 4\cos^2\theta. \tag{5}$$

By symmetry of these inequalities we may assume that $r \geq s$. Also, since $\theta < \pi/2 < 2\pi/3$ we have $1 + 2\cos\theta > 0$. Now since $r^2/s^2 \geq r/s$, for (4) to hold it is sufficient that

$$\frac{r}{s} > 1 - 2\cos\theta,$$

and since $s/r \geq s^2/r^2$, for (5) to hold it is sufficient that

$$\frac{s^2}{r^2} > 1 - 2\cos\theta.$$

Consequently, for the coefficients of $f(x)$ to be SLC and sign-alternating it is sufficient that (3) hold. This completes the proof. ∎

Read and Royle [26] have made extensive calculations of the zeros of chromatic polynomials, and it appears that Theorem 1.3 accounts for $b_j(G)$ being SLC for most of the graphs they consider. However, a similar "balancing act" using quartic factors of $f(x)$ could lead to even stronger results.

Thier [31] has derived the following completely general result on the location of zeros of chromatic polynomials of graphs from Gershgorin's Theorem.

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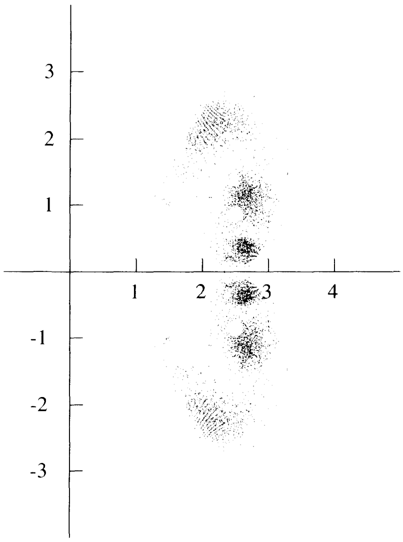

FIGURE 1: THE CHROMATIC ZEROS OF 9-VERTEX 18-EDGE GRAPHS

PROPOSITION 1.4. *Let G be a simple graph with n vertices and e edges. Then the zeros of $P(G;x)$ lie in $U_1 \cap U_2$, where*

$$U_1 = \{z \in \mathbb{C} : |z| \leq e-1 \text{ or } |z-e| \leq e\}$$

and

$$U_2 = \{z \in \mathbb{C} : |z-1| \leq e-1 \text{ or } |z-e+n-2|\,|z-1| \leq e(e-1)\}.$$

It seems that the information given by Proposition 1.4 is rather weak, and that for many graphs the zeros of $P(G;x)$ lie well within this region. Figure 1 illustrates the case for 9 vertices and 18 edges. Note that these zeros are all inside the disc $|z| \leq 5$ while Thier's region contains the disc $|z| \leq 14$.

Biggs, Damerell, and Sands [4] make the following conjecture.

CONJECTURE 1.5. *There is a function $f: \mathbb{N} \longrightarrow \mathbb{R}$ such that if G is a k-regular graph and $P(G;z) = 0$ then $|z| \leq f(k)$. Furthermore, $f(3) = 3$.*

They compute that if G is a 3-regular graph with at most 10 vertices and $P(G;z) = 0$ then $|z| \leq 3$. Read and Royle [26] extend this verification to all 3-regular graphs with at most 16 vertices, as well as some larger graphs. Biggs, Damerell, and Sands also derive formulae for the chromatic polynomials of prisms and Möbius ladders, and use Rouché's Theorem to show that all of their zeros have modulus at most 3 as well. Intriguingly, the set of zeros tends toward a limit set for these two families of graphs, a phenomenon explained by Beraha, Kahane, and Weiss [1, 2, 26]. One may ask for even more information, however. In addition to knowing the limit set of zeros of a sequence of

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polynomials $\{f_n : n \in \mathbb{N}\}$ (provided that it exists), we would also like to know the *limit distribution* of the set of zeros of $\{f_n\}$: that is, the measure μ on $\mathbb{C}$ such that for any Borel set $U \subset \mathbb{C}$, if $m(U;f)$ denotes the number of zeros of f in U counted with multiplicities, then

$$\mu(U) = \lim_{n\to\infty} \frac{m(U;f_n)}{\deg f_n}$$

(provided that it exists). For the recursive sequences of polynomials considered in [1, 2, 4, 26] a satisfactory solution to this problem is still outstanding.

In contrast to Conjecture 1.5, Woodall [41] uses the example of $K_{n,m}$ to show that there can be no upper bound for the modulus of a zero of $P(G;x)$ in terms of any function of the chromatic number $\chi(G)$ of G. In fact for m fixed and $n \longrightarrow \infty$ we have $\chi(K_{n,m}) = 2$, but $P(K_{n,m};x)$ has a real zero arbitrarily close to each integer in the interval $[0, m/2]$.

The results presented in this section were selected because of their relevance to the location of zeros of chromatic polynomials of arbitrary graphs. The class of planar graphs has received much attention as regards location of zeros of the chromatic polynomial [6, 33, 34, 35, 41] but we cannot do justice here to this subject.

2. **A hierarchy of conditions.** Despite the bewildering complexity of the location of zeros of chromatic polynomials, the location of zeros of some closely related polynomials shows really surprising regularities. In this section we discuss these regularity properties; in the next section we begin to see when these properties hold.

Let

$$P(G;x) = \sum_{j=0}^{n} a_j(G)x_{\langle j\rangle} = \sum_{j=0}^{n}(-1)^{n-j}c_j(G)x^{\langle j\rangle} \tag{6}$$

be the chromatic polynomial of a graph G, where $x_{\langle j\rangle}$ is the j-th falling factorial polynomial and $x^{\langle j\rangle} = x(x+1)\cdots(x+j-1)$ is the j-th *rising factorial* polynomial. The σ*-polynomial* of G is

$$\sigma(G;x) = \sum_{j=0}^{n} a_j(G)x^j \tag{7}$$

and the τ*-polynomial* of G is

$$\tau(G;x) = \sum_{j=0}^{n}(-1)^{n-j}c_j(G)x^j. \tag{8}$$

That is, we have $\sigma(G;x) = SP(G;x)$ and $\tau(G;x) = TP(G;x)$ where the linear transformations S and $T\colon \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ are defined by $Sx_{\langle j\rangle} = x^j$ and $Tx^{\langle j\rangle} = x^j$ and linear extension.

The σ-polynomial was introduced by Korfhage [21] (in a slightly different form) and is a special case of the "partition polynomial" considered by Wagner [36]. The other polynomials discussed here were introduced by Brenti [9], although our normalization of $\tau(G;x)$ is slightly different from that in [9]. The reason for this is a symmetry between the transformations S and T which will be exploited in Section 5.

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Standard results on rational generating functions (Corollary 4.3.1 of [29]) show that we may define the *w-polynomial* $w(G;x)$ of G by

$$\sum_{m\in\mathbb{N}} P(G;m)x^m = \frac{\sum_{j=0}^{n} h_j(G)x^j}{(1-x)^{1+n}} = \frac{w(G;x)}{(1-x)^{1+n}} \tag{9}$$

where $n = \#V(G)$ is the degree of $P(G;x)$, and that $w(G;1) = n!$.

The coefficients $a_j(G)$ and $b_j(G)$ have the combinatorial meanings mentioned above and so are nonnegative with no internal zeros. Brenti [9] shows that the coefficients $c_j(G)$ and $h_j(G)$ also have combinatorial interpretations but they are not so straightforward, being related to Stanley's results connecting chromatic polynomials with acyclic orientations of graphs [28]. For a graph G and a subset $B \subseteq V(G)$, let $G|_B$ denote the subgraph of G induced by the vertices B. Also, let $\mathrm{acyc}(G)$ be the number of acyclic orientations of G. For a proof of Proposition 2.1 see Theorem 5.3 of [9].

PROPOSITION 2.1. *For any graph G and $j \in \mathbb{N}$,*

$$c_j(G) = \sum_{\pi} \prod_{B\in\pi} \mathrm{acyc}(G|_B)$$

where the summation is over all partitions of $V(G)$ into exactly j blocks.

Incidentally, Proposition 2.1 shows that $(-1)^n\tau(G;-x)$ is a natural example of a "weighted partition polynomial", as discussed in Section 5 of [36]. We will not state the interpretation of $h_j(G)$, but see Theorem 4.2 of [9]. Nonetheless, this shows that $c_j(G)$ and $h_j(G)$ are nonnegative with no internal zeros, for any graph G. It follows that all the real zeros of $P(G;x)$ and of $\tau(G;x)$ are nonnegative, and that all the real zeros of $\sigma(G;x)$ and of $w(G;x)$ are nonpositive.

It sometimes happens that for a particular graph G, all of the zeros of $P(G;x)$, $\sigma(G;x)$, $\tau(G;x)$ or $w(G;x)$ are in fact real. In this case we say that G is *P-real, σ-real, τ-real*, or *w-real*, respectively. If a graph is not σ-real then it is *σ-unreal*, and so on. If the coefficients $a_j(G)$ of $\sigma(G;x)$ form an SLC sequence then we say that G is *σ-SLC*, and we define *P-*, *τ-*, and *w-SLC* similarly for the sequences $b_j(G)$, $c_j(G)$, and $h_j(G)$. Thus the Read-Hoggar conjecture is that every graph is P-SLC.

The condition that a polynomial has only real nonpositive zeros is deeply connected with the theory of total positivity *via* the characterization of Pólya frequency sequences, see [8, 20, 37]. We limit ourselves to indicating in Figure 2 the implications which hold among the various properties above, as developed in [9]. Analogous properties can be defined for arbitrary polynomials *via* the transformations presented in this section and in this general context there are no implications among the properties which are not implied in Figure 2 (see Section 2.6 of [8]). However, for the graph polynomials of this paper there may well be additional implications imposed by their special character (see, for example [38]). In particular, in view of the Read-Hoggar conjecture it would be very interesting to know if P-SLC were implied by any of the other properties.

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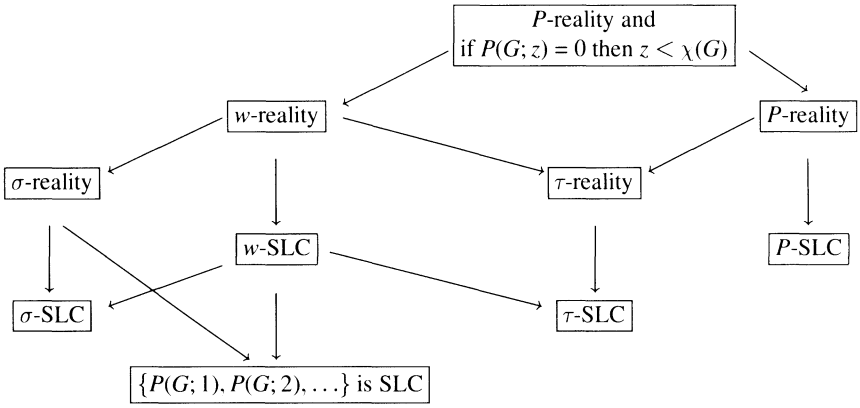

FIGURE 2: A HIERARCHY OF CONDITIONS

3. **Computations.** Computation of chromatic polynomials is in general a very difficult task even for moderately small graphs. However, by using the method of Read [25] we have been able to find and analyse the chromatic polynomials of all graphs on up to 9 vertices, 3-regular graphs on up to 16 vertices, and all other regular graphs on up to 13 vertices (with the exception of those of degree 6 on 13 vertices). The lists of graphs were obtained from various sources. The connected graphs on up to 9 vertices were extracted from the catalogues of Cameron, Colbourn, Read and Wormald [10], the 3-regular graphs from the catalogues of McKay and Royle [23] and the other regular graphs were constructed by Rob Beezer (private communication).

Having computed the chromatic polynomials, the number of complex zeros of each of the P-, σ-, τ- and w-polynomials was determined using the method of Sturm sequences. As these calculations were carried out with exact integer arithmetic we are confident that the validity of our results was not compromised by numerical inaccuracy.

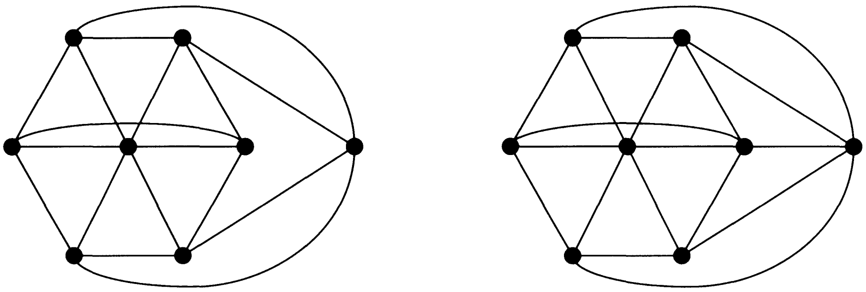

FIGURE 3: THE 8-VERTEX σ-UNREAL GRAPHS

The following tables give the condensed results of these computations; the entries indicate the number of graphs (up to isomorphism) with given properties. We begin

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No. of vertices	3	4	5	6	7	8	9
chordal	2	5	15	58	272	1614	11911
P-real	2	5	15	58	273	1627	12121
P-unreal	0	1	6	54	580	9490	248959
σ-real	2	6	21	112	853	11115	261038
σ-unreal	0	0	0	0	0	2	42
τ-real	2	6	21	112	853	11117	261080
τ-unreal	0	0	0	0	0	0	0
w-real	2	5	18	96	737	9880	238565
w-unreal	0	1	3	16	116	1237	22515

TABLE 1: THE CONNECTED GRAPHS ON UP TO 9 VERTICES

with all connected graphs on up to 9 vertices, in Table 1. (Corollaries 4.6, 5.20, and Proposition 5.12 below explain this restriction to connected graphs.) One prominent feature of this table is that it provides the first example of a σ-unreal graph. The 8-vertex σ-unreal graphs are shown in Figure 3: the graph on the left has $\sigma(G_1; x) = x^8 + 11x^7 + 38x^6 + 36x^5 + 11x^4 + x^3$ which has zeros at $-0.1924 \pm 0.0499i$, and the graph on the right has $\sigma(G_2; x) = x^8 + 10x^7 + 30x^6 + 31x^5 + 10x^4 + x^3$ which has zeros at $-0.2309 \pm 0.0129i$ (to 4 decimal places). The 9-vertex σ-unreal graphs are depicted in the Appendix.

Another prominent feature of Table 1 is the small proportion of P-real graphs. Examination of these graphs shows that the vast majority of them are chordal (see Section 4), although there are a small number of others. Given the stringency of P-reality, it is perhaps surprising that such a large proportion of the graphs in Table 1 are w-real. Most remarkable, however, is the fact that all graphs on up to 9 vertices are τ-real.

Table 2 summarizes the results for the connected regular graphs on 10 to 13 vertices, and 3-regular graphs on up to 16 vertices. These computations produced even more examples of σ-unreal graphs, but again failed to find even a single τ-unreal graph. Notice that while the proportion of w-real graphs in Table 2 is very much lower than in Table 1, among the 3-regular graphs the w-real graphs form a large majority. The 3-regular w-unreal graphs in Table 2 are the same 3-regular graphs of high girth which exhibit extremal behaviour in [26]. All of the graphs in Table 2 are P-unreal, except for the complete graphs and the $(n-3)$-regular graphs on n vertices with complements which have no cycles of length greater than four.

4. σ-polynomials.

We are now concerned with results which state that certain structural conditions on a graph are sufficient to imply that it is σ-real. Some of these results are known, and some are new.

The class of *supersolvable* graphs is the smallest class containing all complete graphs, and such that if G and H are in the class and $G \cap H$ is complete, then $G \cup H$ is also in the class. Supersolvable graphs are also called chordal graphs because of an alternate characterization [14]. Note that Proposition 1.1 implies by induction that if G

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is supersolvable then every zero of $P(G;x)$ is a nonnegative integer; thus, supersolvable graphs satisfy the strongest condition in Figure 2. The converse is false: there are graphs G which are not supersolvable but for which all zeros of $P(G;x)$ are nonnegative integers (see page 35 of [27]).

n	k	σ-real	σ-unreal	τ-real	τ-unreal	w-real	w-unreal
10	3	19	0	19	0	14	5
10	4	59	0	59	0	6	53
10	5	60	0	60	0	0	60
10	6	21	0	21	0	0	21
10	7	5	0	5	0	0	5
10	8	1	0	1	0	0	1
10	9	1	0	1	0	1	0
11	4	265	0	265	0	44	221
11	6	264	2	266	0	0	266
11	8	6	0	6	0	0	6
11	10	1	0	1	0	1	0
12	3	85	0	85	0	80	5
12	4	1544	0	1544	0	318	1226
12	5	8859	89	7848	0	60	7788
12	6	7814	35	7849	0	0	7849
12	7	1489	58	1547	0	0	1547
12	8	92	0	94	0	0	94
12	9	9	0	9	0	0	9
12	10	1	0	1	0	0	1
12	11	1	0	1	0	1	0
13	4	10778	0	10778	0	2643	8135
13	6	?	?	?	?	?	?
13	8	10662	124	10786	0	0	10786
13	10	10	0	10	0	0	10
13	12	1	0	1	0	1	0
14	3	509	0	509	0	503	6
16	3	4060	0	4060	0	4055	5

TABLE 2: THE CONNECTED k-REGULAR GRAPHS ON n VERTICES

A graph $G = (V, E)$ is an *incomparability graph* when there is a partial order $\leq$ on V such that $uv \in E$ if and only if u and v are not comparable in $(V, \leq)$.

If G is a graph and $\mathcal{H} = \{H_v : v \in V(G)\}$ is a collection of pairwise vertex-disjoint graphs, then the *composition of $\mathcal{H}$ into G* is the graph $G[\mathcal{H}]$ with vertex-set $\bigcup\{V(H_v) : v \in V(G)\}$ and with edges ij whenever $ij \in E(H_v)$ for some $v \in V(G)$, or $i \in V(H_u)$ and $j \in V(H_v)$ with $uv \in E(G)$.

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Proposition 4.1 summarizes some results of [9, 36] in this context.

PROPOSITION 4.1. *If G has any of the following properties then G is σ-real:*
(a) G is a supersolvable graph;
(b) the complement G^c of G contains no triangles;
(c) G is an incomparability graph.
(d) Furthermore, if G^c contains no triangles and $\mathcal{H} = \{H_v : v \in V(G)\}$ is a collection of pairwise vertex-disjoint σ-real graphs, then the composition $G[\mathcal{H}]$ is also σ-real.

Part (b) is a rephrasing of the Heilmann-Lieb Theorem on matching polynomials [16, 18]. Part (c) is equivalent to a result of Goldman, Joichi, and White [17]. Part (d) subsumes many constructions: the disjoint union $K_2^c[G,H]$ of G and H, the *join* $K_2[G,H]$ of G and H which is usually denoted by $G \vee H$, the complete multipartite graphs $K_{m_1}^c \vee \cdots \vee K_{m_p}^c$, and so on. It is worth noting that for the join of graphs we have

$$\sigma(G \vee H; x) = \sigma(G; x)\sigma(H; x)$$

(Theorem 3.13 of [9]).

Although we will return to the case of σ-polynomials, some of our results are purely to do with the location of zeros of arbitrary polynomials. Of course, the type of result developed here is motivated by our combinatorial applications. The starting point for our presentation is the following proposition, which is implicit in [36] and can also be derived from the well-known identity

$$x_{\langle i\rangle}x_{\langle j\rangle} = \sum_{k\in\mathbb{N}} k!\binom{i}{k}\binom{j}{k}x_{\langle i+j-k\rangle}$$

(Exercise 23, p. 83 of [12]). We use the notation D for the differentiation operator d/dx.

PROPOSITION 4.2. *Let $S\colon \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ be the linear transformation defined by $Sx_{\langle j\rangle} = x^j$ and linear extension. Then for any $f, g \in \mathbb{R}[x]$ we have $S(fg) = (Sf) * (Sg)$, in which the operation $*\colon \mathbb{R}[x] \times \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ is given by*

$$p * q = \sum_{k\in\mathbb{N}} \frac{x^k}{k!}(D^k p)(D^k q).$$

The $*$-product satisfies a number of interesting identities. Let us say that in algebraic formulae $*$ has a binding strength intermediate between addition and multiplication, so that an expression such as $f + Dg * xh$ is to be parsed as $f + \big((Dg) * (xh)\big)$.

PROPOSITION 4.3. *Let $f, g \in \mathbb{R}[x]$.*
(a) The operation $*$ is commutative, associative, and $\mathbb{R}$-bilinear.
(b) $x^n f * g = x^n\big(f * (1 + D)^n g\big)$, for all $n \in \mathbb{N}$.
(c) $D(f * g) = f * Dg + Df * g + Df * Dg$.
(d) $(1 + D)^n(f * g) = (1 + D)^n f * (1 + D)^n g$, for all $n \in \mathbb{Z}$.

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PROOF. Part (a) is immediate, since $S\colon (\mathbb{R}[x],+,\cdot) \longrightarrow (\mathbb{R}[x],+,*)$ is an $\mathbb{R}$-algebra isomorphism. To prove part (b), just calculate using the product rule $Dx = 1 + xD$ to deduce that $xf * g = x\big(f * (1+D)g\big)$, and then use induction on n. The proof of part (c) is a similar application of the product rule and directly implies part (d) for $n \in \mathbb{N}$. Since the operator $1 + D$ is invertible on $\mathbb{R}[x]$, it follows that (d) holds for all $n \in \mathbb{Z}$. ∎

The next fact is also implicit in [36] (see Theorem 4.5 and the formula at the top of page 153).

PROPOSITION 4.4. *If $f, g \in \mathbb{R}[x]$ are polynomials with only real nonpositive zeros then $f * g$ also has only real nonpositive zeros.*

We may now derive a Complete Cutset Theorem for σ-polynomials.

THEOREM 4.5. *Let G and H be graphs such that $G \cap H$ is complete, and let $m = \#V(G \cap H)$. Then*

$$\frac{\sigma(G\cup H;x)}{x^m} = \frac{\sigma(G;x)}{x^m} * \frac{\sigma(H;x)}{x^m}.$$

PROOF. By Proposition 1.1 we have

$$P(G\cup H;x) = \frac{P(G;x)P(H;x)}{P(G\cap H;x)}$$

since $G\cap H$ is complete. Since $\#V(G\cap H) = m$ we have $P(G\cap H;x) = x_{\langle m\rangle}$; multiplying through by $x_{\langle m\rangle}$ and applying the transformation S we get

$$x^m * \sigma(G\cup H;x) = \sigma(G;x) * \sigma(H;x).$$

Since $G\cap H$ is a complete subgraph of size m of both G and H, we have $a_j(G) = a_j(H) = 0$ for $j = 0,\ldots,m-1$. Thus there are polynomials g, h such that $\sigma(G;x) = x^m g(x)$ and $\sigma(H;x) = x^m h(x)$. That is,

$$x^m * \sigma(G\cup H;x) = x^m g(x) * x^m h(x).$$

From Proposition 4.3 we calculate that

$$x^m * \sigma(G\cup H;x) = x^m(1+D)^m\sigma(G\cup H;x)$$

and that

$$x^m g(x) * x^m h(x) = x^m\big(g(x) * (1+D)^m x^m h(x)\big).$$

Hence

$$\begin{aligned}\sigma(G\cup H;x) &= (1+D)^{-m}\big(g(x) * (1+D)^m x^m h(x)\big)\\ &= (1+D)^{-m} g(x) * x^m h(x)\\ &= x^m\big(g(x) * h(x)\big)\end{aligned}$$

and the result follows. ∎

COROLLARY 4.6. *Let G and H be graphs such that $G\cap H$ is complete. If both G and H are σ-real then $G\cup H$ is also σ-real.*

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PROOF. This follows at once from Proposition 4.4 and Theorem 4.5. ∎

Using Proposition 4.4 we can also derive a condition on $f \in \mathbb{R}[x]$ sufficient to imply that Sf has only real nonpositive zeros. This generalizes Theorem 2.4.2 of [8] and Proposition 3.4 of [36].

THEOREM 4.7. *Let $f \in \mathbb{R}[x]$ and $m \in \mathbb{N}$ be such that all real zeros z of f satisfy $z \leq m$, and that all non-real zeros $s + it$ of f satisfy $4t^2 \leq 1 + 4m - 4s$. Then $Sx_{\langle m \rangle}f$ has only real nonpositive zeros.*

PROOF. The polynomial f factors over $\mathbb{R}$ as $f = p_1 p_2 \cdots p_k q_1 q_2 \cdots q_\ell$, where the p_i are linear and the q_j are quadratic with non-real zeros; hence $Sx_{\langle m \rangle}f = x^m * Sp_1 * \cdots * Sp_k * Sq_1 * \cdots * Sq_\ell$. From Proposition 4.3 we see that for any polynomials $g, h_1, h_2 \in \mathbb{R}[x]$,

$$\begin{aligned} x^m g * h_1 * h_2 &= x^m \big(g * (1+D)^m h_1 \big) * h_2 \\ &= x^m \big(g * (1+D)^m h_1 * (1+D)^m h_2 \big). \end{aligned}$$

It follows that

$$Sx_{\langle m \rangle}f = x^m \big((1+D)^m Sp_1 * \cdots * (1+D)^m Sq_\ell \big).$$

Proposition 4.4 implies that in order to prove the theorem by induction on $k + \ell$ it suffices to check that if $h \in \mathbb{R}[x]$ is a linear or quadratic polynomial satisfying the hypothesis of the theorem then $(1 + D)^m Sh$ has only real nonpositive zeros.

Firstly, if $h(x) = x - r$ with $r \leq m$ then $(1+D)^m S(x-r) = (1+D)^m (x-r) = x-(r-m)$ has only real nonpositive zeros.

Secondly, if $h(x) = \big(x-(s+it)\big)\big(x-(s-it)\big) = x^2 - 2sx + (s^2+t^2)$ with $4t^2 \leq 1+4m-4s$ then

$$\begin{aligned} (1+D)^m Sh(x) &= (1+D)^m [(x^2 + x) - 2sx + (s^2 + t^2)] \\ &= x^2 + (2m + 1 - 2s)x + (s^2 + t^2 - 2ms + m^2). \end{aligned}$$

By the quadratic formula, this has only real zeros if and only if

$$(2m + 1 - 2s)^2 - 4(s^2 + t^2 - 2ms + m^2) \geq 0,$$

which is equivalent to $4t^2 \leq 1 + 4m - 4s$. Now $(1 + D)^m Sh(x)$ has only nonpositive zeros if and only if its coefficients are all nonnegative, and this is easily seen to be the case. ∎

Because of Theorem 4.7 let us use the notation, for any $m \in \mathbb{N}$,

$$\mathcal{R}_S(m) = \{ s + it \in \mathbb{C} : 4t^2 \leq 1 + 4m - 4s \}.$$

COROLLARY 4.8. *For any $f \in \mathbb{R}[x]$ there is an $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, $Sx_{\langle m \rangle}f$ has only real nonpositive zeros. There is also an $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, $(1 + D)^m f$ has only real nonpositive zeros.*

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PROOF. The first statement follows from Theorem 4.7 since $\mathcal{R}_S(m) \subset \mathcal{R}_S(m+1)$ for all $m \in \mathbb{N}$ and the union of the regions $\mathcal{R}_S(m)$ for $m \in \mathbb{N}$ is the whole complex plane. The second statement is equivalent to the first by considering $Sx_{\langle m\rangle}S^{-1}f$. ∎

Specializing to the case of graphs we obtain the following.

PROPOSITION 4.9. *Let the graph G be such that all the non-real zeros $s + it$ of the chromatic polynomial $P(G;x)$ are in the region $R_S\big(\chi(G)\big)$, and all the real zeros z satisfy $z < \chi(G)$. Then G is σ-real.*

PROOF. Take $m = \chi(G)$ and $f(x) = P(G;x)/x_{\langle m\rangle}$ in Theorem 4.7. ∎

We can apply Proposition 4.9 to prove part of Conjecture 7.3 of [9].

PROPOSITION 4.10. *The cycle C_n is σ-real for all $n \geq 3$.*

PROOF. As is well-known, the chromatic polynomial of C_n is $P(C_n;x) = (x-1)^n + (-1)^n(x-1)$, so that all the zeros of $P(C_n;x)$ are in the disc $\{s+it \in \mathbb{C} : (s-1)^2+t^2 \leq 1\}$. This disc is contained in $\mathcal{R}_S(2)$ and the result follows from Proposition 4.9. ∎

Unfortunately, the results of [4, 26] show that the hypothesis of Proposition 4.9 is satisfied neither by prisms nor by Möbius ladders, and seems very rarely to be satisfied in general. An extension of Theorem 4.7 using cubic factors of f may lead to more applicable results, but the requisite calculations (using the discriminant of a cubic to test whether it has only real zeros) are too cumbersome to include here.

Theorem 4.7 also has the following consequence, which shows that in the Complete Cutset Theorem only one factor need be σ-real if it has sufficiently large chromatic number.

THEOREM 4.11. *Let G and H be graphs such that $G \cap H$ is a complete graph. Suppose that G is σ-real, that all real zeros z of $P(H;x)$ satisfy $z \leq \chi(G)$, and that all non-real zeros of $P(H;x)$ are in the region $\mathcal{R}_S\big(\chi(G)\big)$. Then $G \cup H$ is σ-real.*

PROOF. Letting $m = \chi(G)$ we have $\sigma(G;x) = x^m g(x)$ for some $g \in \mathbb{R}[x]$ with only real nonpositive zeros. Furthermore, putting $h(x) = P(H;x)/P(G \cap H;x)$ we have $\sigma(G \cup H;x) = S[P(G;x)h(x)] = x^m g(x) * Sh(x) = x^m\big(g(x) * (1+D)^m Sh(x)\big)$. Now $Sx_{\langle m\rangle}h$ has only real nonpositive zeros, by Theorem 4.7. But $Sx_{\langle m\rangle}h = x^m * Sh = x^m(1+D)^m Sh$, so that $(1+D)^m Sh$ has only nonpositive zeros and the result follows from Proposition 4.4. ∎

The results of Biggs, Damerell, and Sands [4] show that if H is a prism or a Möbius ladder then $\chi(G) \geq 9$ is sufficient in Theorem 4.11. Moreover, if Conjecture 1.5 is true then $\chi(G) \geq 9$ would suffice for all 3-regular graphs H. For any H, an $O\big(\#E(H)^2\big)$ upper bound for the smallest $\chi(G)$ which suffices in Theorem 4.11 can be obtained from Proposition 1.4 but this will be far from sharp.

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5. τ- and w-polynomials. Despite the fact that we have no examples of τ-unreal graphs, the known results which imply τ-reality are weaker than those for σ-reality. There are two reasons for this: the combinatorial meaning of $c_j(G)$ is more involved than that of $a_j(G)$, and the algebraic symmetry between S and T fails to hold up on the combinatorial level, essentially because there is no graph the chromatic polynomial of which is $x^{\langle k\rangle}$. The situation for w-reality is even more unclear.

We begin by describing the connection between S and T in detail. Let $\varepsilon\colon \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ be the $\mathbb{R}$-algebra automorphism defined by $\varepsilon x = -x$ and linear and multiplicative extension, so that for any $f \in \mathbb{R}[x]$ we have $\varepsilon f(x) = f(-x)$. Let Δ be the (backward) finite difference operator, so that for $f \in \mathbb{R}[x]$ we have $\Delta f(x) = f(x) - f(x-1)$. For $f \in \mathbb{R}[x]$ we will also denote by f the operator which acts as multiplication by f on $\mathbb{R}[x]$; no confusion should arise. Proposition 5.1 gives a number of useful computation rules which are for the most part well-known.

PROPOSITION 5.1. *The following operator identities hold:*

(a) $\varepsilon^2 = 1$

(b) $\varepsilon x_{\langle m\rangle}\varepsilon = (-1)^m x^{\langle m\rangle}$

(c) $\varepsilon D\varepsilon = -D$

(d) $\varepsilon S\varepsilon = T$

(e) $\varepsilon(1-\Delta)\varepsilon = (1-\Delta)^{-1}$

(f) $Sx = x(1+D)S$

(g) $Tx = x(1-D)T$

(h) $T(1-\Delta) = (1-D)T$

(i) $\big(x(1-\Delta)\big)^m = x_{\langle m\rangle}(1-\Delta)^m$

(j) $S^{-1}x = x(1-\Delta)S^{-1}$

(k) $Tx_{\langle m\rangle} = \big(x(1-D)^2\big)^m T(1-\Delta)^{-m}$

PROOF. Parts (a), (b), and (c) are immediate. Part (d) follows from (b) by applying $\varepsilon S\varepsilon$ to $x^{\langle m\rangle}$. For part (e) note that $(1-\Delta)f(x) = f(x-1)$. For part (f) $Sxf(x) = x * Sf(x) = x(1+D)Sf(x)$, and part (g) follows from (c), (d), and (f). For part (h) note that $x^{\langle j\rangle} = (x+1)^{\langle j\rangle} - j(x+1)^{\langle j-1\rangle}$, and calculate that $T(1-\Delta)x^{\langle j\rangle} = T(x-1)^{\langle j\rangle} = T[x^{\langle j\rangle} - jx^{\langle j-1\rangle}] = x^j - jx^{j-1} = (1-D)Tx^{\langle j\rangle}$. Since $\{x^{\langle j\rangle}\}$ is a basis for $\mathbb{R}[x]$ the result follows. Part (i) is a straightforward induction on m. For part (j) calculate that $S^{-1}xx^j = x_{\langle j+1\rangle} = x(x-1)_{\langle j\rangle} = x(1-\Delta)S^{-1}x^j$ and use the fact that $\{x^j\}$ is a basis for $\mathbb{R}[x]$. Part (k) follows from (g), (h), and (i) since $Tx_{\langle m\rangle} = T\big(x(1-\Delta)\big)^m(1-\Delta)^{-m} = x(1-D)T(1-\Delta)\big(x(1-\Delta)\big)^{m-1}(1-\Delta)^{-m} = \cdots = \big(x(1-D)^2\big)^m T(1-\Delta)^{-m}$. ∎

Using Proposition 5.1(a,b,c,d), the following facts about T follow from the corresponding ones for S.

PROPOSITION 5.2. *For any $f, g \in \mathbb{R}[x]$ we have $T(fg) = (Tf) \star (Tg)$, in which the operation $\star\colon \mathbb{R}[x] \times \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ is given by*

$$p \star q = \sum_{k\in\mathbb{N}} \frac{(-x)^k}{k!}(D^k p)(D^k q).$$

PROPOSITION 5.3. *Let $f, g \in \mathbb{R}[x]$.*

(a) The operation $\star$ is commutative, associative, and $\mathbb{R}$-bilinear.

(b) $x^n f \star g = x^n(f \star (1-D)^n g)$, for all $n \in \mathbb{N}$.

(c) $D(f \star g) = (f \star Dg) + (Df \star g) - (Df \star Dg)$.

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(d) $(1-D)^n(f \star g) = (1-D)^n f \star (1-D)^n g$, *for all* $n \in \mathbb{Z}$.

PROPOSITION 5.4. *Let* $f, g \in \mathbb{R}[x]$ *be polynomials with only real nonnegative zeros. Then* $f \star g$ *has only real nonnegative zeros.*

For $m \in \mathbb{N}$, let $\mathcal{R}_T(m) = \{s + it : 4t^2 \leq 1 + 4m + 4s\}$.

THEOREM 5.5. *Let* $f \in \mathbb{R}[x]$ *and* $m \in \mathbb{N}$ *be such that all real zeros* z *of* f *satisfy* $z \geq -m$, *and that all non-real zeros of* f *are in the region* $\mathcal{R}_T(m)$. *Then* $Tx^{\langle m \rangle}f$ *has only real nonnegative zeros.*

COROLLARY 5.6. *For any* $f \in \mathbb{R}[x]$ *there is an* $m_0 \in \mathbb{N}$ *such that for all* $m \geq m_0$, $Tx^{\langle m \rangle}f$ *has only real nonnegative zeros. There is also an* $m_0 \in \mathbb{N}$ *such that for all* $m \geq m_0$, $(1-D)^m f$ *has only real nonnegative zeros.*

Specializing to the case of graphs we obtain the following.

PROPOSITION 5.7. *Let the nonempty graph* G *be such that all the non-real zeros of* $P(G;x)$ *are in the region* $\mathcal{R}_T(1)$. *Then* G *is* τ*-real.*

PROOF. Take $m = 1$ and $f(x) = P(G;x)/x$ in Theorem 5.5. ∎

PROPOSITION 5.8. *(a) Every supersolvable graph is* τ*-real.*
(b) The cycle C_n *is* τ*-real for all* $n \geq 3$.

PROOF. Both parts of the result follow from Proposition 5.7. ∎

Part (a) of Proposition 5.8 was first proved in [9] in a completely different way.

PROPOSITION 5.9. *Let* G *and* H *be vertex-disjoint graphs. Then*

$$\tau(G \vee H; x) = \sigma\big(G; x(1-D)^2\big)\tau(H; x).$$

PROOF. From Proposition 5.1(i,j,k) we have $TS^{-1}x^m = \big(x(1-D)^2\big)^m TS^{-1}$. Applying this to the identity $\sigma(G \vee H; x) = \sigma(G;x)\sigma(H;x)$ we get

$$TS^{-1}\sigma(G \vee H; x) = \sigma\big(G; x(1-D)^2\big)TS^{-1}\sigma(H;x)$$

which gives the result. ∎

PROPOSITION 5.10. *Let* G *be* τ*-real. Then* $G \vee K_m$ *is* τ*-real for all* $m \in \mathbb{N}$.

PROOF. From Proposition 5.9 we find that $\tau(G \vee K_m; x) = \big(x(1-D)^2\big)^m \tau(G;x)$. Rolle's Theorem and the Intermediate Value Theorem imply that if $f(x)$ has only real zeros then $(1-D)f(x)$ also has only real zeros. This suffices to prove the result by induction on m. ∎

PROPOSITION 5.11. *For any graph* G *there is an* $m_0 \in \mathbb{N}$ *such that if* $m \geq m_0$ *then* $G \vee K_m$ *is* τ*-real.*

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PROOF. For any fixed G, all the zeros of $P(G; x - m)$ are eventually in the region $\mathcal{R}_T(1)$ for sufficiently large m. Since $P(G \vee K_m; x) = x_{\langle m\rangle} P(G; x - m)$ the result follows from Proposition 5.7. ∎

We also have these partial results toward a Complete Cutset Theorem for τ-polynomials.

PROPOSITION 5.12. *If G and H are τ-real graphs with $\#V(G \cap H) \leq 1$ then $G \cup H$ is also τ-real.*

PROOF. For $G \cap H = \emptyset$ the result follows immediately from Propositions 1.1, 5.2, and 5.4. For $\#V(G \cap H) = 1$ we get

$$x \star \tau(G \cup H; x) = \tau(G; x) \star \tau(H; x)$$

from which it follows that

$$\frac{\tau(G \cup H; x)}{x} = \frac{\tau(G; x)}{x} \star \frac{\tau(H; x)}{x}$$

and the proof follows the same pattern as that of Corollary 4.6. ∎

PROPOSITION 5.13. *Let G and H be graphs such that $G \cap H$ is a complete graph. Suppose that G is τ-real and that all the non-real zeros of $P(H; x)$ are in the region $\mathcal{R}_T(0)$. Then $G \cup H$ is τ-real.*

PROOF. Let $\#V(G \cap H) = k$. From Propositions 1.1 and 5.2 we have $\tau(G \cup H; x) = \tau(G; x) \star Th(x)$, where $h(x) = P(H; x)/x_{\langle k\rangle}$. By our hypothesis on H, $h(x)$ satisfies Theorem 5.5 with $m = 0$, so that $Th(x)$ has only real nonnegative zeros. The result follows from Proposition 5.4. ∎

LEMMA 5.14. *For any $f \in \mathbb{R}[x]$, if $x_{\langle m\rangle}$ divides $T^{-1}f$ then*

$$\frac{1}{x_{\langle m\rangle}} T^{-1} f = (1 - \Delta)^m T^{-1}\big(x(1 - D)^2\big)^{-m} f.$$

PROOF. This follows immediately from Proposition 5.1(k) since, whenever it is applicable, $x_{\langle m\rangle}^{-1} T^{-1}$ is the operator inverse to $Tx_{\langle m\rangle}$. ∎

LEMMA 5.15. *Let ω be any word on the letters x, x^{-1}, $(1 - D)$, and $(1 - D)^{-1}$, and define $e(\omega)$ to be the multiplicity of $(1 - D)$ in ω minus the multiplicity of x in ω. Then for any $f, g \in \mathbb{R}[x]$ such that $\omega f \in \mathbb{R}[x]$,*

$$\omega(f \star g) = \omega f \star (1 - D)^{e(\omega)} g.$$

PROOF. This follows by induction on the length of ω, using Proposition 5.3(b,d). ∎

THEOREM 5.16. *Let G and H be graphs such that $G \cap H \simeq K_m$. Then*

$$\tau(G \cup H; x) = (1 - D)^m \big(x(1 - D)^2\big)^{-m} [\tau(G; x) \star \tau(H; x)].$$

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PROOF. From Proposition 1.1 and various results of this section we calculate that

$$\begin{aligned}
\tau(G\cup H;x) &= T\left[x_{\langle m\rangle}\frac{P(G;x)}{x_{\langle m\rangle}}\frac{P(H;x)}{x_{\langle m\rangle}}\right] \\
&= \left(x(1-D)^2\right)^m T(1-\Delta)^{-m}\left[\left(\frac{1}{x_{\langle m\rangle}}T^{-1}\tau(G;x)\right)\left(\frac{1}{x_{\langle m\rangle}}T^{-1}\tau(H;x)\right)\right] \\
&= (x(1-D)^2)^m\left[\left(x(1-D)^2\right)^{-m}\tau(G;x)\star\left(x(1-D)^2\right)^{-m}\tau(H;x)\right] \\
&= \tau(G;x)\star(1-D)^m\left(x(1-D)^2\right)^{-m}\tau(H;x) \\
&= (1-D)^m\left(x(1-D)^2\right)^{-m}[\tau(G;x)\star\tau(H;x)]
\end{aligned}$$

as was to be shown. ∎

Unfortunately, the operator $(1-D)^m\left(x(1-D)^2\right)^{-m}$ does not in general preserve the property of having only real nonnegative zeros, as can be seen even in the case $m=1$. Thus if Theorem 5.16 can be used to prove the Complete Cutset Theorem for τ-polynomials then some additional structure must be identified.

There are several other formulas relating $\sigma(G;x)$ and $\tau(G;x)$ with $w(G;x)$ as well. From (1) and (9) a simple calculation yields

$$\frac{w(G;x)}{(1-x)^{1+n}}=\frac{1}{1-x}\sum_{j=0}^{n}j!a_j(G)\left(\frac{x}{1-x}\right)^j$$

which motivates the definition of the *augmented σ-polynomial of G*

$$\bar{\sigma}(G;x)=\sum_{j=0}^{n}j!a_j(G)x^j \tag{10}$$

as in [9]. Hence

$$w(G;x)=(1-x)^n\bar{\sigma}\left(G;\frac{x}{1-x}\right). \tag{11}$$

Similarly, let us define the *augmented τ-polynomial of G* by

$$\bar{\tau}(G;x)=\sum_{j=0}^{n}(-1)^{n-j}j!c_j(G)x^j. \tag{12}$$

From (6), (9), and (12) we deduce that

$$w(G;x)=x(1-x)^n\bar{\tau}\left(G;\frac{1}{1-x}\right). \tag{13}$$

Since every real zero of $w(G;x)$ is nonpositive, from (11) one sees that every real zero of $\bar{\sigma}(G;x)$ is in the interval $[-1,0]$. It also follows from (11) that $\bar{\sigma}(G;x)$ has only real zeros if and only if the same is true of $w(G;x)$. Similarly, every real zero of $\bar{\tau}(G;x)$ is in $[0,1]$, and it has only real zeros if and only if $w(G;x)$ does.

Recall that the notation $[t^k]F(s,t)$ denotes the coefficient of t^k when $F(s,t)$ is expanded as a Laurent series in t.

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PROPOSITION 5.17. *The following identities hold.*

(a) $Sx^{\langle m\rangle} = m!\sum_{j=1}^{m}\binom{m-1}{j-1}x^j/j!$

(b) $\sum_k Sx^{\langle k\rangle}y^k/k! = \exp\big(xy/(1-y)\big)$

(c) $\sigma(G;x) = [z^0]\bar{\tau}(G;z^{-1}+1)\exp(xz)$

(d) $j!a_j(G) = \sum_{i=j}^{n}(-1)^{n-i}\binom{i}{j}i!c_i(G)$

(e) $Tx_{\langle m\rangle} = m!\sum_{j=1}^{m}(-1)^{m-j}\binom{m-1}{j-1}x^j/j!$

(f) $\sum_k Tx_{\langle k\rangle}y^k/k! = \exp\big(xy/(1+y)\big)$

(g) $\tau(G;x) = [z^0]\bar{\sigma}(G;z^{-1}-1)\exp(xz)$

(h) $j!c_j(G) = \sum_{i=j}^{n}(-1)^{n-i}\binom{i}{j}i!a_i(G)$

PROOF. We prove parts (a), (b), (c), and (d): the other parts may be proved similarly, or derived from these by Proposition 5.1. Part (a) follows from Proposition 2.1, but for a direct proof we use the well-known formulas $x^{\langle m\rangle} = \sum_{k=0}^{m}c(m,k)x^k$ in which $c(m,k)$ is a signless Stirling number of the first kind, and $x^k = \sum_{j=0}^{k}S(k,j)x_{\langle j\rangle}$ in which $S(k,j)$ is a Stirling number of the second kind. It follows that

$$Sx^{\langle m\rangle} = \sum_{j=0}^{m}x^j\sum_{k=j}^{m}c(m,k)S(k,j).$$

By the combinatorial interpretations of $c(m,k)$ and $S(k,j)$, we see that $[x^j]Sx^{\langle m\rangle}$ is the number of pairs (σ,π) in which σ is a permutation of $\{1,\ldots,m\}$, and π is a partition of the cycles of σ into exactly j blocks. These pairs are in bijective correspondence with the pairs (θ,ρ) in which θ is a partition of $\{1,\ldots,m\}$ into exactly j blocks and ρ is a permutation of $\{1,\ldots,m\}$ which leaves each block of θ invariant. To construct these pairs, choose any permutation $\alpha = a_1a_2\cdots a_m$ of $\{1,\ldots,m\}$ (written in word notation) in one of $m!$ ways, and choose an ordered partition ν of this word into j blocks in one of $\binom{m-1}{j-1}$ ways. If $\beta = a_{p+1}\cdots a_{p+q}$ is a block of ν then let $a_{i_1}\cdots a_{i_q}$ be the elements of β in increasing order. The rule $a_{i_t}\longmapsto a_{p+t}$ defines a unique permutation on the block β. Doing this for each block of ν defines a pair (θ,ρ) as required. Each pair (θ,ρ) is constructed $j!$ times in this manner, as the order of the blocks of ν is irrelevant. This proves part (a).

Part (b) follows from part (a) by standard manipulations with exponential generating functions.

For part (c), notice that

$$\begin{aligned}\sigma(G;x) &= SP(G;x) = \sum_j(-1)^{n-j}c_j(G)Sx^{\langle j\rangle}\\ &= \sum_j(-1)^{n-j}j!c_j(G)[y^j]\exp\big(xy/(1-y)\big)\\ &= [y^0]\sum_j(-1)^{n-j}j!c_j(G)y^{-j}\exp\big(xy/(1-y)\big).\end{aligned}$$

Making the change of variables $z = y/(1-y)$ we obtain

$$\sigma(G;x) = [z^0]\sum_j(-1)^{n-j}j!c_j(G)(z^{-1}+1)^j\exp(xz)$$

which gives the result.

Part (d) now follows from part (c) by equating coefficients of like powers of x. ∎

The polynomial $\bar{\sigma}(G;x)$ is somewhat easier to deal with than $w(G;x)$ because of the relation $\bar{\sigma}(G;x) = BP(G;x)$ in which $B\colon\mathbb{R}[x]\longrightarrow\mathbb{R}[x]$ is defined by $Bx_{\langle m\rangle} = m!x^m$ and

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linear extension. Similarly for $\bar{\tau}(G;x)$, of course. This allows a development parallel to that of Section 4 but in this case the proofs are much more difficult. Propositions 5.18 and 5.19 are proved as Theorems 2.7 and 0.3 of [37].

PROPOSITION 5.18. *Let $B\colon \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ be the linear transformation just defined. Then for any $f, g \in \mathbb{R}[x]$ we have $B(fg) = (Bf)\diamond(Bg)$, in which the operation $\diamond\colon \mathbb{R}[x]\times\mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ is given by*

$$p \diamond q = \sum_{k\in\mathbb{N}} \frac{(x+1)^k x^k}{k!k!}(D^k p)(D^k q).$$

PROPOSITION 5.19. *If both $f, g \in \mathbb{R}[x]$ have all their zeros in the interval $[-1,0]$ then $f\diamond g$ also has all its zeros in the interval $[-1,0]$.*

Corollary 5.20 appears as Theorem 4.20 of [9].

COROLLARY 5.20. *If G and H are vertex-disjoint w-real graphs then $G\cup H$ is also w-real.*

The analogue of Propositions 4.3 and 5.3 for the $\diamond$-product appears in [37], but the complexity of these formulae has prevented us from obtaining analogues of Theorems 4.5, 4.7, and 5.16. However, the next proposition, equivalent to Theorem 4.3.4 of [8], gives us another partial Complete Cutset Theorem for w-polynomials. For a polynomial $g \in \mathbb{R}[x]$ with only real zeros, let $\lambda(g)$ and $\Lambda(g)$ denote the smallest and largest zeros of g, respectively.

PROPOSITION 5.21. *Let $f \in \mathbb{R}[x]$ be such that all the zeros of Bf are in the interval $[-1,0]$, and let the multiplicities of -1 and 0 as zeros of Bf be ℓ and m, respectively. Suppose that $g \in \mathbb{R}[x]$ has only real zeros, and that $g(z) = 0$ for every integer z in the union of intervals $[\lambda(g), -1-\ell] \cup [m, \Lambda(g)]$. Then all the zeros of $B(fg)$ are in the interval $[-1,0]$.*

THEOREM 5.22. *Let G and H be graphs such that $G\cap H$ is a complete graph. Suppose that G is w-real and that H is P-real and all zeros z of $P(H;x)$ satisfy $z < \chi(H)$. Then $G\cup H$ is w-real.*

PROOF. We check that $g(x) = P(G;x)$ and $h(x) = P(H;x)/P(G\cap H;x)$ satisfy the hypothesis of Proposition 5.21; the result then follows from Proposition 1.1. Let $\#V(G\cap H) = k$ and $\chi(H) = p$.

Since G is w-real, all the zeros of $Bg(x) = \bar{\sigma}(G;x)$ are in the interval $[-1,0]$. The multiplicity of 0 as a zero of $\bar{\sigma}(G;x)$ is the smallest j for which $a_j(G) \neq 0$: this is $\chi(G)$. Also, $h(x)$ has only real zeros, $0 \leq \lambda(h)$ and $\Lambda(h) < p$, and $h(z) = 0$ for every integer z in the interval $[\chi(G), p)$ because $x_{\langle p\rangle}/x_{\langle k\rangle}$ divides $h(x)$ and $k \leq \chi(G)$. Now by Proposition 5.21, $\bar{\sigma}(G\cup K;x) = B[P(G;x)P(K;x)/P(G\cap K;x)]$ has all its zeros in the interval $[-1,0]$, which completes the proof. ∎

The special case of Theorem 5.22 when $\#V(G\cap H) \leq 1$ and H is complete appears as Theorem 4.17 of [9]. As an immediate consequence of Theorem 5.22 we obtain the following.

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COROLLARY 5.23. *Let G and H be graphs such that $G \cap H$ is complete. If G is w-real and H is supersolvable then $G \cup H$ is w-real. In particular, any supersolvable graph is w-real.*

The second assertion of Corollary 5.23 is Theorem 4.11 of [9].

Finally, we can also give a condition on the location of zeros of $P(G;x)$ which is equivalent to symmetry of the coefficients of $w(P;x)$. For a proof of the following proposition see Corollary 4.2.4(iii) of [29].

PROPOSITION 5.24. *Let $f(x)$ be a polynomial of degree d, and let*

$$\sum_{m \in \mathbb{N}} f(m)t^m = \frac{t^a G(t)}{(1-t)^{1+d}},$$

where $a \in \mathbb{N}$, $G(t)$ is a polynomial of degree $b \leq d-a$ and $G(0) \neq 0$. Then $G(t) = t^b G(t^{-1})$ if and only if the zeros of $f(x)$ are symmetric through the point $(2a+b-d-1)/2$.

PROPOSITION 5.25. *For a graph G, the nonzero coefficients of $w(G;x)$ are symmetric if and only if the zeros of $P(G;x)$ are symmetric through the point $\big(\chi(G)-1\big)/2$.*

PROOF. Proposition 4.5 of [9] shows that the degree of $w(P;x)$ is $\#V(G)$ and that the multiplicity of 0 as a zero of $w(P;x)$ is $\chi(G)$. The result follows from Proposition 5.24, using $P(G;x)$ in place of $f(x)$. ∎

Specializing to the case of cycles, we obtain the following.

COROLLARY 5.26. *For the cycles C_n, $n \geq 3$, the nonzero coefficients of $w(C_n;x)$ are symmetric if and only if n is odd.*

6. **Speculations.** Regarding the zeros of chromatic polynomials, it seems likely that an extension of Theorem 1.3 using quartic factors could lead to some progress towards proving the Read-Hoggar conjecture. An extension of Theorem 4.7 using cubic factors would also be quite useful for proving σ-reality. In conjunction with these a description of the limit distribution of recursive sequences of polynomials would be very interesting.

We wonder whether Conjecture 1.5 can be strengthened as follows.

QUESTION 6.1. Is there a function $f\colon \mathbb{N} \longrightarrow \mathbb{R}$ such that if G is a graph with maximum degree k and $P(G;z) = 0$ then $|z| \leq f(k)$?

As observed in Section 3, there appears to be some correlation among edge-density, w-reality, and σ-reality. (Recall that the *edge-density* of an n-vertex graph G is $\#E(G)/\binom{n}{2}$.) For high edge-density the situation is problematic, for let G be any σ-unreal graph and note that $\sigma(G \vee K_n;x) = \sigma(G;x)\sigma(K_n;x)$ has non-real zeros and the edge-density of $G \vee K_n$ tends to 1 as $n \longrightarrow \infty$. The situation for low edge-density may be different; note that w-reality seems to imply low edge-density, while low edge-density seems to imply σ-reality.

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QUESTION 6.2. For $n \in \mathbb{N}$ let $\delta(n)$ be the minimum edge-density over all n-vertex σ-unreal graphs. Give a good lower bound for $\delta(n)$. In particular, is there a constant $c > 0$ such that $\delta(n) > c$ for sufficiently large n?

An affirmative answer to Question 6.2 would imply affirmative answers to both of the following questions.

QUESTION 6.3. Is it true that for any $k \in \mathbb{N}$ there are only finitely many σ-unreal graphs with maximum degree k?

QUESTION 6.4. Is it true that for any surface M there are only finitely many σ-unreal graphs embeddable on M?

It is known that even cycles, supersolvable graphs, and incomparability graphs are all examples of perfect graphs. In view of Proposition 4.1(a,c) and Proposition 4.10 it is natural to wonder whether all perfect graphs are σ-real. This is not true: several σ-unreal perfect graphs can be found in the Appendix.

Concerning τ- and w-reality, we are willing to make a few conjectures.

CONJECTURE 6.5. *Let G and H be graphs such that $G \cap H$ is a complete graph. If both G and H are τ-real then $G \cup H$ is τ-real. Also, if both G and H are w-real then $G \cup H$ is w-real.*

CONJECTURE 6.6. *Let G and H be vertex-disjoint graphs. If both G and H are τ-real then the join $G \vee H$ is also τ-real.*

Note that $w(C_4; x) = 14x^4 + 8x^3 + 2x^2$, so that C_4 is w-unreal. Since K_2^c is w-real and $K_2^c \vee K_2^c \simeq C_4$ this shows that w-reality is not preserved by taking joins. However, we have checked all other cycles on up to 20 vertices, and have found them to be w-real.

CONJECTURE 6.7. *The cycle C_n is w-real for all $n \geq 5$.*

Finally, there are three obvious questions which we are still far from being able to answer.

QUESTION 6.8. Let p_n denote the probability that an n-vertex graph (chosen with uniform distribution) is σ-real. Does p_n tend to a limit as $n \longrightarrow \infty$? If so, then what is the value of this limit? In particular, does $p_n \longrightarrow 1$ as $n \longrightarrow \infty$?

QUESTION 6.9. Are all graphs τ-real?

QUESTION 6.10. Let q_n denote the probability that an n-vertex graph (chosen with uniform distribution) is w-real. Does q_n tend to a limit as $n \longrightarrow \infty$? If so, then what is the value of this limit? In particular, does $q_n \longrightarrow 0$ as $n \longrightarrow \infty$?

Of these, the second may be the easiest to answer: if it is true then the Deletion-Contraction Algorithm and a variation of the Sturm sequence techniques of [36, 37] may yield a proof. The other two questions seem to require a "probabilistic Sturm sequence" approach, in whatever sense this can be made meaningful. Alternatively, one could show that the graph properties σ-reality and w-reality obey a zero-one law, in the sense of [11].

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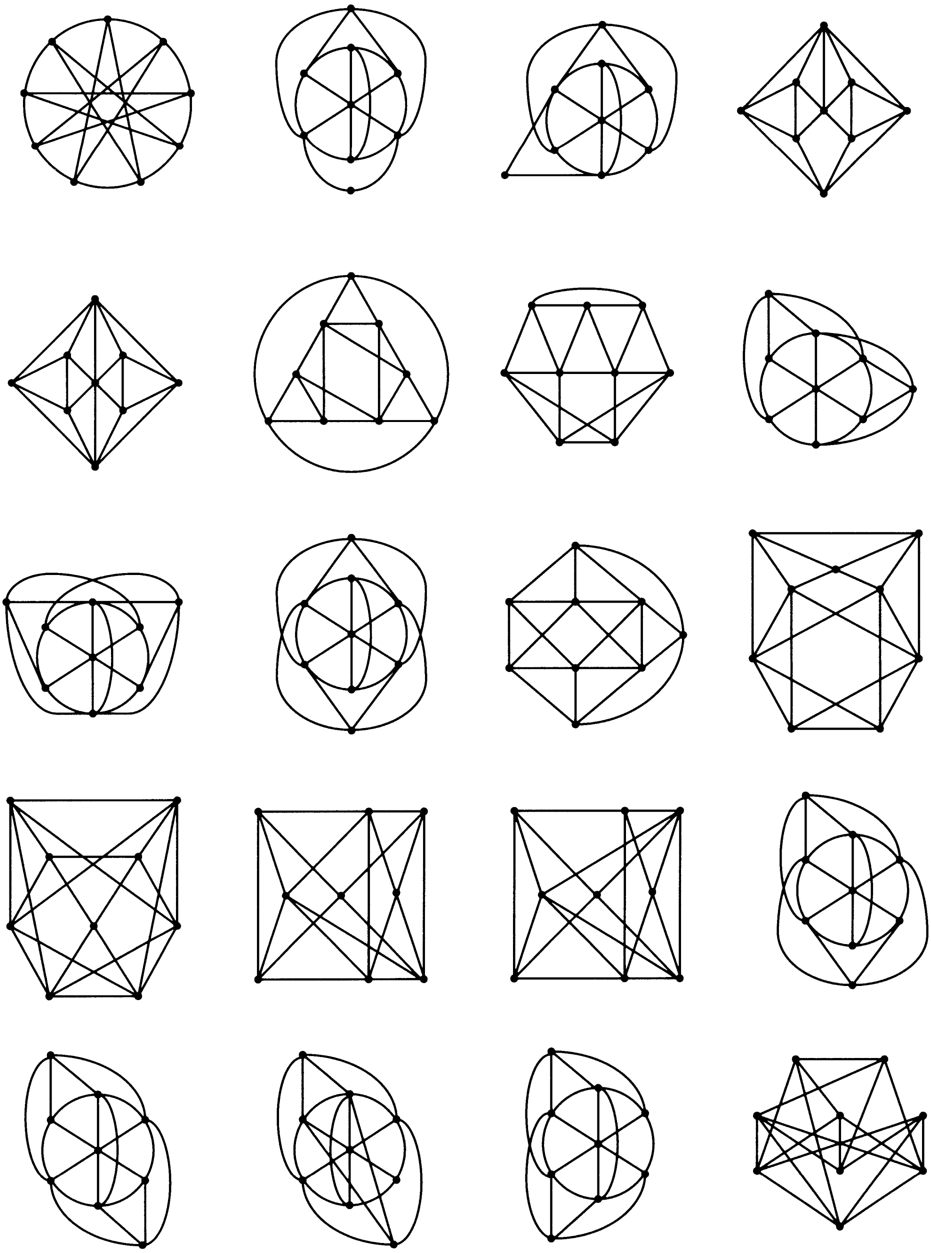
Appendix: the 9-vertex σ-unreal graphs



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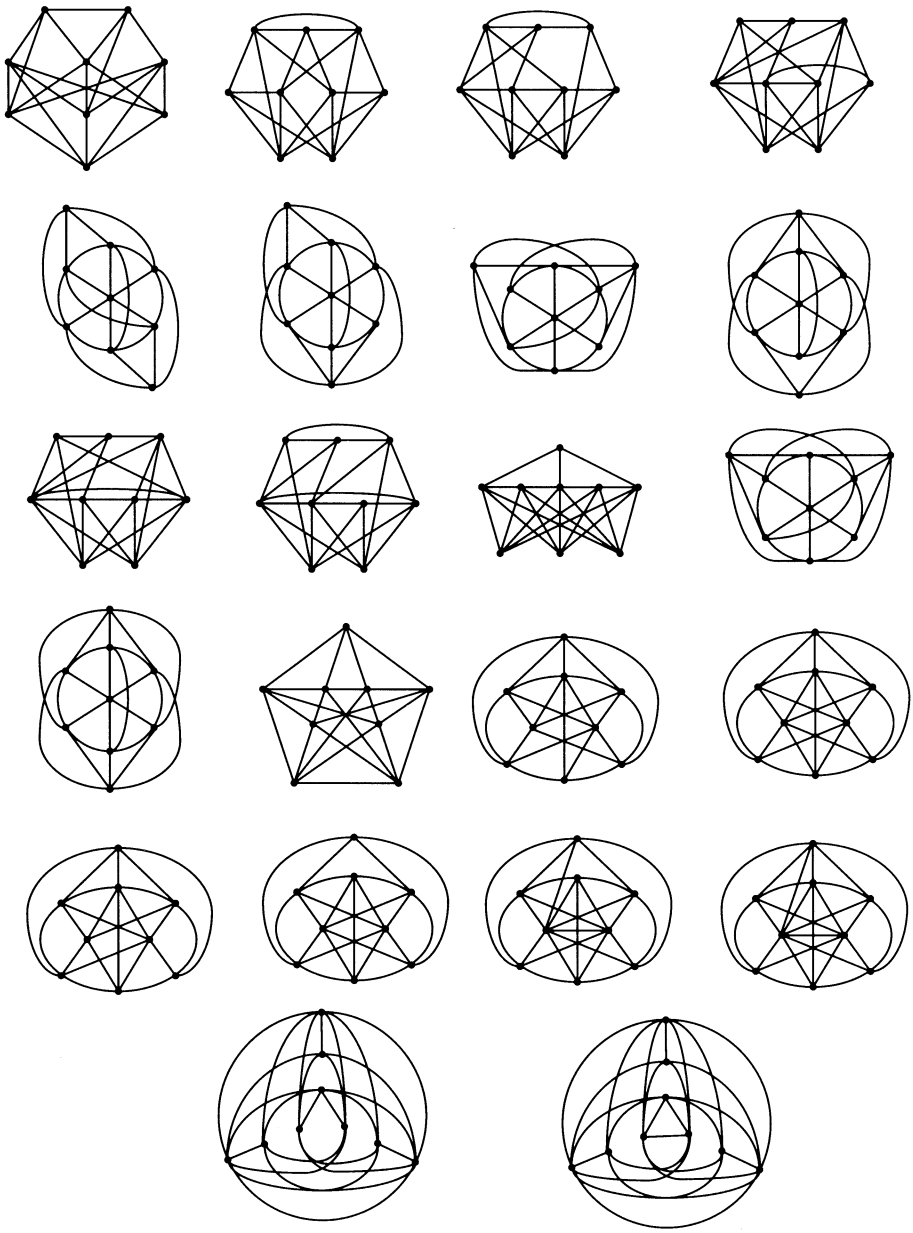

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Dipartimento di Matematica
Universita' di Perugia
Via Vanvitelli 1
I-06123, Perugia
Italy

Department of Computer Science
University of Western Australia
Perth, Western Australia
Australia

Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario
N2L 3G1

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