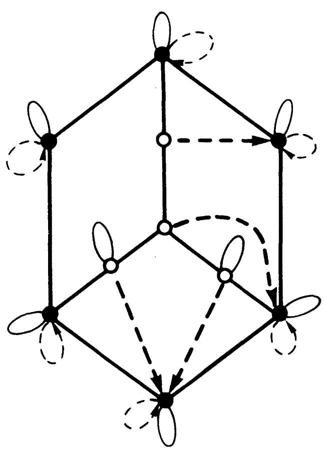
Can. J. Math., Vol. XXXVIII, No. 6, 1986, pp. 1299-1328

A CLASSIFICATION OF REFLEXIVE GRAPHS: THE USE OF "HOLES"

EL MOUSTAFA JAWHARI, MAURICE POUZET AND IVAN RIVAL

The purpose of this article is to develop aspects of a classification theory for reflexive graphs. A first important step was already taken in [**2**]; throughout we follow, at least the spirit, of the classification theory for ordered sets initiated in [**1**].

For a graph G let $V(G)$ denote its vertex set and $E(G) \subseteq V(G) \times V(G)$ its edge set. A graph K is a subgraph of G if $V(K) \subseteq V(G)$ and for $a, b \in V(K)$, $(a, b) \in E(K)$ just if $(a, b) \in E(G)$. The subgraph K of G is a retract of G, and we write $K \lhd G$, if there is an edge-preserving map g of $V(G)$ to $V(K)$ satisfying $g(v) = v$ for each $v \in V(K)$; g is called a *retraction*. A *reflexive* graph is an undirected graph with a loop at every vertex. The reason for a loop at a vertex is that an edge-preserving map can send the two vertices of an adjacent pair to it. The concept is illustrated in Figure 1. From here on, though, we shall for convenience suppress the illustration of the loops in the figures of reflexive graphs.



The subgraph K (with shaded vertices) is a retract of the reflexive graph G.

Figure 1

For reflexive graphs G and H the *direct product* $G \times H$ is the graph with vertex set $V(G) \times V(H)$ and edge set consisting of all pairs $((a, x), (b, y))$ where $(a, b) \in E(G)$ and $(x, y) \in E(H)$ (cf. Figure 2).

Received April 26, 1983 and in revised form December 13, 1984.

https://doi.org/10.4153/CJM-1986-066-9 Published online by Cambridge University Press

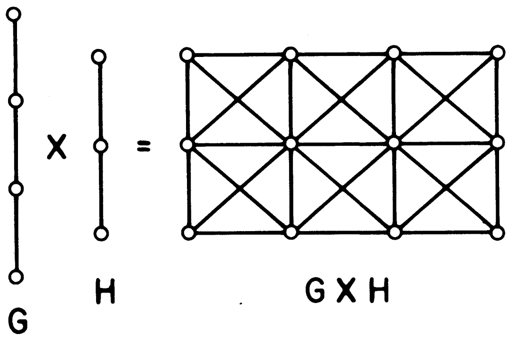
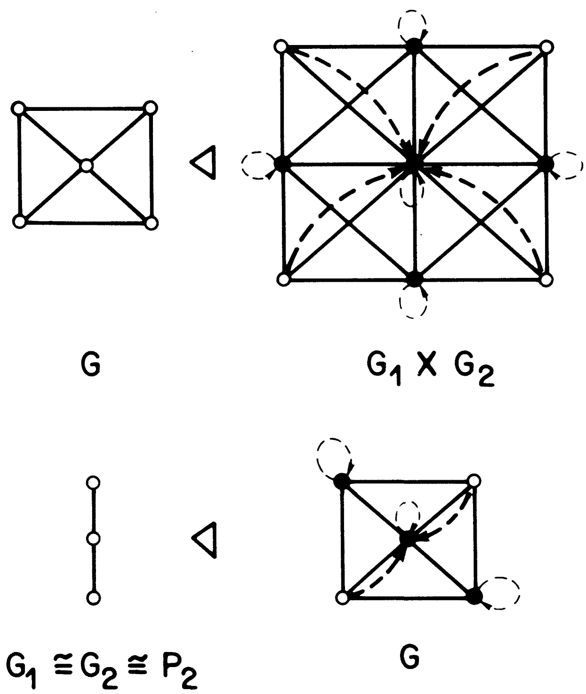


Figure 2

A *representation* of a reflexive graph G is a family $(G_i | i \in I)$ of reflexive graphs such that $G_i \lhd G$ for each $i \in I$, and

$$G \lhd \prod_{i \in I} G_i.$$

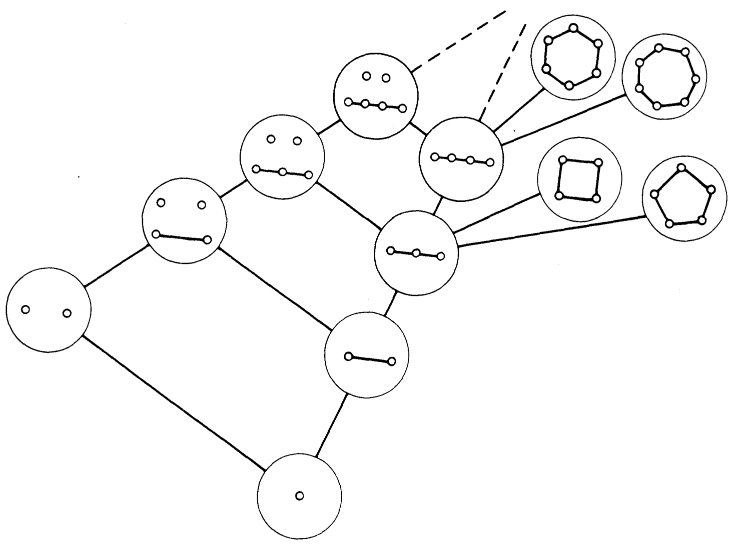
G is *irreducible* if, for every representation $(G_i | i \in I)$ of G, $G \lhd G_i$ for some $i \in I$; otherwise G is *reducible* (cf. Figure 3).



(G_1, G_2) is a representation of G.

Figure 3

A *reflexive graph variety* is a class $\mathscr{V}$ of reflexive graphs which contains all direct products of members of $\mathscr{V}$[$\mathbf{P}(\mathscr{V}) \subseteq \mathscr{V}$] and which contains all retracts of members of $\mathscr{V}$, [$\mathbf{R}(\mathscr{V}) \subseteq \mathscr{V}$]. For a class $\mathscr{K}$ of reflexive graphs let $\mathscr{K}^v$ stand for the smallest reflexive graph variety containing $\mathscr{K}^v$, the variety generated by $\mathscr{K}$. In fact, $\mathscr{K}^v = \mathbf{RP}(\mathscr{K})$. The intersection of a family of reflexive graph varieties is a reflexive graph variety and the class of all reflexive graphs is a reflexive graph variety which contains all others. Therefore, with respect to inclusion, the class of all reflexive graph varieties behaves much as a complete lattice, the *lattice of reflexive graph varieties*. The main results of this article can be expressed fairly accurately in two figures. Figure 4 illustrates an initial segment of the lattice of reflexive graph varieties. The big circles stand for the varieties as lattice elements and the graph(s) within for the reflexive graph(s) generating the variety.



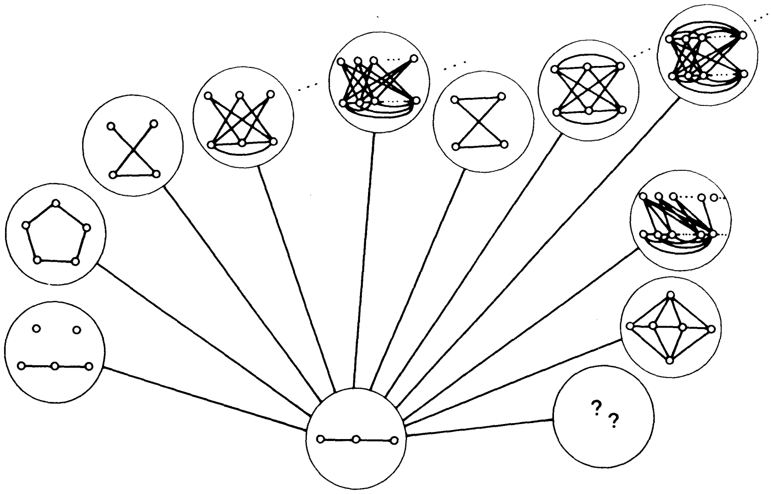
An initial segment of the lattice of reflexive graph varieties.

Figure 4

Figure 5 is an enlargement of a part of Figure 4 showing more of the detail.

The plan of this article is to introduce and illustrate in the next three sections, first the idea of a "hole" in a graph and then, "preserving a hole". We use these two ideas – "hole" and "preserving a hole" to verify the classification illustrated in Figure 4 and Figure 5.

What is a "hole"? Let G be a reflexive graph and let K be a subgraph of G. Just what are the conditions that must be fulfilled in order that the



The reflexive graph varieties known to cover $\{P_2\}^v$.

Figure 5

subgraph K be a retract of G? One condition is this. For any given vertex $w \in V(G)$ there must be a "solution" $x = x(w) \in V(K)$ to the system of inequalities: $d_G(x, v) \leq d_G(w, v)$, $v \in V(K)$, [$d_G(a, b)$ stands for the distance in G between $a, b \in V(G)$, that is, the least length (if it exists) of a path in G joining a to b]. If there is a "retraction" map g of $V(G)$ to $V(K)$ [that is an edge-preserving map g such that $g(v) = v$ for each $v \in V(K)$] then the image $g(w)$ of w must be such a vertex $x = x(w)$ of $V(K)$ which satisfies each of the inequalities. For example, if G is the reflexive graph illustrated in Figure 6 (a) and $K = C_4$ the subgraph consisting of the shaded vertices then the vertex w of G gives rise to the inequalities

$$d_G(x, c_0) \leq d_G(w, c_0) = 1, \quad d_G(x, c_1) \leq d_G(w, c_1) = 2,$$
$$d_G(x, c_2) \leq d_G(w, c_2) = 1, \quad d_G(x, c_3) \leq d_G(w, c_3) = 2.$$

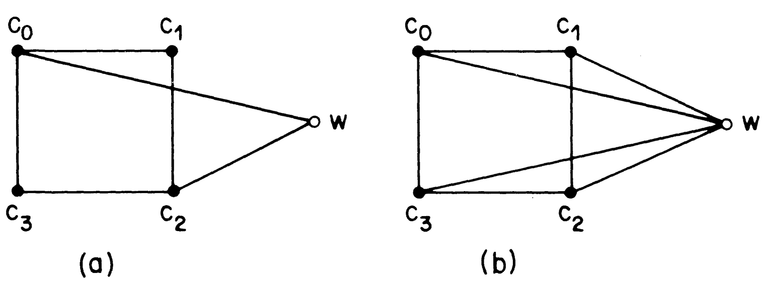


Figure 6

There are two solutions: $x = c_1$ or $x = c_3$. And, for instance, the map g of $V(G)$ to $V(C_4)$ defined by $g(c_i) = c_i$, $i = 0, 1, 2, 3$ and $g(w) = c_1$ is a retraction, so $C_4 \lhd G$. In contrast consider the corresponding subgraph C_4 in Figure 6 (b) with $V(C_4) = \{c_0, c_1, c_2, c_3\}$ again and with inequalities corresponding to $w{:}d_G(x, c_i) \leqq d_G(w, c_i) = 1$, for each $i = 0, 1, 2, 3$. As these inequalities have no simultaneous solutions in C_4 this subgraph C_4 cannot be a retract of G. These examples lead to the idea of a "hole".

Let K be a reflexive graph. A couple (H, δ), where $\emptyset \neq H \subseteq V(K)$ and δ is a function of H to the non-negative integers $\mathbf{N}$, is called a *hole of* K, if H is a subset of $V(K)$ for which there is no $x \in V(K)$ satisfying each of the inequalities

$$d_K(x, v) \leq \delta(v), \quad v \in H.$$

If we let $D_K(v, k)$ stand for the disk in K with centre v and radius k, [that is, $D_K(v, k) = \{u \in V(K) | d_K(u, v) \leq k\}$] then (H, δ) is a hole in K if $\emptyset \neq H \subseteq V(K)$ satisfies:

$$\bigcap_{v \in H} D_K(v, \delta(v)) = \emptyset.$$

We say that a hole (H, δ) of K is a *minimal hole* if, for each $H' \subsetneq H$ such that $|H'| < |H|$ there is some $u' \in V(K)$ satisfying:

$$u' \in \bigcap_{v \in H'} D_K(v, \delta(v)).$$

We can illustrate this by reference to Figure 6. First, take C_4 to be the subgraph with $V(C_4) = \{c_0, c_1, c_2, c_3\}$ of the graph in Figure 6 (a) and think of the vertex w as defining a function δ of $H = V(C_4)$ to $\mathbf{N}$:

$$\delta(c_1) = 2 = \delta(c_3) \quad \text{and} \quad \delta(c_0) = 1 = \delta(c_2).$$

As

$$\bigcap_{i=0}^{3} D'_{C_4}(c_i, \delta(c_i)) = \{c_1, c_3\}$$

(H, δ) is not a hole of C_4. In contrast, take C_4 from Figure 6 (b) to be the subgraph with the same vertex set $V(C_4) = \{c_0, c_1, c_2, c_3\}$ and again put $H = V(C_4)$ and let δ of $H = V(C_4)$ to $\mathbf{N}$ be induced by the inequalities associated with $w{:}\delta(c_i) = 1$ for each $i = 0, 1, 2, 3$. Then

$$\bigcap_{i=0}^{3} D_{C_4}(c_i, 1) = \emptyset$$

so (H, δ) is a hole of C_4 (in fact, a minimal hole).

LEMMA 1. *Let G be a reflexive graph, let K be a subgraph and let (H, δ) be a hole of K. If K is a retract of G then (H, δ) is a hole of G, too.*

Proof. Let G be an edge-preserving map of $V(G)$ to $V(K)$ such that $g(v) = v$ for each $v \in V(K)$. If (H, δ) is not a hole of G then there is

$$w \in \bigcap_{v \in H} D(v, \delta(v)).$$

As g is edge-preserving it follows that

$$d_K(g(a), g(b)) \leq d_G(a, b) \quad \text{for each } a, b \in V(G).$$

Then

$$d_K(g(v), g(w)) = d_K(v, g(w)) \leq d_G(v, w)$$

and in particular

$$g(w) \in \bigcap_{v \in H} D(v, \delta(v))$$

with $g(w) \in V(K)$. This is a contradiction.

Examples of holes.

Paths. For $n \in \mathbf{N}$ let P_n stand for the reflexive graph with vertex set $\{a_0, a_1, a_2, \ldots, a_n\}$ and edges joining consecutive vertices. P_n is called a *path*, and n is its *length.*

Even the path P_1 has a hole: define $\delta(a_0) = 0 = \delta(a_1)$. On the other hand, this is the only minimal hole of P_1, for if $\delta(a_0) = 0$ and $\delta(a_1) > 0$ then

$$D_{P_1}(a_0, 0) \cap D_{P_1}(a_1, \delta(a_1)) \neq \emptyset.$$

For $n \geq 1$, $\delta(a_0) = 0 = \delta(a_n)$ defines a hole $(\{a_0, a_n\}, \delta)$ using only the endpoints of P_n. For P_2, $\delta(a_0) = 0$ and $\delta(a_2) = 1$ defines a hole, while $\delta(a_0) = 1 = \delta(a_1)$ would not. In general, for P_n, the function $\delta(a_0) = 0$ and $\delta(a_n) = n - 1$ defines a minimal hole.

Holes in paths provide a natural setting for the idea of "isometry" in graphs. A subgraph K of a reflexive graph G is *isometric in* G if for each $a, b \in V(K)$, $d_K(a, b) = d_G(a, b)$.

LEMMA 2. *Let K be a subgraph of a reflexive graph G. Then K is isometric in G if and only if, any hole (H, δ) of K, for which some $a \in H$ has $\delta(a) = 0$, is also a hole of G.*

Proof. Suppose K is isometric in G and let (H, δ) be a hole of K with $\delta(a) = 0$ for some $a \in H \subseteq V(K)$. If there is

$$w \in \bigcap_{v \in H} D_G(v, \delta(v))$$

then, in particular, $w \in D_G(a, 0)$ so $w = a$. As (H, δ) is a hole of K there must be some $v \neq a$ such that

$$a \in D_G(v, \delta(v)) - D_K(v, \delta(v)).$$

Then $d_G(a, v) \leq \delta(v)$ while $d_K(v, a) \nleq \delta(v)$ which contradicts the assumption that K is isometric in G. Therefore, (H, δ) must be a hole of G, too. Conversely, if K is not isometric in G then there are $a, b \in V(K)$ such that $d_G(a, b) < d_K(a, b)$. Putting $\delta(a) = 0$ and $\delta(b) = d_G(a, b)$ gives a hole $(\{a, b\}), \delta)$ of K. As

$$a \in D_G(a, 0) \cap D_G(b, \delta(b))$$

this is not a hole of G.

It follows from Lemma 1 and Lemma 2 that

COROLLARY 3. *Let K be a subgraph of a reflexive graph G. If K is a retract of G then K is isometric in G.*

The converse of this holds if G contains no "cycles" (see **[2]**).

Cycles. For each integer $n \geq 3$ let C_n stand for the reflexive graph with vertex set $\{c_0, c_1, c_2, \dots, c_{n-1}\}$ and edges joining consecutive vertices and also c_{n-1} to c_0. (We read the indices modulo n.)

Define $\delta(c_i) = m - 1$ for each $i = 0, 1, 2, \dots, 2m - 1$. Then $(V(C_{2m}), \delta)$ is a hole of C_{2m} for each $m \geq 2$. For $i = 0, 1, 2, \dots, m - 1$,

$$c_{m+i} \in D_{C_{2m}}(c_i, m - 1)$$

and

$$c_i \in D_{C_{2m}}(c_{m_i}, m - 1),$$

so

$$\bigcap_{i=0}^{2m-1} D_{C_{2m}}(c_i, m - 1) = \emptyset$$

(see Figure 7 (b) for the case $m = 2$). For $m \geq 3$, a different hole in C_{2m} is this. Put

$$\delta(c_0) = \delta(c_{2m-1}) = 1 \quad \text{and} \quad \delta(c_{m-1}) = \delta(c_m) = m - 1.$$

Then $(\{c_0, c_{m-1}, c_m, c_{2m-1}\}, \delta)$ is a minimal hole of C_{2m}, since:

$$D_{C_2}(c_0, 1) \cap D_{C_{2m}}(c_{2m-1}, 1) = \{c_0, c_{2m-1}\} \quad \text{and}$$

$$D_{C_{2m}}(c_{m-1}, m - 1) \cap D_{C_2}(c_{2m-1}, m - 1) = \{c_1, c_2, \dots, c_{2m-2}\}.$$

For C_3, $\delta(c_0) = 0 = \delta(c_2)$ gives this hole $(\{c_0, c_2\}, \delta)$ of C_3. In general, for $m \geqq 2$, if $\delta(c_i) = m - 1$ for each $i \neq m - 1, m + 1$ then (H, δ) with $H = V(C_{2m+1}) - \{c_{m-1}, c_{m+1}\}$ is a hole of C_{2m+1}. It need not be a minimal hole though (see Figure 13).

(H, δ), with $H = \{c_0, c_1, c_3, c_5, c_6\}$ and $\delta(c_i) = 2$ for each $i = 0, 1, 3, 5, 6$, is not a minimal hole of C_7. $(\{c_1, c_3, c_5\}, \delta| \{c_1, c_3, c_5\})$ is a minimal hole.

For the case of the cycles C_{2n+1} a minimal hole can, however, be defined as follows. Put

$$H = V(C_{2n+1}) - \{c_i | i \equiv 1(2)\}$$

and

$$\delta(c_j) = n - 1$$

for each $c_j \in H$. Then (H, δ) is a minimal hole of C_{2n+1}.

The graph D_6. The reflexive graph D_6 is illustrated in Figure 15. Define $\delta(a) = \delta(d) = \delta(r) = \delta(s) = 1$. Then $(\{a, d, r, s\}, \delta)$ is a minimal hole of D_6.

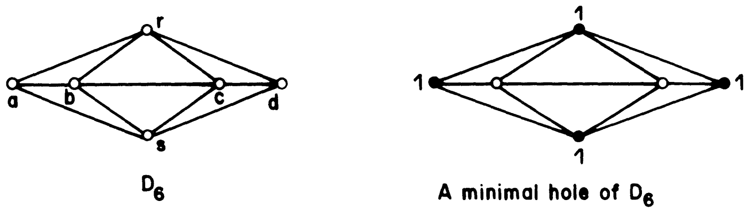


Figure 7

The graphs (J_n) *and* (L_n). The reflexive graph J_n, $n \geq 2$, has vertex set $\{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\}$. The subset $A_n = \{a_1, a_2, \ldots, a_n\}$ forms a complete n-element subgraph of J_n, the subset $B_n = \{b_1, b_2, \ldots, b_n\}$ has no edges, and otherwise, each pair of vertices $a_i \in A_n$, $b_j \in B_n$ is joined by an edge except if $i = j$. Note that $J_2 \cong P_3$. The reflexive graph L_n, $n \geq 1$ has the vertex set $\{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\}$. Both subsets $A_n = \{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ form complete n-element subgraphs of L_n, and further, for each $i \neq j$ there is an edge joining a_i and b_j. Note that $L_2 \cong C_4$.

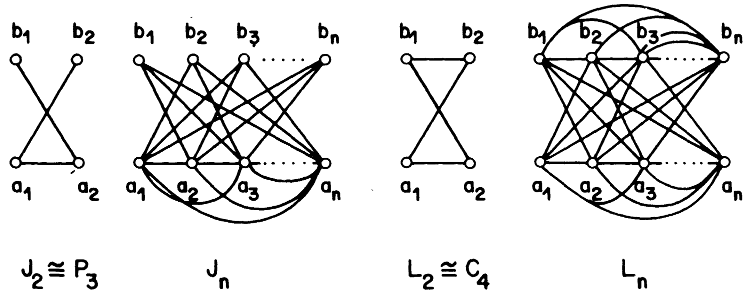


Figure 8

For J_n, define $\delta(b_i) = 1$, $i = 1, 2, \ldots, n$. Then (B_n, δ) is a minimal hole of J_n. It is enough to note that $b_i \notin D_{J_n}(b_j, 1)$ whenever $i \neq j$, and, for each i, $a_i \notin D_{J_n}(b_i, 1)$. For L_n, define $\delta(v) = 1$ for each $v \in V(L_n)$. Then $(V(L_n), \delta)$ is a minimal hole of L_n.

Which functions δ of a set H to $\mathbf{N}$ can be minimal holes?

PROPOSITION 4. *Let H be a set and let δ be a function of H to* $\mathbf{N}$. *There is a* *connected reflexive graph K in which (H, δ) is a minimal hole if and only if, either $|H| = 2$ or $|H| \geq 3$ and $\delta(v) > 0$ for each $v \in H$.*

Proof. Suppose that (H, δ) is a minimal hole of the reflexive graph K. The case $|H| = 1$ is of course impossible since, if $H = \{a\}$ then $a \in D_K(a, \delta(a))$, even if $\delta(a) = 0$. Suppose that $|H| \geq 3$ and that $\delta(a_0) = 0$ for some $a_0 \in H$. As

$$\bigcap_{a \in H} D_K(a, \delta(a)) = \emptyset$$

there must be $a_1 \in H$, $a_1 \neq a_0$, such that

$$a_0 \notin D_K(a_1, \delta(a_1)).$$

It follows that

$$d_K(a_0, a_1) > \delta(a_1)$$

so (H, δ) cannot be a minimal hole of K.

Conversely, let

$$H = \{a_1, a_2, \dots a_n, \dots\} \text{ and } H' = \{b_1, b_2, \dots, b_n, \dots\}$$

with $|H'| = |H|$. For each $i, j = 1, 2, \dots, n, \dots, i \neq j$ let P_{ij} stand for a path of length $\delta(a_i)$ and with endpoints a_i and b_j. In contrast to edges in a graph we use a perforated line segment to illustrate such a path. Suppose $|H| = 2$. Then the reflexive graph K with vertex set $V(P_{12}) \cup V(P_{21})$ and edge set $E(P_{12}) \cup E(P_{21})$ together with an edge joining b_1 and b_2 has (H, δ) as a minimal hole. Suppose then that $|H| \geq 3$. We construct a graph K with vertex set

$$V(K) = \bigcup_{i \neq j} V(P_{ij})$$

and edge set

$$E(K) = \bigcup_{i \neq j} E(P_{ij})$$

such that, for $i \neq j$ and $j \neq k$,

$$V(P_{ij}) \cap V(P_{ik}) = \{a_i\},$$

and

$$V(P_{ij}) \cap V(P_{kj}) = \{b_j\}.$$

Then it is straightforward to verify that (H, δ) is a minimal hole of K.

The graph M_ω. The reflexive graph M_ω has vertex set $\{a_1, a_2, \dots\} \cup \{b_1, b_2, \dots\}$. The subset $A = \{a_1, a_2, \dots\}$ forms a complete graph, the subset $B = \{b_1, b_2, \dots\}$ has no edges at all, and otherwise, each $a_i \in A$ is joined by an edge to each $b_j \in B$ satisfying $j < i$. Notice that in M_ω a subset H of $V(M_\omega)$ together with the function $\delta(v) = 1$ for each $v \in H$

https://doi.org/10.4153/CJM-1986-066-9 Published online by Cambridge University Press

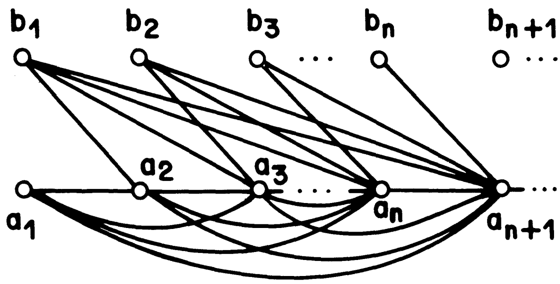


Figure 9

determines a hole (H, δ) of M_ω just if H contains any infinite collection of the b_j's, that is,

$$\bigcap_{v \in H} D_{M_\omega}(v, 1) = \emptyset$$

if and only if $|H \cap B|$ is infinite. In fact, each such (H, δ) with infinitely many of the b_j's is a minimal hole of M_ω.

"Preserving" holes. Let (H, δ) be a hole of a direct product

$$G = \prod_{i \in I} G_i$$

of reflexive graphs G_i, $i \in I$. Let π_i stand for the ith projection map of $V(G)$ to $V(G_i)$. This map π_i is, of course, edge-preserving. Suppose for each $i \in I$ there is $u_i \in V(G_i)$ satisfying

$$d_{G_i}(u_i, \pi_i(v)) \leq \delta(v) \quad \text{for each } v \in H.$$

Then the vertex $u \in V(G)$ defined by $\pi_i(u) = u_i$ $(i \in I)$ satisfies

$$d_G(u, v) = \sup\{d_{G_i}(u_i, \pi_i(v)) \,|\, i \in I\} \leq \delta(v)$$

for each $v \in H$ which contradicts the assumption that (H, δ) is a hole of G. Therefore, there must be some $i \in I$ for which no vertex $u_i \in V(G_i)$ exists satisfying

$$d_{G_i}(u_i, \pi_i(v)) \leq \delta(v).$$

This fact we shall summarize by saying that each hole of $\prod_{i \in I} G_i$ is preserved by a projection map.

In general, if G, K are reflexive graphs and (H, δ) is a hole of G then we say that the hole (H, δ) of G is *preserved by* K (or K *preserves* the hole (H, δ) of G), if there is an edge-preserving map f of $V(G)$ to $V(K)$ such that there is no vertex w of K satisfying

$$d_K(w, f(v)) \leq \delta(v),$$

for each $v \in V(G)$. The map f is also called a *hole-preserving* map of (H, δ) in K (or f *preserves* the hole (H, δ) of G in K). For example each hole (H, δ) of $\prod_{i \in I} G_i$ is preserved by some G_i and the hole-preserving maps may be chosen from among the projection maps.

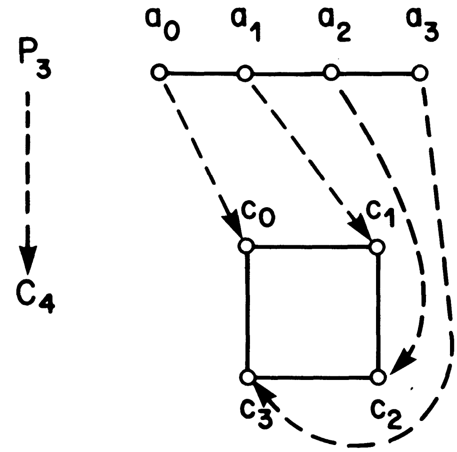
Let F be a retract of a reflexive graph G. According to Lemma 1, any hole (H, δ) of F is a hole of G. Suppose that this hole (H, δ) of G is preserved by some reflexive graph K. Then this hole (H, δ) of F is also preserved by K. For, if f is the hole-preserving map of (H, δ) (of G) to K and g is the retraction map of $V(G)$ to $V(F)$ then the edge-preserving map $f \circ g$ of $V(G)$ to $V(K)$ preserves the hole (H, δ) of F in K (remember that $H \subseteq V(K)$ and $f \circ g|V(F) = f|V(F)$).

Here is another way to formulate this idea of a hole-preserving map f of $V(G)$ to $V(K)$. Define the map δ_f of $f(H)$ to $\mathbf{N}$ by

$$\delta(u) = \min\{\delta(v) \,|\, f(v) = u\}.$$

Then $(f(H), \delta_f)$ is a hole of K, if (H, δ) is a hole of G and f is a hole-preserving map.

We consider some particular examples. Consider the edge-preserving map f of P_3 to C_4. Then the hole (H, δ) of P_3, where $H = \{a_0, a_3\}$ and $\delta(a_0) = \delta(a_3) = 1$ is not preserved by f in C_4 (see Figure 10).



f does not preserve the hole ($\{a_0, a_3\}$, $\delta(a_0) = \delta(a_3)$) of P_3 in C_4.

Figure 10

Actually, this hole of P_3 cannot be preserved by C_4 at all. In fact, if f is an edge-preserving map of $V(P_3)$ to $V(K)$, for some graph K, and f preserves this hole, then

$$d_K(f(a_0), f(a_3)) \leqq 3,$$

since f is edge-preserving and

$$d_K(f(a_0), f(a_3)) \geq 3,$$

for otherwise

$$D_K(f(a_0), 1) \cap D_K(f(a_3), 1) \neq \emptyset.$$

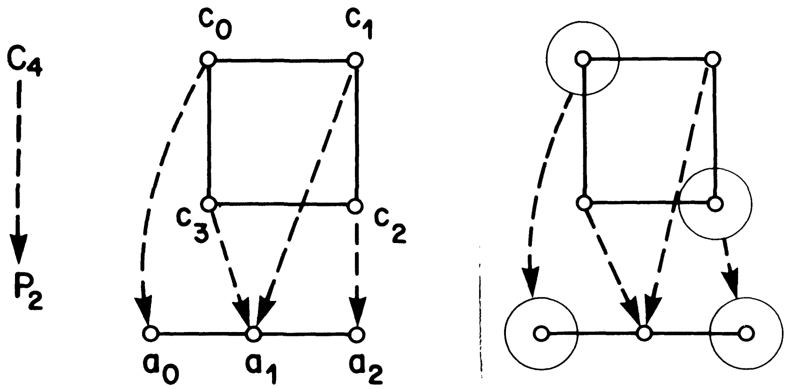
There is, of course, no pair c_i, c_j of vertices in C_4 satisfying

$$d_{C_4}(c_i, c_j) = 3.$$

What about the hole $(V(C_4), \delta)$ of C_4, where $\delta(c_i) = 1$, for each $i = 0, 1, 2, 3$? To preserve this hole requires an edge-preserving map f to a graph K and, the map must be one-to-one as well. It is easy to verify that the subgraph determined by $f(K)$ in K must be isomorphic to C_4. In particular, this hole of C_4 cannot be preserved by P_3. In contrast the hole

$$(\{c_0, c_2\}, \delta(c_0) = 0 = \delta(c_2))$$

of C_4 can be preserved by P_2 (see Figure 11).



f preserves the hole $(\{c_0, c_2\}, \delta(c_0) = 0 = \delta(c_2))$ of C_4 in P_2.

Figure 11

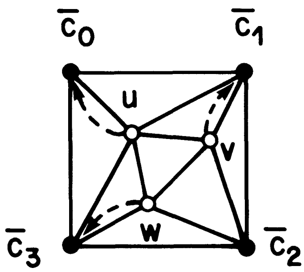
We consider holes in C_6 and C_7. First $\delta(c_i) = 2$, for each $c_i \in V(C_6)$ defines a hole $(H, \delta) = (V(C_6), \delta)$ of C_6. Also, $\delta'(c_i') = 2$ for each $c_i' \in V(C_7) - \{c_2, c_4\}$ defines a hole

$$(H', \delta') = (V(C_7) - \{c_2, c_4\}, \delta')$$

of C_7, too. Now let f be an edge-preserving map of $V(C_6)$ to $V(C_7)$. As f cannot be onto, its image must be a path of length at most three. In particular, this hole of C_6 cannot be preserved by C_7. Now let f' be an edge-preserving map of $V(C_7)$ to $V(C_6)$. As f' cannot be one-to-one C_6 cannot preserve this hole of C_7.

Now take this hole of C_6:$H = \{c_0, c_3\}$ and $\delta(c_0) = \delta(c_3) = 1$. This hole is preserved by P_3 using the map $f(c_0) = a_0$, $f(c_0) = f(c_5) = a_1$, $f(c_2) = f(c_4) = a_3$, and $f(c_3) = a_3$.

The graph D_6 (cf. Figure 7) has the hole (H, δ) where $H = \{a, d, e, f\}$ and $\delta(a) = \delta(d) = \delta(e) = \delta(f) = 1$. It makes sense to try preserving this hole of D_6 in C_4. If there were such an edge-preserving map f of $V(D_6)$ to $V(C_4)$ then f would be onto and without less of generality, $f(a) = c_0$, $f(d) = c_2, f(r) = c_1$ and $f(s) = c_3$. Now, $f(b)$ must be adjacent to c_0, c_1, c_3 (since f is edge-preserving) so $f(b) = c_0$ and similarly $f(c)$ must be adjacent to c_1, c_2, c_3 so $f(c) = c_2$. But $f(c)$ must also be adjacent to $f(b)$. Therefore, C_4 cannot preserve this particular hole of D_6.



A subgraph of $\prod_{i \in I} G_i$ with $\bar{c}_0, \bar{c}_1, \bar{c}_2, \bar{c}_3\} \cong C_4 \lhd \prod_{i \in I} G$.

Figure 12

Consider the reflexive graph J_3 (see Figure 8). Can it preserve the hole $(V(C_4), \delta)$ of C_4 with $\delta(c_i) = 1$, for each $c_i \in V(C_4)$? Suppose f is an edge-preserving map of $V(C_4)$ to $V(J_3)$ which preserves this hole of C_4. Then each $b_i \in f(V(C_4))$ which, however, is impossible since f is edge-preserving and the b_i's are pairwise non-adjacent. Moreover this same hole of C_4 cannot be preserved by L_3 either or, for that matter, by any J_n or L_n, $n \geq 3$. It follows that C_4 cannot be a retract of a direct product of reflexive graphs each isomorphic to a J_n $(n \geq 3)$ or to an L_n $(n \geq 3)$. For if

$$C_4 \lhd \prod_{i \in I} G_i$$

then, by Lemma 1, $(V(C_4), \delta)$ is a hole of $\prod_{i \in I} G_i$ and so some G_i must preserve this hole, and this is impossible if $G_i \cong J_n$ or $G_i \cong L_n$, $n \geq 3$. In other terms, C_4 does not have a representation using only the J_n's and L_n's.

To show that C_4 does not have a representation using a family of graphs each isomorphic to D_6 is more difficult, because the hole $(V(C_4), \delta)$, $\delta(c_i) = 1$, $i = 0, 1, 2, 3$, is preserved by D_6; just take $f(c_0) = a, f(c_1) = r$, $f(c_2) = d, f(c_3) = s$. Suppose that

$$C_4 \lhd \prod_{i \in I} G_i$$

where each $G_i \cong D_6$. Let g be the retraction map of $V(\prod_{i \in I} G_i)$ to $V(C_4)$. We may suppose that C_4 is a subgraph of $\prod_{i \in I} G_i$ with vertices labelled $\bar{c}_0, \bar{c}_1, \bar{c}_2, \bar{c}_3$ (and all of their coordinates chosen from among $V(D_6)$). We shall construct vertices u, v, w in $\prod_{i \in I} G_i$ which, with $V(C_4)$, determine, a subgraph in $\prod_{i \in I} G_i$ as indicated in Figure 23.

Not all of the edges, as illustrated in Figure 9, can exist though, since consider the effect of the retraction map $g:g(c_i) = c_i$, $i = 0, 1, 2, 3$ so $g(u) = c_0$ and then $g(v) = c_1$ and $g(w) = c_3$, although $g(v)$ and $g(w)$ should be adjacent. We construct the vertices u, v, w by prescribing their ith coordinates u_i, v_i, w_i, for each $i \in I$. For our purposes there are two kinds of projection maps: $i \in I_0$, if π_i preserves the hole $(V(C_4), \delta)$ in $G_i \cong H$; $i \in I_1$, otherwise. For $i \in I_0$, the coordinates of u, v, w are prescribed according to the values in Table 1. For $i \in I_1$, there is

$$t \in \bigcap_{j=0}^{3} D(\pi_i(c_j), 1),$$

and we put $u_i = v_i = w_i = t$ in this case. Then the vertices u, v, w given by $\pi_i(u) = u_i$, $\pi_i(v) = v_i$, $\pi_i(w) = w_i$ if $i \in I_0$ and $\pi_i(u) = \pi_i(v) = \pi_i(w) = t$ if $i \in I_1$ are pairwise adjacent and moreover, u is adjacent to $\bar{c}_0$, $\bar{c}_1$, $\bar{c}_3$, v to $\bar{c}_1$, $\bar{c}_2$, and w to $\bar{c}_2$, $\bar{c}_3$. In other terms we have shown that C_4 is not a retract of any direct product of D_6's or, equivalently, $C_4 \notin \{D_6\}^{\nu}$.

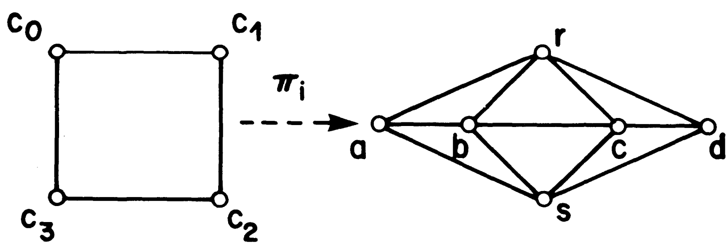


Figure 13

This is a convenient fact: the "image" of a minimal hole by a hole-preserving map is a minimal hole (see [3]).

TABLE 1

$\pi_i(c_0)$	$\pi_i(c_2)$	$\pi_i(c_1)$	$\pi_i(c_3)$	u_i	v_i	w_i
a	d	s	r	b	c	c
a	d	r	s	b	c	c
d	a	s	r	c	b	b
d	a	r	s	c	b	b
r	s	a	d	r	b	c
r	s	d	a	r	c	b
s	r	a	d	s	b	c
s	r	d	a	s	c	b

LEMMA 5. *Let G, K be reflexive graphs, let (H, δ) be a minimal hole in G and let f be an edge-preserving map of $V(G)$ to $V(K)$ which preserves this hole. Then $(f(H), \delta_f)$, where*

$$\delta_f(u) = \min\{\delta(v) \mid f(v) = u\}$$

is a minimal hole in K.

Proof. According to the definition of a hole-preserving map, ($\mathcal{U} = f(H)$, δ_f) is a hole in K. Is it minimal? If not, there is $\mathcal{U}' \subsetneq \mathcal{U}$ such that

$$|\mathcal{U}'| < |\mathcal{U}| \quad \text{and} \quad \bigcap_{u \in \mathcal{U}'} D_K(u, \delta_f(u)) = \emptyset.$$

Now construct a subset H' of H consisting of those vertices $v' \in H$ such that

$$f(v') \in \mathcal{U}' \quad \text{and} \quad \delta(f(v')) = \min\{\delta(v) \mid f(v) = f(v')\}.$$

Then $H' \subsetneq H$ and $|H'| < |H|$. Therefore there is

$$u' \in \bigcap_{v' \in H'} D_G(v', \delta(v'))$$

and so

$$f(u') \in \bigcap_{v' \in H'} D_K(f(v'), \delta(v')) = \bigcap_{u \in \mathcal{U}'} D_K(u, \delta_f(u))$$

which is a contradiction.

What about graphs with "infinitary" holes? For instance, M_ω has a minimal hole (H, δ) for which H is infinite. Therefore, any graph which preserves this hole must itself have an infinite hole. It follows that M_ω cannot be a retract of any direct product of finite graphs, no matter how large the index set of this direct product.

Irreducible reflexive graphs. Our purpose is to show that, for each $n \geq 3$, each of the reflexive graphs P_n, C_n, D_6, J_n, L_n *and* M_ω is irreducible. However, first we record an observation already implicit in the calculations above.

LEMMA 6. *If* $(G_i | i \in I)$ *is a representation of the reflexive graph G and* (H, δ) *is a hole of G, then this hole is preserved by some* G_i.

Let $(G_i | i \in I)$ be any representation of the path P_n; that is, each $G_i \lhd P_n$ and $P_n \lhd \prod_{i \in I} G_i$. Consider the hole (H, δ) of P_n, with $H = \{a_0, a_n\}$ and $\delta(a_0) = 0$, $\delta(a_n) = n - 1$. According to Lemma 6 this hole is preserved by some G_i. Now, each $G_i \lhd P_n$, so G_i must be a path P_m, say, where $m \leq n$. But to preserve this hole (H, δ) of P_n, $m = n$, that is, $G_i \cong P_n$ therefore, P_n is irreducible.

In practice we use Lemma 6 in this form (cf. [3]).

COROLLARY 7. *Each hole of a reducible reflexive graph is preserved by a proper retract.*

Suppose the cycle C_{2m} $(m \geq 2)$ is reducible. Let (H, δ) be this hole of C_{2m}:$H = V(C_{2m})$ and $\delta(c_i) = m - 1$ for each $c_i \in V(C_{2m})$. Now, any proper subgraph of C_{2m} which is a retract must be connected whence it must be a path P_k and $k \leq m$. But P_k cannot preserve this hole of C_{2m}. For the cycle C_{2m+1} use the hole (H, δ) with

$$H = V(C_{2m+1}) - \{c_{m-1}, c_{m+1}\} \quad \text{and} \quad \delta(c_i) = m - 1.$$

Again, any proper retract of C_{2m+1} must be a path P_k and $k \leq m$. But P_k cannot preserve this hole. This shows that each cycle $C_n(n \geq 3)$ is irreducible.

Consider the hole (H, δ) of D_6 given by $H = \{a, d, r, s\}$ and $\delta(a) = \delta(d) = \delta(r) = \delta(s) = 1$. Now, any proper retract D of D_6 must contain a vertex adjacent to all other vertices of D, so it could not preserve this hole of D_6. It follows that D_6 must be irreducible.

Consider the hole (H, δ) of L_n given by $H = V(L_n)$ and $\delta(v) = 1$ for each $v \in V(L_n)$. Again, any proper subgraph L of L_n contains a vertex adjacent to all other vertices of L, so it could not preserve this hole. Therefore, L_n is irreducible.

Consider the hole (H, δ) of J_n given by $H = \{b_1, b_2, \dots, b_n\}$ and $\delta(b_i) = 1$, for each $i = 1, 2, \dots, n$. Suppose $J \lhd J_n$, $J \supsetneqq J_n$, preserves this hole. Then each $b_i \in V(J)$ for otherwise J contains a vertex adjacent to all others of J and so J could not preserve this hole (H, δ). But then it is simple to verify that J must also contain each a_i, for, a given a_i is the only vertex adjacent to all other vertices different from b_i.

Finally, the reflexive graph M_ω. Suppose M_ω is reducible. Then there is a retract G of M and an edge-preserving map f of $V(M_\omega)$ to $V(G)$ which preserves the hole (H, δ) of M, where $H = B$ and $\delta \equiv 1$ (cf. Figure 9). Evidently G must be infinite and we are to suppose that M_ω is itself not a retract of G. Now, we may treat G as a subgraph of M_ω and so G must contain infinitely many of the b_j's in $B \subseteq V(M_\omega)$. In fact, we may suppose that there is an increasing sequence $\sigma(1) < \sigma(2) < \dots$ of indices such that each of $b_{\sigma(1)}, b_{\sigma(2)}, \dots$ belongs to $V(G)$ and so that $(\{b_{\sigma(1)}, b_{\sigma(2)}, \dots\}, \delta)$ with $\delta(b_{\sigma(i)}) = 1$ for each i, is a hole of G. Now, set $b'_1 = b_{\sigma(1)}$ and choose $a'_1 \in A \cap V(G)$ such that $(a'_1, b'_1) \in E(G)$. (Note that such a vertex a'_1 exists in $G{:}g(a_{\sigma(1)})$ is such a vertex, where g is the retraction map of M_ω to G.) Let b'_2 be the first $b_{\sigma(i)}$ not adjacent to a'_1. Then choose $a'_2 \in V(G)$ which is adjacent to b'_2 (and therefore b'_1 too). (It exists, right?) Then choose b'_3 the first from among the $b_{\sigma(i)}$'s not adjacent to a'_2; then $a'_3 \in V(G)$, etc. In this way we construct a subgraph of G isomorphic to M_ω itself. In fact this subgraph is a retract of G itself. To see this it is convenient to relabel the vertices a'_i, b'_i of G according to their label in

$$M_\omega{:}a'_i = a_{\tau(i)},\ b'_i = b_{\rho(i)}.$$

Then we can define a map h of $V(G)$ to this subgraph of G by these rules

$$h(a_j) = h(b_j) = a_{\tau(1)}$$

for all $a_j \in V(G)$ such that $j \leq \tau(1)$ and for all $b_j \in V(G)$ such that $j < \tau(1)$;

$$h(a_j) = h(b_j) = a_{\tau(n)}$$

for all $a_j \in V(G)$ such that $\tau(n - 1) < j \leqq \tau(n)$ and for all $b_j \in V(G)$ such that $\tau(n - 1) < j < \tau(n)$;

$$h(b_j) = b_j$$

if $j = \tau(n)$. It is straightforward to verify that h is a retraction, that is, $M_\omega \lhd G$ and so M_ω is irreducible after all.

Varieties of reflexive graphs. Our purpose is to justify Figure 4 and Figure 5 concerning the lattice of reflexive graph varieties. We shall prove these results.

THEOREM A. *The lattice of reflexive graph varieties contains an infinite chain. In fact,*

$$\{P_0\}^\nu \prec \{P_1\}^\nu \prec \{P_2\}^\nu \prec \ldots \prec \{P_n\}^\nu \prec \ldots .$$

THEOREM B. *The lattice of reflexive graph varieties contains an infinite antichain. In fact, for distinct positive integers n, $m \geq 4$,*

$$\{C_n\}^\nu \nleqq \{C_m\}^\nu$$

and

$$\{C_m\}^\nu \nleqq \{C_n\}^\nu.$$

Moreover,

$$\{P_n\}^\nu \prec \{C_{2n}\}^\nu$$

and

$$\{P_n\}^\nu \prec \{C_{2n+1}\}^\nu.$$

In the lattice of reflexive graph varieties $\{P_0\}^\nu$ is the least element. Let $\mathscr{V}$ be any variety which contains a member $G \in \mathscr{V}$ with $|V(G)| > 1$. Suppose G contains an adjacent pair of vertices u, v. Then the subgraph on $\{u, v\}$ is isomorphic to P_1 and it is easy to construct a retraction for $P_1 \lhd G$. Therefore, $P_1 \in \mathscr{V}$. If $E(G) = \emptyset$ then any pair of distinct vertices u, v forms a subgraph called A_2 and again it is easy to provide a retraction for $A_2 \lhd G$. Therefore $A_2 \in \mathscr{V}$. In summary the least variety $\{P_0\}^\nu$ has precisely two covers: $\{P_1\}^\nu$ and $\{A_2\}^\nu$. Let $\mathscr{V}$ be any variety satisfying $\mathscr{V} > \{P_1\}^\nu$ or, $\mathscr{V} > \{A_2\}^\nu$. Then $\mathscr{V}$ contains a member G such that $G \notin \{P_1\}^\nu$ or $G \notin \{A_2\}^\nu$. Evidently, $|V(G)| > 2$. If G is a complete graph, that is, every vertex is adjacent to every other vertex, then

$$G \lhd \prod_{i \in I} G_i$$

with each $G_i \cong P_1$, and so $G \in \{P_1\}^\nu$. If $E(G) = \emptyset$ then

$$G \lhd \prod_{i \in I} G_i$$

with each $G_i \cong A_2$, and so $G \in \{A_2\}^{\nu}$. Suppose $E(G) \neq \emptyset$ and, yet, not all vertices are adjacent to all others. If G contains a subgraph isomorphic to P_2 then it is not hard to verify that $P_2 \lhd G$. Otherwise, G consists of components (maximal connected subsets) each of which is a complete graph. In this case, $P_1 \lhd G$ and $A_2 \lhd G$. This is the substance of the fact that $\{P_1\}^{\nu}$ has precisely two covers, $\{P_2\}^{\nu}$ and $\{P_1, A_2\}^{\nu}$, and $\{A_2\}^{\nu}$ has precisely one cover, $\{P_1, A_2\}^{\nu}$. More generally, for each n, $\{P_{n+1}, A_2\}^{\nu}$ covers both $\{P_n, A_2\}^{\nu}$ and $\{P_{n+1}\}^{\nu}$. That much about the lattice of reflexive graph varieties was fairly straightforward.

THEOREM C. *In the lattice of reflexive graph varieties each of the graphs* P_3, C_5, D_6, J_n $(n \geq 3)$, L_n $(n \geq 3)$, *and* M_{ω}, *generates a distinct reflexive graph variety which covers* $\{P_2\}^{\nu}$.

In a sense the heart of these results lies in this lemma. (A similar technique is used in [**1**], see especially Lemma 6.12.)

LEMMA 8. *Let*

$$\mathscr{K} = \{P_n, C_{n+1}, J_{n+1}, L_{n+1} | n \geq 2\},$$

let $K \in \mathscr{K}$ *and let* $G \in \{\mathscr{K}\}^{\nu}$. *Then any edge-preserving map* g *of* $V(G)$ *onto* $V(K)$ *is a retraction.*

Proof. Let $K = P_n$. Suppose $G \in \{P_n\}^{\nu}$ and let g be an edge-preserving map of $V(G)$ to $V(P_n)$. (We shall use about G only the hypothesis that P_n preserves each hole of G which, of course, follows from $G \in \{P_n\}^{\nu}$.) For each $i = 0, 1, 2, \ldots, n$ let $A_i = g^{-1}(\{a_i\})$. Once we show that there is a system of representatives $v_i \in A_i$ such that the subgraph $\{v_0, v_1, \ldots, v_n\} \cong P_n$, with each v_i adjacent to v_{i+1}, then $P_n \lhd G$ by identifying $\{v_0, v_1, \ldots, v_n\}$ with P_n. As g is onto, each $A_i \neq \emptyset$. Choose $v_0 \in A_0$ and $v_n \in A_n$. Is $(\{v_0, v_n\}, \delta)$, with $\delta(v_0) = 0$ and $\delta(v_n) = n$, a hole of G? If it were then P_n would preserve it and that is impossible. Therefore,

$$D_G(v_0, 0) \cap D_G(v_n, n) \neq \emptyset$$

and this means that

$$d_G(v_0, v_n) \leq n.$$

As g is an edge-preserving map of $V(G)$ onto $V(P_n)$,

$$d_G(v_0, v_n) = n,$$

and each minimal path of length n in G must meet each A_i.

Let $K = C_{2m}$, where $m \geq 2$, let $G \in \{C_{2m}\}^{\nu}$ and suppose g is an edge-preserving map of $V(G)$ onto $V(C_{2m})$. Put

$$A_i = g^{-1}(\{c_i\}) \quad \text{for each } c_i \in V(C_{2m}).$$

Choose $v_0 \in A_0$, $v_{m-1} \in A_{m-1}$, and $v_m \in A_m$. Define

$$\delta(v_0) = 1, \delta(v_{m-1}) = \delta(v_m) = m - 1.$$

Now $(\{v_0, v_{m-1}, v_m\}, \delta)$ cannot be a hole of G since C_{2m} cannot preserve such a hole. (Recall, that $(\{c_0, c_{m-1}, c_m, c_{2m-1}\}, \delta')$ with

$$\delta'(c_0) = \delta'(c_{2m-1}) = 1 \quad \text{and} \quad \delta'(c_{m-1}) = \delta'(c_m) = m - 1$$

is a minimal hole of C_{2m}.) Therefore, there is a vertex $u \in V(G)$ such that

$$d_G(u, v_0) \leq 1, d_G(u, v_{m-1}) \leq m - 1, \quad \text{and} \quad d_G(u, v_m) \leq m - 1.$$

But g is edge-preserving and

$$d_{C_{2m}}(c_0, c_m) = m,$$

so there is a path of length m in G with endpoints v_0 and v_n and passing through v_{n-1}. By symmetry there is a path of length m in G with endpoints v_0 and v_n and passing through $v_{2m-1} \in A_{2m-1}$. These two paths must meet each block and form a subgraph of G isomorphic to C_{2m}. Once we identify it with C_{2m} we have that g is indeed a retraction. The case $K = C_{2m+1}$, where $m > 2$ is similar. Choose $v_0 \in A_0$, $v_m \in A_m$ with $\delta(v_0) = 0$, $\delta(v_m) = m$. Then $(\{v_0, v_m\}, \delta)$ is not a hole of G since C_{2m+1} cannot preserve it. Therefore, there is a path $v_0, v_1, \dots, v_m$ of length m in G with endpoints v_0 and v_m, and passing through $v_1 \in A_1$, say. Choose $v_{m+1} \in A_{m+1}$ and apply the same argument to construct a path of length m passing through $A_1, A_2, \dots, A_{m+1}$ and having endpoints v_1 and v_{m+1}. This path together with another one with endpoints v_{m+1} and v_0 gives a subgraph of G isomorphic to C_{2m+1} and once identified with C_{2m+1}, g is a retraction.

Let $K = L_n$, (cf. Figure 8), let $G \in \{L_n\}^\nu$ and let g be an edge-preserving map of $V(G)$ onto $V(L_n)$. Put

$$A_i = g^{-1}(\{a_i\}) \quad \text{and} \quad B_i = g^{-1}(\{b_i\}), i = 1, 2, \dots, n.$$

Choose $v_i \in A_i$ and $u_i \in B_i$. Let

$$H_i = \{v_j | j = 1, 2, \dots, n\} \cup \{u_j | j = 1, 2, \dots, i - 1, i + 1, \dots, n\}$$

and let $\delta_i(w) = 1$ for each $w \in H_i$. Evidently, (H_i, δ_i) cannot be a hole of G so there is a vertex in

$$\bigcap_{w \in H_i} D_G(w, 1) \neq \emptyset.$$

Such a vertex must belong to A_i and we may suppose that it is $v_i \in A_i$. In fact, we may suppose that the v_j's are so chosen that

$$v_j \in \bigcap_{w \neq u_j} D_G(w, 1)$$

https://doi.org/10.4153/CJM-1986-066-9 Published online by Cambridge University Press

and, by symmetry, that the u_j's are so chosen that

$$u_j \in \bigcap_{w \neq v_j} D_G(w, 1).$$

Then the subgraph

$$\{v_j, u_j | j = 1, 2, \ldots, n\} \cong L_n$$

and g must be a retraction.

Let $K = J_n$, (cf. Figure 8), let $G \in \{J_n\}^v$ and let g be an edge-preserving map of $V(G)$ onto $V(J_n)$. Again, set

$$A_i = g^{-1}(\{a_i\}) \quad \text{and} \quad B_i = g^{-1}(\{b_i\}), i = 1, 2, \ldots, n.$$

Let $v_i \in A_i$, $u_i \in B_i$ be chosen. As g is edge-preserving there are no edges between distinct u_i's and no edges joining v_i and u_i, for each $i = 1, 2, \ldots, n$. Put

$$H_i = \{u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n\}$$

and $\delta_i(u) = 1$ for each $u \in H_i$. Then (H_i, δ_i) cannot be a hole of G. In fact, we may even suppose that

$$v_i \in \bigcap_{j \neq i} D_G(u_j, 1).$$

Now, put

$$H^i = \{v_j, u_j | j \neq i\} \quad \text{and} \quad \delta^i(w) = 1 \text{ for each } w \in H^i.$$

Suppose that (H^i, δ^i) is a hole of G. Then there is an edge-preserving map f^i of $V(G)$ to $V(J_n)$ which preserves this hole. If some $b_j \notin f^i(H^i)$ then a_j is adjacent to each vertex of $f^i(H^i)$ and the hole is not preserved. Therefore, each $b_j \in f^i(H^i)$. Then, for some v_r, $f^i(v_r) = b_k$, say, and, as f^i is edge-preserving

$$f^i(u_j) \in \{a_1, a_2, \ldots, a_n\} \quad \text{for each } j \neq r, i.$$

Again, for some u_s, $f^i(u_s) = b_l$ and evidently, $s = r$. f^i is edge-preserving so

$$f^i(v_j) \in \{a_1, a_2, \ldots, a_n\} \quad \text{for } j \neq r, i.$$

As $n \geq 3$, there is $b_m \neq f^i(u_r), f^i(v_r)$, and so

$$a_m \in \bigcap_{w \in H^i} D_{J_n}(f^i(w), 1).$$

Therefore, (H^i, δ^i) is not a hole of G. In particular

$$\bigcap_{w \in H^i} D_G(w, 1) \neq \emptyset$$

and the common vertex must belong to A_i. We may suppose it is v_i. In summary,

$$v_i \in \bigcap_{j=1}^{n} D_G(v_j, 1) \cap \bigcap_{j \neq i} D_G(u_j, 1).$$

Then $\{u_i, v_i | i = 1, 2, \ldots, n\}$ determines a subgraph of G isomorphic to J_n. Then g is a retraction.

Insofar as all of our principal results pertain to matters concerning covers of "path" varieties, we make use of this basic result of [**3**].

LEMMA 9. *Let $\mathcal{K}$ be any set of paths. Then a reflexive graph G belongs to $\mathcal{K}^v$ if and only if each hole of G can be preserved by some path in $\mathcal{K}$.*

Proof of Theorem A. First, since each $P_i \lhd P_{i+1}$ we know that

$$\{P_0\}^v \leq \{P_1\}^v \leq \ldots \leq \{P_i\}^v \leq \{P_{i+1}\}^v \leq \ldots.$$

But the hole $(\{a_0, a_{i+1}\}, \delta(a_0) = 0, \delta(a_{i+1}) = i)$ of P_{i+1} cannot be preserved by P_j for any $j \leq i$, so P_{i+1} cannot be a retract of a direct product of graphs G_i each isomorphic to some P_j, $j \leq i$. Therefore,

$\{P_{i+1}\}^v \nleq \{P_i\}^v$ and it follows that

$$\{P_0\}^v < \{P_1\}^v < \ldots < \{P_i\}^v < \{P_{i+1}\}^v < \ldots.$$

Now, let $\mathcal{V}$ be any variety satisfying

$$\{P_i\}^v < \mathcal{V} \leqq \{P_{i+1}\}^v.$$

Then there is a graph $G \in \mathcal{V}$ and $G \notin \{P_i\}^v$. In the light of Lemma 9, G must have a hole (H, δ) which cannot be preserved by P_i although it can be preserved by P_{i+1}. Let f be an edge-preserving map of $V(G)$ to $V(P_{i+1})$ which preserves the hole (H, δ). Now G is connected so $f(V(G))$ must be a path. If $f(V(G)) \subsetneq P_{i+1}$ then, in effect, P_i preserves this hole. Therefore,

$$f(V(G)) = V(P_{i+1}).$$

From Lemma 8 it now follows that $P_{i+1} \lhd G$. In particular,

$$\{P_{i+1}\}^v \leq \{G\}^v \leq \mathcal{V} \leq \{P_{i+1}\}^v,$$

so $\{P_{i+1}\}^v = \mathcal{V}$. In summary, $\{P_i\}^v$ is covered by $\{P_{i+1}\}^v$; in symbols,

$$\{P_0\}^v \prec \{P_1\}^v \prec \{P_2\}^v \prec \ldots \prec \{P_n\}^v \prec \{P_{n+1}\}^v \prec \ldots.$$

For the proof of Theorem B we shall make use of this fact from [**2**].

LEMMA 10. *Let G be a reflexive graph and let T be an isometric subgraph. If T contains no cycles then $T \lhd G$.*

Proof of Theorem B. We shall first verify the relations

https://doi.org/10.4153/CJM-1986-066-9 Published online by Cambridge University Press

$$\{P_n\}^v < \{C_{2n}\}^v \quad \text{and} \quad \{P_n\}^v < \{C_{2n+1}\}^v.$$

As $P_n \lhd C_{2n}$ and $P_n \lhd C_{2n+1}$,

$$\{P_n\}^v \leq \{C_{2n}\}^v \quad \text{and} \quad \{P_n\}^v \leq \{C_{2n+1}\}^v.$$

Also, the hole $(V(C_{2n}), \delta(c_0) = \delta(c_1) = \ldots = \delta(c_{2n}) = n - 1)$ cannot be preserved by P_n so

$$C_{2n} \notin \{P_n\}^v.$$

Similarly, the hole

$$(V(C_{2n+1}) - \{c_{n-1}, c_{n+1}\}, \delta(c_0) = \delta(c_1) = \ldots = \delta(c_{n-2}) = \delta(c_n)$$

$$= \delta(c_{n+2}) = \ldots = \delta(c_{2n}) = n - 1)$$

cannot be preserved by P_n, so $C_{2n+1} \notin \{P_n\}^v$, too. Therefore,

$$\{P_n\}^v < \{C_{2n}\}^v \quad \text{and} \quad \{P_n\}^v < \{C_{2n+1}\}^v.$$

Let $\mathscr{V}$ be any variety satisfying

$$\{P_n\}^v < \mathscr{V} \leq \{C_{2n}\}^v.$$

Then there is $G \in \mathscr{V}$ such that $G \notin \{P_n\}^v$. According to Lemma 9, G must have a hole (H, δ) which cannot be preserved by P_n. As $G \in \{C_{2n}\}^v$ though, this hole can be preserved by C_{2n}. Let f be an edge-preserving map of $V(G)$ to $V(C_{2n})$ which preserves this hole. Suppose

$$f(V(G)) \subsetneqq V(C_{2n}).$$

Now G is connected so $f(V(G))$ must be a path P_k. If $k \leq n$ then this hole can be preserved by P_n, which is impossible. Otherwise, $k > n$. This implies that G contains vertices

$$a \in f^{-1}(\{a_0\}), \quad b \in f^{-1}(\{a_k\})$$

(where a_0, a_k are the endpoints of P_k) satisfying $d_G(a, b) = k$. Then G itself contains an isometric path P_k of length k. According to Lemma 10, $P_k \lhd G$, so $P_k \in \{C_{2n}\}^v$, which would mean that the hole

$$(\{a_0, a_k\}, \delta(a_0) = 0, \delta(a_k) = k - 1)$$

can be preserved by C_{2n}. This is impossible since $k > n$. We conclude that $f(V(G)) = V(C_{2n})$, that is, f is onto. From Lemma 8, we have that $C_{2n} \lhd G$, so

$$\{C_{2n}\}^v \leq \{G\}^v \leq \mathscr{V} \leq \{C_{2n}\}^v$$

and then $\{C_{2n}\}^v = \mathscr{V}$. A similar argument shows that $\{P_n\}^v \prec \{C_{2n+1}\}^v$, too.

To show that $\{C_n\}^v$ is noncomparable with $\{C_m\}^v$ for each $n \neq m$ $n, m \geq 4$ we consider the usual hole in the cycle depending on its parity. For instance, if $n = 2r$ and $m = 2s + 1$ let (H_r, δ_r) be the hole of C_{2r} with $H_r = V(C_{2r})$ and $\delta(c_i) = r - 1$ for each $i = 0, 1, 2, \dots, 2r - 1$, and let (H_s, δ_s) be the hole of C_{2s+1} with $H_s = V(C_{2s+1})$ and $\delta(c_i) = s - 1$ except if $i = s - 1$ and $i = s + 1$. Then it is straightforward to verify that neither can the hole (H_r, δ_r) be preserved by C_{2s+1} nor can the hole (H_s, δ_s) be preserved by C_{2r}.

Proof of Theorem C. In view of Theorem A and Theorem B it remains to prove that the varieties $\{D_6\}^v$, $\{J_n\}^v$, $\{L_n\}^v$ $(n \geq 3)$, and $\{M_\omega\}^v$, are all distinct and that each covers $\{P_2\}^v$. We treat first the cases of $\{J_n\}^v$ and $\{L_n\}^v$, for $n \geq 3$.

As $P_2 \lhd J_n$ it follows that $\{P_2\}^v \leq \{J_n\}^v$. But the hole

$$(\{b_1, b_2, \dots, b_n\}, \delta(b_1) = \delta(b_2) = \dots = \delta(b_n) = 1)$$

cannot be preserved by P_2, so from Lemma 9, $\{P_2\}^v \leq \{J_n\}^v$. Let $\mathscr{V}$ be any variety satisfying

$$\{P_2\}^v < \mathscr{V} \leq \{J_n\}^v.$$

Then there is $G \in \mathscr{V}$ such that $G \notin \{P_2\}^v$. There must be a hole (H, δ) of G which cannot be preserved by P_2. As $G \in \{J_n\}^v$ this hole can be preserved by J_n. Let f be an edge-preserving map of $V(G)$ to $V(J_n)$ which preserves this hole. Suppose some $b_i \in f(H)$. Then $a_i \in V(J_n)$ is adjacent to each vertex of $f(H)$. Therefore, this hole (H, δ) of G must contain some $v_0 \in H$ with $\delta(v) = 0$. From Proposition 4, $|H| = 2$, say $H = \{v_0, v_1\}$. If $\delta(v_1) \geq 2$ then there is an isometric path joining v_0 and v_1 of length $\delta(v_1) + 1 \geq 3$. According to Lemma 10, $P_3 \lhd G$, so $P_3 \in \{J_n\}^v$ which is impossible since the hole

$$(\{a_0, a_3\}, \delta(a_0) = 0, \delta(a_3) = 2)$$

cannot be preserved by J_n. Therefore, $\delta(v_1) \leq 1$ and, in any event, this constitutes a hole which can be preserved by P_2. We may therefore suppose that each $b_i \in f(H)$. Recall that

$$(\{b_1, b_2, \dots, b_n\}, \delta(b_1) = \delta(b_2) = \dots = \delta(b_n) = 1)$$

is a hole of J_n and that, from Lemma 5, $(f(H), \delta_f)$ is a hole of J_n. Now $n \geq 3$ so $|H| \geq 3$ and each $\delta(v) > 0$ for $v \in H$ (Proposition 4). It follows that $\delta(v) = 1$ for each $v \in H$, $|H| = n$. Suppose

$$H = \{v_1, v_2, \dots, v_n\} \quad \text{with } f(v_i) = b_i.$$

Suppose now that some $a_i \notin f(V(G))$. Then

$$f^{-1}(\{a_i\}) = \emptyset$$

and $(H - \{v_i\}, \delta|H - \{v_i\})$ is also a hole of G which is impossible by the minimality of H. We conclude that f must be onto. According to Lemma 8, $J_n \lhd G$ so

$$\{J_n\}^v \leq \{G\}^v \leq \mathscr{V} \leq \{J_n\}^v$$

https://doi.org/10.4153/CJM-1986-066-9 Published online by Cambridge University Press

and then $\mathscr{V} = \{J_n\}^v$. Now for distinct $n, m \geq 3$, $\{J_n\}^v$ is noncomparable to $\{J_m\}^v$. This is because the hole

$$(\{b_1, b_2, \ldots, b_m\}, \delta(b_1) = \delta(b_2) = \ldots = \delta(b_m) = 1)$$

of J_m cannot be preserved by J_n.

The analysis of the varieties $\{L_n\}^v$ is much the same. Furthermore, the varieties $\{J_n\}^v$ are all distinct from the varieties $\{L_n\}^v$ because as usual the L_n's cannot separate the holes of the J_m's and vice versa.

We turn next to the variety $\{D_6\}^v$. Obviously $P_2 \lhd D_6$ so $\{P_2\}^v \leq \{D_6\}^v$. On the other hand, $\{P_2\}^v < \{D_6\}^v$ since D_6 has a hole which cannot be preserved by P_2. Let $\mathscr{V}$ be any variety satisfying

$$\{P_2\}^v < \mathscr{V} \leq \{D_6\}^v.$$

Let $G \in \mathscr{V}$ such that $G \notin \{P_2\}^v$. Then there is a hole (H, δ) of G which cannot be preserved by P_2 but there is an edge-preserving map f of $V(G)$ to $V(D_6)$ which does preserve this hole. If $\{a, d, r, s\} \subsetneq f(H)$ then as D_6 contains a vertex adjacent to all vertices of $f(H)$ it follows that there is $v_0 \in H$ satisfying $\delta(v_0) = 0$. By Proposition 4, $|H| = 2$. Let $H = \{v_0, v_1\}$. If $\delta(v_1) \leq 1$ then (H, δ) can be separated by P_2. Otherwise $\delta(v_1) \geq 2$ and, from Lemma 10, $P_3 \lhd G$. Hence, $P_3 \in \{D_6\}^v$ which is impossible. We conclude that $\{a, d, r, s\} \subseteq f(H)$ and, from Proposition 4, $\delta(v) > 0$ for all $v \in H$ since $|H| \geq 3$. This in turn, means that $\{a, d, r, s\} = f(H)$. Furthermore, for $(\{a, d, r, s\}, \delta^f)$ to be a hole of D_6 (see Lemma 5) $\delta^f(u) = 1$ for each $u \in \{a, d, r, s\}$, so $\delta(v) = 1$ for each $v \in H$, too. Let

$$A = f^{-1}(\{a\}), B = f^{-1}(\{b\}),$$
$$C = f^{-1}(\{c\}), D = f^{-1}(\{d\}),$$
$$R = f^{-1}(\{r\}) \text{ and } S = f^{-1}(\{s\}).$$

We know that $A \neq \emptyset$, $D \neq \emptyset$, $R \neq \emptyset$ and $S \neq \emptyset$.

To proceed we make use of the fact that $G \in \{D_6\}^v$ implies

$$G \lhd \prod_{i \in I} G_i$$

where each $G_i \cong D_6$, $i \in I$. Let I_0 stand for all those i for which the ith projection π_i preserves the hole (H, δ) in G_i; let $I_1 = I - I_0$. Now, let

$$H = \{x_1, x_2, x_3, x_4\}$$

where $f(x_1) = a, f(x_2) = d, f(x_3) = r, f(x_4) = s$, say, and we may suppose that G is a subgraph of $\prod_{i \in I} G_i$. Now, for each $i \in I_0$,

$$\{\pi_i(x_1), \pi_i(x_2), \pi_i(x_3), \pi_i(x_4)\} = \{a, d, r, s\}$$

and there are eight cases in all. Otherwise, for each $i \in I_1$,

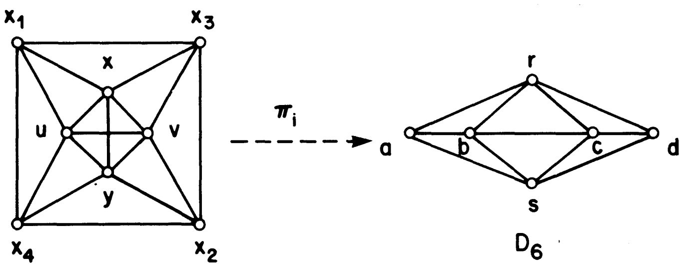
$$\{\pi_i(x_1), \pi_i(x_2), \pi_i(x_3), \pi_i(x_4)\}$$

https://doi.org/10.4153/CJM-1986-066-9 Published online by Cambridge University Press

must miss at least one of the values a, d, r, s. The possibilities are tabulated in Table 2. Our immediate aim is to construct four vertices u, v, x, y in $\prod_{i \in I} G_i$ with adjacencies as illustrated in Figure 25. We do this by prescribing the projections case by case (see Table 3). Then we can verify that the adjacencies as illustrated in Figure 25 are obtained.

TABLE 2

		$\pi_i(x_1)$	$\pi_i(x_2)$	$\pi_i(x_3)$	$\pi_i(x_4)$
$(i \in I_0)$	1	d	a	s	r
	2	d	a	r	s
	3	a	d	s	r
	4	a	d	r	s
	5	s	r	a	d
	6	s	r	d	a
	7	r	s	a	d
	8	r	s	d	a
$(i \in I_1$	9	$\neq a$	$\neq a$	$\neq a$	$\neq a$
	10	$\neq d$	$\neq d$	$\neq d$	$\neq d$
	11	$\neq s$	$\neq s$	$\neq s$	$\neq s$
	12	$\neq r$	$\neq r$	$\neq r$	$\neq r$



A subgraph of $G \lhd \prod_{i \in I} G_i$

Figure 14

TABLE 3

		$\pi_i(u)$	$\pi_i(v)$	$\pi_i(x)$	$\pi_i(y)$	$\pi_i(z)$	$\pi_i(w)$
$i \in I_0$	1	c	b	c	b	s	—
	2	c	b	c	b	s	—
	3	b	c	b	c	r	—
	4	b	c	b	c	r	—
	5	c	b	b	c	b	s
	6	b	c	c	b	b	s
	7	c	b	b	c	c	s
	8	b	c	c	b	c	s
$i \in I_1$	9	c	c	c	c	c	c
	10	b	b	b	b	b	b
	11	r	r	r	r	r	1
	12	s	s	s	s	s	0

The vertices x_1, x_2, x_3, x_4 belong to G in $\prod_{i \in I} G_i$. What about the newly manufactured vertices u, v, x, y? In fact, if g is the retraction of $\prod_{i \in I} G_i$ to G then by analysing the possible images for u, v, x, y under g we conclude that

$$\{g(u), g(v), g(x), g(y)\} \cap \{x_1, x_2, x_3, x_4\} = \emptyset$$

and the vertices $g(u), g(v), g(x), g(y)$ are all distinct in G. For simplicity we shall in the sequel suppose $g(u) = u$, $g(v) = v$, $g(x) = x$, $g(y) = y$.

Let $i \in I_0$. Suppose

$$\{\pi_i(x_1), \pi_i(x_2)\} = \{a, d\} \quad \text{and} \quad \{\pi_i(x_3), \pi_i(x_4)\} = \{r, s\}.$$

These are the cases 1, 2, 3, 4. Construct a vertex

$$z \in V\left(\prod_{i \in I} G_i\right)$$

as in Table 3. Then z is adjacent to x_1, x_2, u, x, y, z in $\prod_{i \in I} G_i$ and, in particular,

$$g(z) \in \{g(x_1), g(x_2), g(x_3), g(x_4), g(u), g(v), g(x), g(y)\}.$$

Let us simply write $g(z) = z$. By symmetry we may treat just the first of these four cases (see Figure 15). Then the elements

$$x_1 \in \pi_i^{-1}(\{d\}), u \in \pi_i^{-1}(\{c\}), y \in \pi_i^{-1}(\{b\}), x_2 \in \pi_i^{-1}(\{a\}),$$
$$z \in \pi_i^{-1}(\{s\}) \text{ and } x_4 \in \pi_i^{-1}(\{r\})$$

is a system of representatives of the blocks of π_i^{-1} and forms a subgraph of G isomorphic to D_6; in particular $D_6 \lhd G$ so in this case $\mathscr{V} = \{D_6\}^{\nu}$.

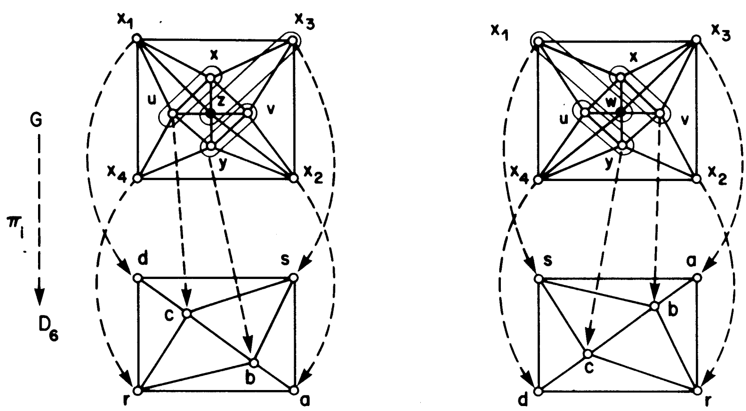


Figure 15

Hence we may suppose that if $i \in I_0$ then it must occur as case 5, 6, 7 or 8. Now, consider the vertex

https://doi.org/10.4153/CJM-1986-066-9 Published online by Cambridge University Press

$$w \in V\left(\prod_{i \in I} G_i\right)$$

prescribed as in Table 3. Then w is adjacent to x_3, x_4, x, u, v, y and again

$$g(w) \notin \{g(x_1), g(x_2), g(x_3), g(x_4), g(x), g(u), g(v), g(y)\}$$

and again let us simply suppose that $g(w) = w$. By symmetry we may treat just the case 5 (see Figure 23). This time $\{x_3, v, y, x_4, x_2, w\}$ is a system of representatives of the blocks of π_i^{-1} and it is isomorphic to D_6. Therefore, $D_6 \lhd G$ and again $\mathscr{V} = \{D_6\}^v$.

Finally, $\{D_6\}^v$ is different from the other covers of $\{P_2\}^v$ recorded earlier. This follows by examining the possible preservation of holes that equality would entail. The arguments are similar to these recorded earlier.

Finally, we turn to the matter of the variety $\{M_\omega\}^v$. It is evident that, at any rate $\{P_2\}^v < \{M_\omega\}^v$. Suppose that $G \in \{M_\omega\}^v$ and that G has a minimal hole (H, δ) which is preserved by M_ω but not by P_2. Let f be an edge-preserving map of $V(G)$ to $V(M_\omega)$ which preserves this hole. Every "finite" hole of M_ω is preserved by P_2 and this implies that (H, δ) must be an "infinite" hole of G. Let

$$A_i = f^{-1}(\{a_i\}) \quad \text{and} \quad B_i = f^{-1}(\{b_i\}).$$

Then H must intersect infinitely many of the blocks $B_1, B_2, \ldots$, say, $B_{\sigma(1)}$, $B_{\sigma(2)}, \ldots$ where $\sigma(1) < \sigma(2) < \ldots$. Let $b'_1 \in B_{\sigma(1)}$ and let a'_1 be chosen from $\cup_i A_i$ (it exists) such that a'_1 is joined to b'_1 by an edge. Let b'_2 be chosen from the first $B_{\sigma(i)}$ such that b'_2 is not adjacent to a'_1. Then choose $a'_2 \in \cup_i A_i$ such that a'_2 is adjacent to b'_1 and to b'_2. In fact,

$$a'_2 \in D_G(b'_1, 1) \cap D_G(b'_2, 1)$$

which is nonempty because (H, δ) is a minimal "infinite" hole of G. Let b'_2 be so chosen that the first $B_{\sigma(i)}$ such that $a'_2 \in D(b'_2, 1)$. Continuing in this way we produce vertices $b'_1, b'_2, \ldots$, and $a'_1, a'_2, \ldots$ which all together form a subgraph of G isomorphic to M_ω. Now it is easy to check that $M_\omega \lhd G$ and that means that $\{G\}^v \geq \{M_\omega\}^v$ and so $\{M_\omega\}^v \succ \{P_2\}^v$, which completes the proof.

Remarks. The problem of finding all of the reflexive graph varieties which cover $\{P_2\}^v$ remains unresolved. We have said above that the technique launched by Lemma 8 lies at the heart of Theorems A, B and C. How far can this technique be exploited to settle this problem? We shall present an example below whose point is this: either further scrutiny of the example itself will indicate how to exploit the technique of Lemma 8 or the example marks a limitation on the usefulness of this technique. In either case some fresh insights are needed.

The remainder of this article is concerned to prove this fact: there is a reflexive graph G for which there is an edge-preserving map of $V(G)$ onto $V(C_4)$ yet none of the reflexive graphs A_2, P_3, C_4, C_5, J_n, L_n $(n \geq 3)$, D_6 and M_ω is a retract of G.

The graph G has as its vertices the integers $\mathbf{Z}$. Two integers x, y are adjacent in G just if one of these conditions holds:

$$x - y \equiv 0(4) \quad \text{or} \quad |x - y| = 1 \quad \text{or} \quad x = y + 3 + 4k$$

for some positive integer k. This graph is illustrated schematically in Figure 16. This graph is fairly symmetric; note that the map φ_k of $V(G)$ onto $V(G)$ defined by $\varphi_k(x) = x + k$ is actually an isomorphism. Also notice that there is an edge-preserving map of $V(G)$ onto $V(C_4)$; namely, $f(x) = a_i$ for each

$$x \in A_i = \{i + 4s | s \in \mathbf{Z}\} \quad i = 0, 1, 2, 3.$$

We aim now to establish the important properties of this graph by examining its minimal holes. Let (H, δ) be a minimal finite hole of G. Let us suppose though that $|H| \geqq 4$ and that $H \cap A_i \neq \emptyset$ for each $i = 0, 1, 2, 3$.

Let $v_i \in H \cap A_i$. On account of the minimality of the set H there

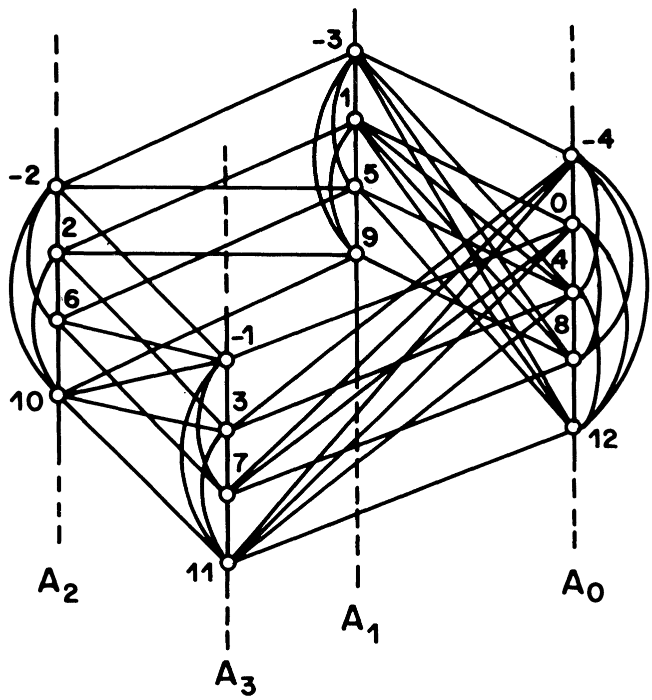


Figure 16

https://doi.org/10.4153/CJM-1986-066-9 Published online by Cambridge University Press

are vertices

$$u_0 \in \bigcup_{v \in H - \{v_0\}} D_G(v, \delta(v)) \quad \text{and} \quad u_2 \in \bigcup_{v \in H - \{v_2\}} D_G(v, \delta(v)).$$

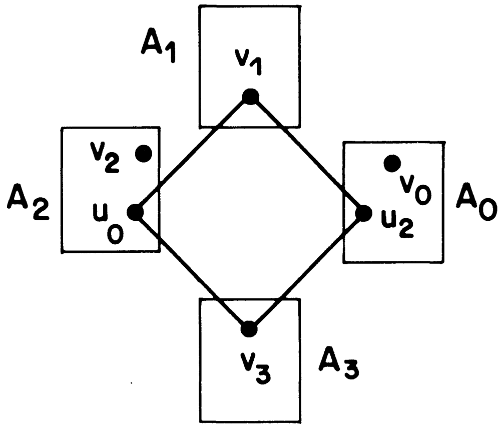


Figure 17

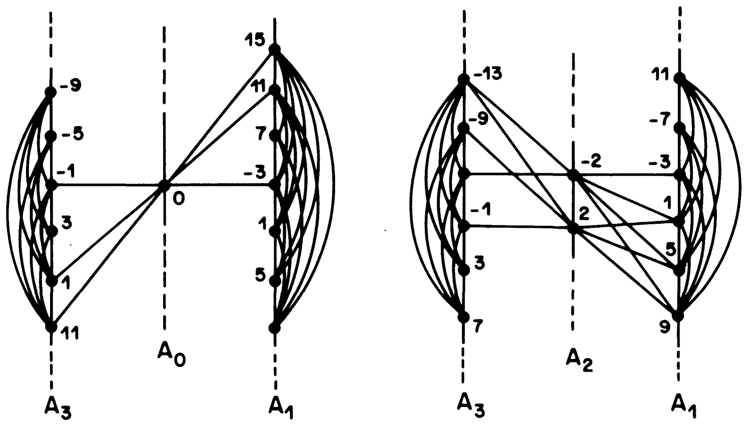


Figure 18

Then $u_0 \in A_2$, $u_2 \in A_0$, and:

$$d_G(v_1, u_0) = d_G(v_3, u_0) = d_G(v_1, u_2) = d_G(v_3, u_2) = 1.$$

By applying the isomorphism we can suppose that $u_2 = 0$. Then

$$v_1 \in E = \{1, -7, -11, -15, \dots\} \quad \text{and}$$

$$v_3 \in F = \{-1, 7, 11, 15, \dots\} = -E.$$

Now, $u_0 \neq -2$ since $d_G(u_0, x) = 2$ for each $x \in E$, and also $u_0 \neq 2$ since

$d_G(u_0, x) = 2$ for each $x \in F$. Applying the isomorphism φ_{4k}, again, we can conclude that $u_0 \in A_2$.

It follows that if $|H| \geq 4$ one of the blocks A_i does not meet H at all. Suppose that $H \cap A_3 = \emptyset$. Then $H \cap A_2 \neq \emptyset$ for otherwise, for large enough $|x|$ the vertex $x < 0$ satisfies $d_G(x, v) = 1$ for all $v \in H$. Similarly, $H \cap A_0 \neq \emptyset$.

Suppose that $H \cap A_1 \neq \emptyset$. Then according to the minimality of H, $|H \cap A_1| = 1$, say $H \cap A_1 = \{v_1\}$. It now follows that

$$|H \cap A_0| = 1 = |H \cap A_1|,$$

too, say,

$$H \cap A_0 = \{v_0\} \quad \text{and} \quad H \cap A_2 = \{v_2\}.$$

In summary, if (H, δ) is a minimal hole of G such that $|H| \geq 4$, then the vertices of H are situated in three "consecutive" blocks in such a way that if the "middle" block is nonempty then $|H| \leq 3$, each block containing a vertex.

It is now technically straightforward to verify that none of the graphs A_2, P_3, C_5, J_n, L_n $(n \geq 3)$, D_6 and M_ω is a retract of G.

References

1. D. Duffus and I. Rival, *A structure theory for ordered sets*, Discrete Math. *35* (1981), 53-118.
2. R. J. Nowakowski and I. Rival, *A fixed edge theorem for graphs with loops*, J. Graph Theory *3* (1979), 339-350.
3. ——— *The smallest graph variety containing all paths*, Discrete Math. *43* (1983), 223-234.

Université Claude-Bernard,
Villeurbanne, France;
University of Ottawa,
Ottawa, Ontario