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Alternating groups and rational functions on surfaces

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Abstract

Let X be a smooth complex projective surface and let $\mathbf{C}(X)$ denote the field of rational functions on X. In this paper, we prove that for any $m > M(X)$, there exists a rational dominant map $f\colon X \to Y$, which is generically finite of degree m, into a complex rational ruled surface Y, whose monodromy is the alternating group A_m. This gives a finite algebraic extension $\mathbf{C}(X)\colon \mathbf{C}(x_1, x_2)$ of degree m, whose normal closure has Galois group A_m.

1. Introduction

Let F be an extension field of L, we denote by $G(F\colon L)$ the Galois group of the extension $F\colon L$, which consists of all automorphisms of the field F which fix L elementwise. If $F\colon L$ is finite and separable, its normal closure $N\colon L$ is a Galois extension, see [Gar86]. Set $M(F, L) = G(N\colon L)$. Let X be an irreducible complex algebraic variety, we can associate to it the field $\mathbf{C}(X)$ of rational functions on X. This gives a one-to-one correspondence between birational classes of irreducible complex algebraic varieties and finitely generated extensions of $\mathbf{C}$. Let X and Y be irreducible complex algebraic varieties of the same dimension n. Let $f\colon X \to Y$ be a generically finite dominant morphism of degree d. The field $\mathbf{C}(X)$ is a finite algebraic extension of degree d of the field $\mathbf{C}(Y)$, the group $M(\mathbf{C}(X), \mathbf{C}(Y))$ is called the *Galois group of the morphism* f, see [Har79]. There is an isomorphism between the Galois group of f and the monodromy group $M(f)$, associated to the topological covering induced by f, see § 2.1. Fix an irreducible variety X of dimension n, $\mathbf{C}(X)$ is a finite algebraic extension of $\mathbf{C}(\mathbf{P}^n) = \mathbf{C}(x_1, x_2, \ldots, x_n)$, see [Zar58, Zar60]. The study of possible monodromy groups for X is a classic, algebraic and geometric problem. In general, $M(f)$ is a subgroup of the symmetric group S_d. It is interesting to see in which cases $M(f)$ is a subgroup of the alternating group A_d; if this happens we say that f has *even monodromy*.

If $n = 1$: let X be a compact Riemann surface of genus g. Any non-constant meromorphic function $f \in \mathbf{C}(X)$, of degree d, gives a holomorphic map $f\colon X \to \mathbf{P}^1$, which is a ramified covering of degree d. f is indecomposable if and only if the group $M(f)$ is a primitive subgroup of S_d. There are several results on even monodromy of such maps: first of all by Riemann's existence theorem, for all $g \geqslant 0$ and for all $d \geqslant 2g + 3$, there are Riemann surfaces of genus g admitting maps with monodromy group A_d, see [Fri89]. Actually, for a generic Riemann surface X of genus $g \geqslant 4$, for any indecomposable map the monodromy group is either A_d or the symmetric group S_d, see [GN95] and [GM98]. Finally, a generic compact Riemann surface of genus 1 admits meromorphic functions with monodromy group A_d, for $d \geqslant 4$, see [FKK01]. This result has been recently generalized to any compact Riemann surface X of genus g for $d \geqslant 12g + 4$, see [AP05]. This implies that every extension field $F\colon \mathbf{C}$, with trascendence degree 1, can be realized as a finite algebraic extension of degree d, $F\colon L$, with $L \simeq \mathbf{C}(x)$ and monodromy group $M(F, L) = A_d$.

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In higher dimension there are many various results concerning the monodromy of branched coverings $f\colon X \to Y$ of a variety Y (multiple planes theory, braid groups, Chisini problem, fundamental groups of the complement of a divisor, etc., see [Chi42, Sev46, Cat86, AbS01, Nor83], etc.). On the other hand, not much seems to be known when X is fixed, for instance, the existence of maps f with $M(f)$ solvable is also unknown for projective surfaces of degree $d \geqslant 6$. It is easy to produce, by general linear projections, finite maps $X \to \mathbf{P}^n$ with monodromy the full symmetric group S_d, see [Sev48]. So it is interesting to see whether other primitive groups can be realized as monodromy of X. In this paper, we deal with surfaces and even monodromy groups. Our result is the following.

Theorem 1. *Let F be an extension field of $\mathbf{C}$, with trascendence degree 2. Then there exists an integer $M(F)$ with the following property: for any $m > M(F)$, F admits a subfield $L \simeq \mathbf{C}(x_1, x_2)$ such that $F\colon L$ is a finite algebraic extension of degree m and the group $M(F, L)$ is the alternating group A_m.*

We will deduce Theorem 1 from the following geometric result.

Theorem 2. *Let S be a smooth, complex, projective surface and K_S denote a canonical divisor on S. Let H be a very ample divisor on S, with $H^2 \geqslant 5$ and such that $(S, O_S(H))$ does not contain lines or conics. Set $g = p_{\mathrm{a}}(2H + K_S)$. Then, for any $m > 16g + 7$, there exist a smooth complex projective surface X, in the birational class of S, and a generically finite surjective morphism $f\colon X \to Y$, of degree m, into a smooth complex rational ruled surface Y, such that the monodromy group $M(f)$ is the alternating group A_m.*

Let us describe briefly the method we use in proving this result. Let H be a very ample divisor on S: under our assumptions, which are actually verified by almost all H, we can find a Lefschetz pencil P in the linear system $|2H + K_S|$, whose elements are all irreducible, see § 4.1. By blowing up the base points of P, we produce a smooth, complex, projective surface $\hat{S}$, and a surjective morphism $\phi\colon \hat{S} \to \mathbf{P}^1$, with fibre F_t, see § 4.2. The pull-back of $O_S(H + K_S)$ defines a natural spin bundle L_t on each smooth fibre of ϕ. Following the method of [AP05], for each smooth fibre F_t, we can introduce the variety $\mathcal{H}(F_t, D_t)$, parametrizing a family of meromorphic functions f_t on F_t with even monodromy, related to L_t, see § 4.3. As t varies on $\mathbf{P}^1$, we have a family $p\colon \mathcal{H} \to \mathbf{P}^1$ of projective varieties. Our aim is to glue these meromorphic functions in a suitable way. This can be done by producing a section of p. As for any $t \in \mathbf{P}^1$ the fibre $p^{-1}(t)$ is a normal rationally connected variety, we can apply the following result: *every family of rationally connected varieties over a smooth curve admits a section*. This property, conjectured by Kollár, has been recently proved by Graber, Harris and Starr, and by de Jong and Starr (see [GHS03] and [DS03]). The existence of a section allows us to produce a generically finite surjective morphism $f\colon X \to Y$, where X is birationally equivalent to S, Y is a smooth complex rational ruled surface, such that the restriction $f_{|F_t}$ to a general smooth fibre has monodromy A_m. To conclude our proof, we show that the monodromy of a general smooth fibre completely induces the monodromy of f. For this, we use a topological result of Nori, see § 5.1.

Finally, we apply our result to surfaces of general type with ample canonical divisor, see § 5.3. We conjecture that Theorem 1 holds for any finitely generated extension F of the complex field.

2. Preliminaries

2.1 Monodromy

Let X and Y be irreducible complex algebraic varieties of the same dimension n. Let $f\colon X \to Y$ be a generically finite dominant morphism of degree d. We recall the definition of *monodromy group $M(f)$ of f*, see [Har79]. Let G be the Galois group of the morphism f, see § 1, G acts faithfully on the general fibre $f^{-1}(y)$ and so can be seen as a subgroup of $\mathrm{Aut}(f^{-1}(y)) \simeq S_d$. Let $U \subset Y$

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be an open dense subset such that the restriction $f\colon f^{-1}(U) \to U$ is a covering of degree d in the classical topology (i.e. non-ramified). For any point $y \in U$, let $f^{-1}(y) = \{x_1, \dots, x_d\}$, we have the monodromy representation of the fundamental group $\pi_1(U, y)$:

$$\rho(f, y)\colon \pi_1(U, y) \to \mathrm{Aut}(f^{-1}(y)),$$

sending $[\alpha] \to \sigma(\alpha)$, where $\sigma(\alpha)$ is the automorphism which sends x_i to the end-point of the lift of α at the point x_i. Let $M(f, y) = \rho(f, y)(\pi_1(U, y))$. It is easy to verify that $M(f, y)$ is isomorphic to the Galois group G of f, and so does not depend on the choice of the open subset U. The monodromy group $M(f)$ is defined as the conjugacy class of the transitive subgroups $M(f, y)$.

2.2 Rational connectedness

Let X be a proper complex algebraic variety of dimension n. We recall that X is *rationally connected* if and only if for very general closed points p, $q \in X$ there is an irreducible rational curve $C \subset X$ which contains p and q, see [Kol96].

In the sequel we will need the following.

Proposition 2.2.1. *Let $X \subset \mathbf{P}^N$ be a complex irreducible variety of codimension m which is the complete intersection of m hypersurfaces $Q_1, Q_2, \dots, Q_m$, of degree $d_1, d_2, \dots, d_m$. Let $h = \dim(\mathrm{Sing}(X))$ and $h = -1$ if X is smooth. If*

$$\sum_{i=1}^{m} d_i \; + h + 1 \leqslant N \tag{1}$$

then X is rationally connected. In particular, a complete intersection X of m quadrics with $h \leqslant N - 2m - 1$ is rationally connected.

Proof. If X is smooth, then $\sum_{i=1}^{m} d_i \leqslant N$ implies that X is a Fano variety, hence it is rationally connected (see [Kol96, p. 240]).

So we can assume that $\dim(\mathrm{Sing}(X)) = h \geqslant 0$. Let $H \in (\mathbf{P}^N)^*$ be a general hyperplane: the hyperplane section $Y = X \cap H \subset \mathbf{P}^{N-1}$ is a complete intersection, irreducible and non-degenerate, of m hypersurfaces of $\mathbf{P}^{N-1}$ of degree $d_1, \dots, d_m$. Moreover, $\dim(\mathrm{Sing}(Y)) = h - 1$ and Y satisfies inequality (1). Then it follows, by induction on h, that Y is rationally connected. Finally, for general points p and $q \in X$, there exists a rationally connected hyperplane section Y containing p and q, hence an irreducible rational curve $C \subset X$ connecting the two points. This concludes the proof. ☐

We remark that, in the previous proof, one can intersect X with a general linear space of dimension $N - h - 1$ to get a smooth Fano variety connecting two general points of X.

An important property of rational connectedness is given by the following result, see [GHS03, DS03, KMM92].

Theorem 2.2.2. *Let $p\colon X \to B$ be a proper flat morphism from a complex projective variety into a smooth complex projective curve, assume that p is smooth over an open dense subset U of B. If the general fibre of p is a normal and rationally connected variety, then p has a section. Moreover, for any arbitrary finite set $A \subset U$ and for any section $\sigma_1\colon A \to p^{-1}(A)$, there exists a section $\sigma\colon B \to X$ such that $\sigma_{|A} = \sigma_1$.*

2.3 Notation

Let S be a smooth, complex, connected, projective surface: we denote by O_S the structure sheaf and by K_S a canonical divisor of S, so that $O_S(K_S)$ is the sheaf of the holomorphic 2-forms. Let $q(S) = h^1(S, O_S)$ be the irregularity of S, let $p_g(S) = h^2(S, O_S) = h^0(S, O_S(K_S))$ be the geometric genus of S, finally let $p_n(S) = h^0(S, O_S(nK_S))$, $n \geqslant 1$, be the plurigenera of S. We denote by $k(S)$

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the Kodaira dimension of S. A minimal surface S is said *of general type* if $k(S) = 2$. Let $C \subset S$ be an irreducible curve on S, we denote by $p_a(C) = h^1(C, O_C)$ the arithmetic genus of C, then $p_a(C) = 1 + \frac{1}{2}(C^2 + C \cdot K_S)$. If C is smooth, $p_a(C) = g(C)$ is the geometric genus of C, and $O_S(K_S + C)_{|C} = \omega_C$ the canonical line bundle on C. A g^r_d on a smooth curve C is a linear series (not necessarily complete) on C of degree d and dimension exactly r.

2.4 Very ample line bundles

Let L be a line bundle on S, L is said k-spanned for $k \geqslant 0$ (i.e. it defines a kth order embedding), if for any distinct points $z_1, z_2, \dots, z_t$ on S and any positive integers $k_1, k_2, \dots, k_t$ with $\sum_{i=1}^t k_i = k+1$, the natural map $H^0(S, L) \to H^0(Z, L \otimes O_Z)$ is onto, where (Z, O_Z) is a zero-dimensional subscheme such that at each point z_i: $I_Z O_{S,z_i}$ is generated by $(x_i, y_i^{k_i})$, with (x_i, y_i) local coordinates at z_i on S. Note that $k = 0, 1$ means, respectively, L globally generated, L very ample (see [BFS89]). In the sequel, we will need the following.

Lemma 2.4.1. *Let S be a smooth complex projective surface and K_S be a canonical divisor on it. Let H be a very ample divisor on S, such that $H^2 \geqslant 5$ and $(S, O_S(H))$ does not contain lines and conics. Then we have the following properties:*

(a) *the divisor $2H + K_S$ is also very ample;*

(b) *let $R \subset |2H + K_S|$ be the locus of reducible curves, then R is a closed subset of codimension at least 2;*

(c) *a general pencil $P \subset |2H + K_S|$ has all irreducible elements and the singular curves of P have a unique node as singularities.*

Proof. Note that as H is very ample, to prove property (a), it is enough that $O_S(H + K_S)$ is a line bundle globally generated on S. This is true for any pair $(S, O_S(H))$ which is not a scroll or $(\mathbf{P}^2, O_{\mathbf{P}^2}(i))$, $i = 1, 2$, see [SV87]. Let us examine (b). Let $S^* \subset |2H + K_S|$ be the locus of singular curves, then S^* is an irreducible variety and $\operatorname{codim} S^* \geqslant 1$ (see [Har92]), moreover $R \subset S^*$ (see [Hart77, Corollary III 7.9]). So property (b) means either $\operatorname{codim} S^* \geqslant 2$ or R is a proper closed subset of S^*. Let $p \in S$ be any point, let $\epsilon\colon X \to S$ be the blow-up of S at the point p with exceptional divisor E: assume that there exists a smooth irreducible curve in the linear system $|\epsilon^*(2H + K_S) - 2E|$, this would give us an irreducible curve having a unique node in p in the linear system $|2H + K_S|$, which implies that $R \neq S^*$. For this it is enough to request that $\epsilon^*(2H + K_S) - 2E$ is ample and globally generated, which is of course true if it is very ample. This last property is achieved for every point p, whenever $2H + K_S$ defines a third-order embedding, (i.e. it is 3-spanned), see [BS96, Proposition 3.5]. In particular, if $H^2 \geqslant 5$, $2H + K_S$ is 3-spanned unless there exist an effective divisor F on S such that either $H \cdot F = 1$ and $F^2 = 0, -1, -2$ or $H \cdot F = 2$ and $F^2 = 0$, see [BFS89]. As we assumed that there are no curves embedded by H as lines or conics, this concludes property (b). Actually, we have also proved that a general element of S^* is an irreducible curve with a unique node, which implies property (c). □

Remark. Note that on any surface S we can easily find very ample line bundles satisfying the assumptions of the lemma: for any very ample H, it is enough to take nH with $n \geqslant 3$.

3. Odd ramification coverings of smooth curves

Let X be a smooth, irreducible, complex projective curve of genus g. Let $f \in \mathbf{C}(X)$ be a non-constant meromorphic function on X of degree d, then it defines a holomorphic map $f\colon X \to \mathbf{P}^1$, which is a ramified covering with branch locus $B \subset \mathbf{P}^1$ and ramification divisor $R \subset X$. Let $M(f)$ be the monodromy group of f, see § 2.1. We say that f is an *odd ramification covering* if all

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ramification points of f have odd index. Note that if f is an odd ramification covering, then it has even monodromy, in fact all of the generators of the group $M(f)$ can be decomposed in cycles of odd length.

3.1 Constructing map with even monodromy

We recall the method used in [AP05] to produce odd ramification coverings. A line bundle L on X is said a *spin bundle* if $L^2 = K_X$, where K_X denotes the canonical line bundle on X. Fix three distinct points p_1, p_2, p_3 on X and define the divisor

$$D = n_1 p_1 + n_2 p_2 + n_3 p_3, \quad n_i \in \mathbf{N}, \quad n_1 > n_2 > n_3 \geqslant 0; \tag{2}$$

set $d = \deg D = n_1 + n_2 + n_3$ and denote by $[D]$ the support of D, we have $\deg[D] = k$ with $k = 2$ or 3. Let L be a spin bundle on X: we consider the line bundle $L(D)$. Note that if s is a global section in $H^0(X, L(D))$, then s^2 can be identified with a meromorphic form ω on X having poles at the points of $[D]$. If ω were an exact form, then there would be a non-constant meromorphic function $f \in H^0(X, O_X(2D - [D]))$ on X, such that $\omega = df$. It is easy to verify that $f\colon X \to \mathbf{P}^1$ would be a ramified covering with odd ramification index at every point. Let us define, set-theoretically,

$$\mathcal{A}(X, D) = \{s \in H^0(X, L(D))\colon s^2 \text{ is exact}\}, \tag{3}$$

$$\mathcal{F}(X, D) = \{f \in \mathbf{C}(X)\colon df = s^2,\ s \in \mathcal{A}(X, D)\}. \tag{4}$$

Note that $\mathcal{A}(X, D)$ is actually the zero scheme of the following map:

$$\psi\colon H^0(X, L(D)) \to H^1(X - [D], \mathbf{C}) \tag{5}$$

sending each global section s into the De Rham cohomology class $[s^2]$ of the form $\omega = s^2$. Actually we will consider the projectivization of $\mathcal{A}(X, D)$

$$\mathcal{H}(X, D) = \{(s) \in \mathbf{P}(H^0(X, L(D)))\colon s^2 \text{ is exact}\}. \tag{6}$$

We have the following results.

Proposition 3.1.1. *Let X be a smooth complex projective curve of genus g, let D be a divisor as in (2) with degree d and support of degree k. We assume that: $d > 8g + 3k - 4$ and, moreover, if $k = 2$, then $2n_i > 3g + 2$ for $i = 1, 2$; if $k = 3$, then $2n_i > 3g + 3$, $i = 1, 2, 3$. Then $\mathcal{H}(X, D) \subset \mathbf{P}^{d-1}$ is a complex projective variety with the following properties:*

(i) *$\mathcal{H}(X, D)$ is an irreducible variety of dimension $d - 2g - k$ and its singular locus $\operatorname{Sing}(\mathcal{H}(X, D))$ has dimension $h < 4(g - 1) + k$;*

(ii) *$\mathcal{H}(X, D) \subset \mathbf{P}^{d-1}$ is a complete intersection of $2g + k - 1$ linearly independent quadrics;*

(iii) *$\mathcal{H}(X, D)$ is a normal rationally connected variety.*

Proof. Note that ψ factors through the natural linear map

$$\theta\colon \operatorname{Sym}^2 H^0(X, L(D)) \to H^1(X - [D], \mathbf{C}), \tag{7}$$

defined as $\theta(s \otimes t) = [s \cdot t]$, the De Rham cohomology class of $s \cdot t$. This implies that $\mathcal{H}(X, D)$ is the zero locus of homogeneous polynomials of degree 2. Actually, $\mathcal{H}(X, D)$ can be seen as the zero locus of a global section σ of the following vector bundle of rank $2g + k - 1$ on $\mathbf{P}(H^0(X, L(D))) = \mathbf{P}^{d-1}$:

$$E = H^1(X - [D], \mathbf{C}) \otimes O_{\mathbf{P}^{d-1}}(2), \tag{8}$$

see [Pir98, Proposition 2.1]. Note that the ideal sheaf $\mathcal{I}_{\mathcal{H}(X,D)}$ is the image of the dual map $\sigma^*\colon E^* \to O_{\mathbf{P}^{d-1}}$, hence it is locally generated by $2g + k - 1$ elements. By studying the tangent map we can obtain that, under the above assumptions, actually $\mathcal{H}(X, D)$ is irreducible of dimension $d - 2g - k$ and, moreover, $\dim \operatorname{Sing}(\mathcal{H}(X, D)) = h < 4(g-1) + k$, see [Pir98, Proposition 5.1 and Corollary 5.3].

This also implies that $\mathcal{H}(X,D)$ is a complete intersection of $2g+k-1$ quadrics and concludes the proofs of properties (i) and (ii). The variety $\mathcal{H}(X,D)$ is normal as from property (i) it is regular in codimension 1 (see [Hart77, p. 186]). Finally, as $\mathcal{H}(X,D) \subset \mathbf{P}^{d-1}$ is an irreducible complete intersection of $2g+k-1$ quadrics, by Proposition 2.2.1, it is rationally connected if we have

$$h \leqslant d - 4g - 2k,$$

this immediately follows from property (i), as we assumed $d > 8g + 3k - 4$. □

Let $(s) \in \mathcal{H}(X,D)$: it defines a unique linear series $g^1_m(s)$ on X as follows:

$$g^1_m(s) = \{\lambda f + \mu \ = 0\}_{(\lambda,\mu)\in\mathbf{P}^1},$$

where $f \in \mathbf{C}(X)$ and $df = s^2$. We have the following result.

Proposition 3.1.2. *Let X be a smooth complex projective curve of genus g, let D be a divisor with degree d and support of degree k as in (2). Assume that: $d > 6g+2k-3$, if $k = 2$, then $2n_i > 3g+3$ for $i = 1,2$, if $k = 3$, then $2n_i > 3g+4$ for $i = 1,2,3$, moreover, the triple $(2n_1-1, 2n_2-1, 2n_3-1)$ is given by relatively prime integers. Then for general $(s) \in \mathcal{H}(X,D)$ the linear series $g^1_m(s)$ is base-points free and defines an indecomposable finite morphism $F\colon X \to \mathbf{P}^1$ of degree $m = 2d-k$ with monodromy $M(F) = A_m$.*

For the proof see [AP05, Proposition 3 and Theorem 1].

4. Main constructions

In this section we introduce some basic constructions that, we will need in proving our main theorem.

4.1 Lefschetz pencil

Let S be a smooth complex projective surface and let K_S be a canonical divisor on S. Let H be a very ample divisor on S such that $H^2 \geqslant 5$ and $(S, O_S(H))$ does not contain lines or conics. Set $g = p_{\mathrm{a}}(2H+K_S)$ and $N = (2H+K_S)^2 \geqslant H^2 \geqslant 5$. By Lemma 2.4.1(c), we can choose a general pencil $P = \{C_t\}_{t\in\mathbf{P}^1}$ in the linear system $|2H+K_S|$, with the following properties:

- (i) every curve in P is irreducible;
- (ii) the generic curve in P is a smooth, irreducible, complex projective curve of genus g;
- (iii) there are at most finitely many singular curves in P and they have a unique node as singularities;
- (iv) every pair of curves C_t and $C_{t'}$ of P intersect transversally, so that P has N distinct base points, $p_1,\dots,p_N$.

We will call P *a Lefschetz pencil of irreducible curves on S of genus g*. Starting from these data (S,H,P) we will introduce the following constructions.

4.2 Construction 1

Let $\hat{S}$ be the smooth complex projective surface obtained by blowing up the base points of the pencil P:

$$\hat{S} = B_{p_1,p_2,\dots,p_N}(S). \tag{9}$$

Let us denote by $\epsilon\colon \hat{S} \to S$ the blow up map, by $E_1,\dots,E_N$ the exceptional curves, such that $E_i^2 = -1$ and $E_i \cdot E_j = 0$, for $i \neq j$, then $K_{\hat{S}} = \epsilon^* K_S + E_1 + \cdots + E_N$. Note that the strict transforms of the curves of the pencil P satisfies $\tilde{C}_t \cdot \tilde{C}_{t'} = 0$, for any $t \neq t'$. Hence, the pencil P

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induces a surjective morphism

$$\phi\colon \hat{S} \to \mathbf{P}^1, \tag{10}$$

with fibre $F_t = \tilde{C}_t$, for any $t \in \mathbf{P}^1$, $C_t \in P$. Moreover, ϕ is actually a flat morphism and the exceptional curves $E_1, \ldots, E_N$ in $\hat{S}$ turn out to be sections of the morphism ϕ. We will define on $\hat{S}$ the line bundle

$$L = \epsilon^* O_S(H + K_S). \tag{11}$$

Note that if F_t is any singular fibre of ϕ, then its dualizing sheaf ω_{F_t} is a line bundle, as we have $\omega_{F_t} = (\omega_{\hat{S}} + F_t)_{|F_t}$, as for smooth fibres. It is easy to verify that for any fibre F_t we have

$$L^2_{|F_t} \simeq \omega_{F_t}, \tag{12}$$

so we say that L is *a spin bundle relatively to* ϕ. We denote by $U \subset \mathbf{P}^1$ the open subset corresponding to smooth fibres of ϕ, set $\hat{S}_U = \phi^{-1}(U)$, then $\phi\colon \hat{S}_U \to U$ is a smooth morphism. We have proved the following.

Claim 1. The smooth complex projective surface $\hat{S}$ is endowed with a surjective morphism $\phi\colon \hat{S} \to \mathbf{P}^1$, with smooth fibre F_t of genus g, and a line bundle L which is a spin bundle relatively to ϕ.

4.3 Construction 2

Now let us choose three distinct exceptional curves E_1, E_2, E_3 on the surface $\hat{S}$, and fix integers $n_1 > n_2 > n_3 \geqslant 0$: we will consider on $\hat{S}$ the line bundle

$$L(n_1E_1 + n_2E_2 + n_3E_3). \tag{13}$$

As each E_i is a section of the morphism $\phi\colon \hat{S} \to \mathbf{P}^1$, for any fibre F_t of ϕ we have

$$L(n_1E_1 + n_2E_2 + n_3E_3)_{|F_t} = L_t(D_t), \tag{14}$$

where $D_t = n_1p_1^t + n_2p_2^t + n_3p_3^t$, with $p_i^t = E_{i|F_t}$ and $L_t = L_{|F_t}$ is a spin bundle on F_t. Set $d = \deg(D_t) = n_1 + n_2 + n_3$ and $[D_t] =$ the support of D_t, with $\deg[D_t] = k$, $k = 2$ or 3. We assume that: $d > 8g + 3k - 4$, if $k = 2$, then $2n_i > 3g + 3$ for $i = 1, 2$; if $k = 3$, then $2n_i > 3g + 4$ for $i = 1, 2, 3$, finally $(2n_1 - 1, 2n_2 - 1, 2n_3 - 1)$ are relatively prime integers. For such (d, k), for any smooth fibre F_t, by Proposition 3.1.1, we can introduce the irreducible projective variety:

$$\mathcal{H}(F_t, D_t) \subset \mathbf{P}(H^0(F_t, L_t(D_t))) = \mathbf{P}_t^{d-1}. \tag{15}$$

Claim 2. There exists a complex projective variety $\mathcal{H}$ and a surjective morphism $p\colon \mathcal{H} \to \mathbf{P}^1$, with the following property: let $U \subset \mathbf{P}^1$ be the open subset corresponding to smooth fibres F_t of ϕ, for any $t \in U$, the fibre $p^{-1}(t)$ is the projective variety $\mathcal{H}(F_t, D_t)$.

Let us consider on $\hat{S}$ the line bundle $L(n_1E_1 + n_2E_2 + n_3E_3)$ and look at its restriction $L_t(D_t)$ to any fibre F_t. As for all t, F_t is irreducible and lies on a smooth surface, then $\deg(L_t(D_t)) > 2p_{\mathrm{a}} - 2$, implies that $h^1(F_t, L_t(D_t)) = 0$, see [CF96], so we can apply Riemann Roch theorem and obtain $h^0(F_t, L_t(D_t)) = d$. As $\phi\colon \hat{S} \to \mathbf{P}^1$ is a flat morphism, by Grauert 's theorem (see [Hart77, p. 288]), the sheaf

$$\mathcal{F} = \phi_*(L(n_1E_1 + n_2E_2 + n_3E_3)) \tag{16}$$

is a locally free sheaf of rank d on $\mathbf{P}^1$. So we can introduce the associated projective space bundle $\mathbf{P}(\mathcal{F})$ and the following smooth morphism

$$\pi\colon \mathbf{P}(\mathcal{F}) \to \mathbf{P}^1, \tag{17}$$

whose fibre at t is the projective space $\mathbf{P}(H^0(F_t, L_t(D_t))) = \mathbf{P}_t^{d-1}$. Let $O_{\mathcal{F}}(1)$ be the tautological line bundle on $\mathbf{P}(\mathcal{F})$, i.e. $O_{\mathcal{F}}(1)_{|\mathbf{P}(\mathcal{F}_t)} = O_{\mathbf{P}_t^{d-1}}(1)$. Let $U \subset \mathbf{P}^1$ be the open subset where ϕ is smooth and $\hat{S}_U = \phi^{-1}(U)$. Set $W = \hat{S}_U - \{E_1, E_2, E_3\}$, we can consider the restriction

$$\bar{\phi} = \phi_{|W} \colon W \to U, \tag{18}$$

with fibre $\bar{\phi}^{-1}(t) = F_t - [D_t]$, for any $t \in U$. As for any smooth fibre F_t, we have $h^1(F_t - [D_t], \mathbf{C}) = 2g + k - 1$, the sheaf $R^1\bar{\phi}_*(\mathbf{C})$ is actually a vector bundle on U with fibre $H^1(F_t - [D_t], \mathbf{C})$, set

$$\mathcal{G}_U = R^1\bar{\phi}_*(\mathbf{C}). \tag{19}$$

Let $\mathrm{Sym}^2 \mathcal{F}$ be the 2 symmetric power of $\mathcal{F}$, we have the following natural maps:

$$\alpha \colon O_{\mathcal{F}}(-2)_{|U} \to \mathrm{Sym}^2 \mathcal{F}_{|U} \tag{20}$$

$$\Theta \colon \mathrm{Sym}^2 \mathcal{F}_{|U} \to \mathcal{G}_U, \tag{21}$$

see the proof of Proposition 3.1.1. By composition we obtain a non-zero global section τ of the vector bundle $\mathcal{G}_U \otimes O_{\mathcal{F}}(2)_{|U}$. We define the projective variety

$$\mathcal{H}_U \subset \mathbf{P}(\mathcal{F})_{|U}, \tag{22}$$

as the zero locus of the section τ. It admits a natural surjective morphism $p_U \colon \mathcal{H}_U \to U$, whose fibre at t is actually the projective variety $\mathcal{H}(F_t, D_t)$. It' s easy to verify that p_U turns out to be a proper flat morphism. Finally, let $\mathcal{H}$ be the scheme-theoretic closure of $\mathcal{H}_U$ into the projective variety $\mathbf{P}(\mathcal{F})$, then $\mathcal{H}$ is a complex projective variety, moreover, as $U = \mathbf{P}^1 - \{t_1, \ldots, t_Q\}$, then there exists a flat morphism $p \colon \mathcal{H} \to \mathbf{P}^1$, which extends p_U (see [Hart77, p. 258]).

Claim 3. The variety $\mathcal{H}$ admits a section σ.

Look at the surjective flat morphism $p \colon \mathcal{H} \to \mathbf{P}^1$: for any $t \in U$, the fibre $p^{-1}(t) = \mathcal{H}(F_t, D_t)$ is a normal rationally connected variety, see Proposition 3.1.1. This allows us to apply Theorem 2.2.2 to p and to conclude that p has a section, let us denote it by σ,

$$\sigma \colon \mathbf{P}^1 \to \mathcal{H}, \tag{23}$$

with the following property: for general $t \in U$, the linear series $g^1_m(t)$, defined by $\sigma(t)$, on the smooth fibre F_t, is base points free of degree $2d - k$. So, by Proposition 3.1.2, the associated map $F_t \to \mathbf{P^1}$ is indecomposable with monodromy group A_{2d-k}. Note that under the assumptions made in § 4.3, m is even and $m > 16g + 2$ if $k = 2$, while m is odd and $m > 16g + 7$ if $k = 3$.

4.4 Construction 3

There exists a smooth, complex, rational ruled surface Y and a finite rational map of degree $m = 2d - k$, $\delta \colon \hat{S} \to Y$ with the following property: for a general smooth fibre F_t the restriction $\delta_{|F_t}$ is given by the linear series $g^1_m(t)$ on F_t and the following diagram commutes:

$$\begin{array}{ccc} \hat{S} & \xrightarrow{\delta} & Y \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \xrightarrow{\mathrm{id}} & \mathbf{P}^1 \end{array}$$

where the vertical arrows are respectively the morphism ϕ and the ruling π of Y.

As $\phi \colon \hat{S}_U \to U$ is a smooth morphism, we can consider the quasi projective variety $\hat{S}^{(m)}_U$ parametrizing the symmetric products $F^{(m)}_t$ of the smooth fibres F_t of ϕ. There is natural map induced by ϕ, which is a smooth morphism

$$\phi_m \colon \hat{S}^{(m)}_U \to U, \tag{24}$$

with smooth fibre $F_t^{(m)}$. The existence of σ, allows us to define a quasi projective variety $\mathcal{I}$ as follows:

$$\mathcal{I} = \{(A, t) \in \hat{S}_U^{(m)} \times U \colon A \in g_m^1(t)\}. \tag{25}$$

Let $\pi_1 \colon \mathcal{I} \to U$ the natural projection, then ${\pi_1}^{-1}(t) \simeq \mathbf{P}^1$ is the linear series $g_m^1(t)$. So $\mathcal{I}$ is a quasi-projective surface endowed with a rational ruling π_1. Then let Y be a smooth rational ruled surface whose ruling

$$\pi \colon Y \to \mathbf{P}^1, \tag{26}$$

that restricts to U is π_1, let F_t^Y denote the fibre of π at t. Finally, we define the rational map δ: let $x \in \hat{S}_U$, then there exists a unique smooth fibre F_t through x, assume that x is not a base point of the series $g_m^1(t)$, then $\delta(x)$ is the unique divisor in $g_m^1(t)$ passing through the point x. It is easy to see that δ is a rational map. Let $t \in U$ be a general point, then the fibre F_t is smooth and the linear series $g_m^1(t)$ is base-points free of degree $m = 2d - k$, see Claim 3. The restriction $\delta_{|F_t}$ is actually the morphism associated to $g_m^1(t)$:

$$\delta_{|F_t} \colon F_t \to F_t^Y \simeq \mathbf{P}^1, \tag{27}$$

so the map induced on the $\mathbf{P}^1$ must be the identity. Moreover, by Claim 3, for general $t \in U$, the monodromy group $M(\delta_{|F_t})$ is the alternating group A_m.

4.5 Construction 4

The rational map $\delta \colon \hat{S} \to Y$ can be resolved with a finite number of blow-ups as follows. Let $V \subset \hat{S}$ be an open subset where δ is defined. Let $\Gamma_\delta \subset \hat{S} \times Y$ be the closure of the graph of the morphism $\delta_{|V}$. Γ_δ is a projective variety, and it has two natural projections $\pi_1 \colon \Gamma_\delta \to \hat{S}$, which is a birational morphism, and $\pi_2 \colon \Gamma_\delta \to Y$, which is a generically finite surjective morphism of degree m. Then there exist a smooth surface X and a birational morphism $r \colon X \to \Gamma_\delta$ which is a resolution of singularities of Γ_δ, see [Hir63]. Hence, we have:

(i) X is a smooth complex projective surface in the birational class of S;

(ii) there exists a surjective morphism $\eta = \phi \cdot \pi_1 \cdot r \colon X \to \mathbf{P}^1$, whose general smooth fibre is isomorphic to a general smooth fibre F_t of ϕ;

(iii) there exists a generically finite surjective morphism, $f = \pi_2 \cdot r \colon X \to Y$, of degree m, such that the restriction $f_{|F_t}$ is actually $\delta_{|F_t}$, for a general smooth fibre F_t.

So we have proved the following.

Claim 4. We have a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \xrightarrow{\text{id}} & \mathbf{P}^1 \end{array}$$

where the vertical arrows are, respectively, the morphism η and the ruling π of Y, such that for a general smooth fibre F_t, the monodromy group $M(f_{|F_t})$ is the alternating group A_m.

5. The main result

5.1 Technical lemma

We start with a basic lemma, which is an easy application of a topological result of Nori (see [Nor83, Lemma 1.5]).

Lemma 5.1.1. *Let X be a smooth complex projective surface endowed with a surjective morphism $\eta\colon X \to \mathbf{P}^1$ with general smooth fibre F_t. Let Y be a smooth complex rational ruled surface with ruling $\pi\colon Y \to \mathbf{P}^1$, and fibre $F_t^Y \simeq \mathbf{P}^1$. Assume that $f\colon X \to Y$ is a generically finite dominant morphism of degree m, such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \xrightarrow{\text{id}} & \mathbf{P}^1 \end{array}$$

then the restriction of f to general smooth fibres of η completely induces the monodromy group $M(f)$, i.e.

$$M(f,x) \simeq M(f_{|F_t}, x),$$

for a general smooth fibre F_t and $x \in f(F_t)$, not a branch point of f.

Proof. Let us consider the morphism $f\colon X \to Y$, let $R \subset X$ be the ramification divisor and $B \subset Y$ the branch locus of f. The following map

$$q = f_{|X - f^{-1}(B)}\colon X - f^{-1}(B) \to Y - B, \tag{28}$$

is a covering of degree m in the classic topology. Look at the restriction to a general smooth fibre F_t of η, by the above commutative diagram, we have

$$q_{|F_t} = f_{|F_t - (F_t \cap f^{-1}(B))}\colon F_t - (F_t \cap f^{-1}(B)) \to F_t^Y - (B \cap F_t^Y),$$

as $R \cap F_t$ is actually the ramification divisor of $f_{|F_t}$ and $B \cap F_t^Y$ is the branch locus of $f_{|F_t}$, then $q_{|F_t}$ is also a covering of degree m. Now let us also restrict π to $Y - B$:

$$\pi_{|Y-B}\colon Y - B \to \mathbf{P}^1, \tag{29}$$

by the above commutative diagram, as the induced map on the $\mathbf{P}^1$ is the identity, B cannot contain a complete fibre. This allows us to conclude that the restriction $\pi_{|Y-B}$ is also surjective. Moreover, note that as π is a ruling of a rational ruled surface, it admits a section: so it cannot have multiple fibres, that is every fibre must have a reduced component. So by [Nor83, Lemma 1.5(c)], we have the following exact sequence between the fundamental groups:

$$\pi_1(F_t^Y - (B \cap F_t^Y)) \to \pi_1(Y - B) \to \pi_1(\mathbf{P}^1), \tag{30}$$

for a general smooth fibre F_t^Y. As $\pi_1(\mathbf{P}^1) = 0$, this gives us a surjective map s_t:

$$s_t\colon \pi_1(F_t^Y - (B \cap F_t^Y)) \to \pi_1(Y - B). \tag{31}$$

Let $x \in Y - B$ be a point such that $x \in F_t^Y = f(F_t)$, for a general smooth fibre F_t. We recall that the monodromy representation is the group homomorphism

$$\rho(f,x)\colon \pi_1(Y - B) \to \operatorname{Aut}(f^{-1}(x)), \tag{32}$$

whose image is $M(f,x)$. The surjectivity of s_t immediately implies that

$$M(f,x) = M(f_{|F_t}, x), \tag{33}$$

for a general smooth fibre F_t and for any $x \in f(F_t)$, $x \notin B$. This concludes the proof. ∎

5.2 Proof of Theorem 2

Let $m > 16g + 7$ be any integer, we can find a pair of integers (d,k) satisfying the following properties:

$$k = 2 \text{ or } 3, \quad d > 8g + 3k - 4, \quad 2d - k = m.$$

As $H^2 \geqslant 5$ and the pair $(S, O_S(H))$ does not contain lines and conics, see § 4.1, we can choose a Lefschetz pencil P of irreducible curves of genus g, in the linear system $|2H + K_S|$. We can apply all constructions of § 4 to the data (S, H, P), where (d, k) are given as above. So we produce the following situation: X is a smooth complex projective surface, birationally equivalent to S, endowed with a surjective morphism $\eta\colon X \to \mathbf{P}^1$, with smooth fibre F_t of genus g, Y is a smooth complex rational ruled surface, with ruling $\pi\colon Y \to \mathbf{P}^1$, $f\colon X \to Y$ is a generically finite morphism of degree $m = 2d - k$, finally the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \xrightarrow{\text{id}} & \mathbf{P}^1 \end{array}$$

where the vertical maps are, respectively, η and π. Moreover, for a general smooth fibre F_t of η, the monodromy group $M(f_{|F_t})$ is the alternating groupA_m. Note that all the assumptions of Lemma 5.1.1 are verified, hence we have

$$M(f, x) = M(f_{|F_t}, x), \tag{34}$$

for a general smooth fibre F_t and a point $x \in f(F_t)$, which is not a branch point. As $M(f_{|F_t}) = A_m$, for a general smooth fibre F_t, we can finally conclude that the monodromy group $M(f)$ is actually the alternating group A_m.

Remark. Note that the above theorem works under the following more general hypothesis: let H be an ample divisor, such that $2H + K_S$ is very ample and $2H + K_S$ defines a third-order embedding, see Lemma 2.4.1.

5.3 Surfaces of general type

We would like to apply the above result to surfaces of general type. Let S be a minimal, smooth complex projective surface of general type with ample canonical divisor K_S. As is well known, for some $n > 0$ the pluricanonical map ϕ_{nK_S} is an embedding; in order to apply Theorem 2, we will be interested in the smallest n such that ϕ_{nK_S} is actually a third-order embedding. In fact, in this situation, if $n = 2t + 1 \geqslant 3$, we can find a Lefschetz pencil P, of irreducible curves in the linear system $|nK_S|$, see § 4.1, and apply our constructions of § 4 to the data $(S, O_S(tK_S), P)$. At this point, we will use the following result.

Lemma 5.3.1. *Let S be a minimal surface of general type with ample cannonical divisor K_S.*

(i) *If $n \geqslant 5$, the divisor nK_S is very ample, if $p_g \geqslant 3$ and $K_S^2 \geqslant 3$, then $3K_S$ is also very ample;*

(ii) *If $n \geqslant 5$ and $K_S^2 \geqslant 3$, then nK_S defines a third-order embedding; moreover, if $K_S^2 > 5$, then $3K_S$ defines a third-order embedding unless there exists an effective divisor F on S such that $K_S \cdot F = 2$ with $F^2 = 0$.*

For the proofs, see [Cat85] for part (i) and [BFS89] for part (ii).

As an immediate consequence of our result, we have the following.

Theorem 5.3.2. *Let S be a minimal, smooth, complex, connected, projective surface of general type with ample canonical divisor K_S, with $K_S^2 > 3$. Then for any $m > 16(1 + 15K_S^2) + 7$, there exist a smooth complex projective surface X, in the birational class of S, and a generically finite surjective morphism, of degree m:*

$$f\colon X \to Y,$$

into a smooth complex rational ruled surface Y such that the monodromy group $M(f)$ is the alternating group A_m.

Moreover, if $p_g \geqslant 3$ and $K_S^2 > 5$, and S does not contain any effective divisor F described in Lemma 5.3.1, then $m > 16(1 + 6K_S^2) + 7$.

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