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TRANSLATION COMPLEMENTS OF C-PLANES : (I)

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Narayana Rao, Rodabaugh, Wilke and Zemmer constructed a new class of finite translation planes from exceptional near-fields described by Dickson and Zassenhaus. These planes referred to as C-planes are not coordinatized by the generalized André systems. In this paper we compute the translation complement of the C-plane corresponding to the C-system III-1. It is found that the translation complement is of order 6912 and it divides the set of ideal points into two orbits of lengths 2 and 48.

1. Introduction.

Examples of finite near-fields were given by Dickson in 1905. Zassenhaus [12] constructed an infinite class of near-fields that can be constructed from $GF(p^r)$, p a prime and r a positive integer. Apart from these, Zassenhaus had shown that there exist exactly seven other near-fields. These seven near-fields of order 5^2, 11^2, 7^2, 23^2, 11^2,

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29^2 and 59^2 are referred to as the exceptional near-fields. Narayana Rao, Rodabaugh, Wilke and Zemmer [6] constructed quasifields from these exceptional near-fields and showed that these quasifields give rise to nine non-isomorphic translation planes which are not coordinatised by the generalized André systems of Foulser (λ-systems).

The nine *C*-systems are denoted by *I-1, I-2, II-1, III-1, III-3, III-4, V-1, V-2* and *VI-1* . The reader is referred to [6] for the notation and nomenclature used in this paper. Ostrom [9] remarked that the translation complements of these *C*-planes and their actions on the sets of ideal points of these planes have not so far been completely determined. However Lueder [4] has determined the action of the translation complements of the *C*-planes corresponding to the two of the *C*-systems namely *I-1* and *III-4* . Narayana Rao and Satyanarayana [8] have also determined the translation complement of the plane corresponding to the *C*-system *I-2* and established that one of the planes of Walker [11] is isomorphic to the *C*-plane. The translation complements of the remaining six planes are yet to be investigated. In this paper we investigate the translation complement of the plane corresponding to the *C*-system *III-1* . The translation complements of the remaining planes are under investigation and the results will be reported in due course.

2. Construction of the *C*-plane corresponding to the *C*-system *III-1*.

Zassenhaus [12] described the structure of the exceptional near-field *III* of order 7^2 in terms of 2×2 matrices over $GF(7)$. The reader is referred to Marshall Hall [3] for the description of all the exceptional near-fields. The group of non-zero elements of the exceptional near-field *III* is generated by the 2×2 matrices $\{\begin{pmatrix} 0 & 1 \\ 6 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}\}$.

An examination of the non-zero matrices of the exceptional near-field reveals that they are of the following type:

$$\begin{pmatrix} 0 & a \\ 6a^{-1} & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} a & a \\ 2a^{-1} & 3a^{-1} \end{pmatrix}, \begin{pmatrix} a & 2a \\ a^{-1} & 3a^{-1} \end{pmatrix}$$

$$\begin{pmatrix} a & 3a \\ 6a^{-1} & 5a^{-1} \end{pmatrix}, \begin{pmatrix} a & 4a \\ 4a^{-1} & 3a^{-1} \end{pmatrix}, \begin{pmatrix} a & 5a \\ 5a^{-1} & 5a^{-1} \end{pmatrix}, \begin{pmatrix} a & 6a \\ 3a^{-1} & 5a^{-1} \end{pmatrix}$$

$$a = 1, 2, 3, 4, 5 \text{ and } 6 .$$

The set of these 48 matrices together with the zero matrix forms a 1-spread set [2] over $GF(7)$ defining the near-field $(F, +, .)$ where $F = \{(x,y) | x,y \in GF(7)\}$. Addition is defined as vector addition. Multiplication is defined by $(x,y).(a,b) = (a,b)\ D(x,y)$ where $D(x,y)$ is the unique matrix in the 1-spread set associated with (x,y) in the near-field.

The C-system III-1 is constructed from the exceptional near-field $(F, +, .)$ in the following way. In what follows, the C-system means the C-system III-1 and C-plane is the plane π coordinatized by the C-system.

Let T be the additive automorphism given by $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Let $G = \langle xT.x^{-1} \rangle$ where $x \in F - \{0\}$. Let $(F, +, o)$ be the structure defined by

i) $(a,b) + (c,d) = (a + c,\ b + d)$ for all $a,b,c,d \in GF(7)$.

ii) $(x,y)o(a,b) = (x,y).(a,b)T^{\lambda(x,y)}$ where

$$\lambda(x,y) = \begin{cases} 0 & \text{if } (x,y) \in G \\ 1 & \text{if } (x,y) \notin G\ ,\ (x,y) \neq (0,0) \end{cases}$$

iii) $(0,0)o(a,b) = (0,0)$.

This is the C-system III-1 described in [6]. The structure of the C-system as a 1-spread set is obtained in the following way:

Let $D\begin{pmatrix} x & y \\ p & q \end{pmatrix}$ be the unique matrix associated with (x,y) in the near-field. Let $M(x,y)$ be the unique matrix associated with (x,y) in the C-system. Since

$$\begin{aligned} (x,y)o(a,b) &= (x,y).(a,b)\ T^{\lambda(x,y)} \\ &= (a,b)\ T^{\lambda(x,y)}\ D(x,y) \quad \text{for all } a,b \in GF(7) \end{aligned}$$

we obtain that

$$M(x,y) = T^{\lambda(x,y)}\ D(x,y)\ . \quad \text{That is}$$

$$M(x,y) = \begin{cases} D(x,y) & \text{if } (x,y) \in G \\ \begin{pmatrix} x & y \\ 2p & 2q \end{pmatrix} & \text{if } (x,y) \notin G\ . \end{cases}$$

Narayana Rao, Rodabaugh, Wilke and Zemmer [6] have established that G is generated by the two elements $\left\{\begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\right\}$ and obtained the result that G acts as both left nucleus N_l and middle nucleus N_m [5] for F. The element $(0,1) \not\in G$ and the associated matrix in the near-field for $(0,1)$ is $\begin{pmatrix} 0 & 1 \\ 6 & 0 \end{pmatrix}$. Then the associated matrix for $(0,1)$ in the C-system is $\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}$. Hence the 1-spread set C for the C-system can be written as

$$C = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\} \cup G \cup \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} G = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\} \cup G \cup G\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} .$$

For the sake of elegance we give the general forms of the matrices in C. They are, apart from $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad \begin{pmatrix} a & 3a \\ 6a^{-1} & 5a^{-1} \end{pmatrix}, \quad \begin{pmatrix} a & 5a \\ 5a^{-1} & 5a^{-1} \end{pmatrix}, \quad \begin{pmatrix} a & 6a \\ 3a^{-1} & 5a^{-1} \end{pmatrix}$$

$$\begin{pmatrix} 0 & a \\ 5a^{-1} & 0 \end{pmatrix}, \quad \begin{pmatrix} a & a \\ 4a^{-1} & 6a^{-1} \end{pmatrix}, \quad \begin{pmatrix} a & 2a \\ 2a^{-1} & 6a^{-1} \end{pmatrix}, \quad \begin{pmatrix} a & 4a \\ a^{-1} & 6a^{-1} \end{pmatrix},$$

$$a = 1, 2, 3, 4, 5 \text{ and } 6 .$$

It may not be out of place to mention here that G consists of elements of the first four forms and $\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} G$ consists of the elements of the next four forms. The matrices of C along with their characteristic polynomials are listed in Table 1. Here the entry (a,b) under the heading $C.P$ indicates that $\lambda^2 + a\lambda + b$ is the characteristic polynomial of the corresponding matrix.

3. Some Collineations of the C-plane.

Let $V_i = \{(a,b,c,d) \mid a,b \in GF(7), (c,d) = (a,b)M_i, M_i \in C\}$, $0 \le i \le 48$ and $V_{49} = V_\infty = \{(0, 0, c, d) \mid c,d \in GF(7)\}$ be subspaces of $V(4,7)$, the four dimensional vector space over $GF(7)$. The incidence structure V_i, $0 \le i \le 49$, and its cosets in the additive group of $V(4,7)$ as lines and the vectors of $V(4,7)$ as points with the inclusion as incidence relation is the C-plane π whose translation complement we will be determining. It is customary to denote the ideal point corresponding to V_i by (i). The ideal point corresponding to V_{49} is denoted by (49) or (∞). It is known that

TABLE 1

	M_i	C.P	i	M_i	C.P	i	M_i	C.P	i	M_i	C.P
0	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	(0,0)	13	$\begin{pmatrix}5&4\\1&1\end{pmatrix}$	(1,1)	25	$\begin{pmatrix}1&1\\4&6\end{pmatrix}$	(0,2)	37	$\begin{pmatrix}3&3\\0&2\end{pmatrix}$	(2,2)
1	$\begin{pmatrix}3&2\\2&4\end{pmatrix}$	(0,1)	14	$\begin{pmatrix}5&2\\2&1\end{pmatrix}$	(1,1)	26	$\begin{pmatrix}1&2\\2&6\end{pmatrix}$	(0,2)	38	$\begin{pmatrix}3&5\\5&2\end{pmatrix}$	(2,2)
2	$\begin{pmatrix}3&1\\4&4\end{pmatrix}$	(0,1)	15	$\begin{pmatrix}6&0\\0&6\end{pmatrix}$	(2,1)	27	$\begin{pmatrix}1&4\\1&6\end{pmatrix}$	(0,2)	39	$\begin{pmatrix}3&6\\3&2\end{pmatrix}$	(2,2)
3	$\begin{pmatrix}3&4\\1&4\end{pmatrix}$	(0,1)	16	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	(5,1)	28	$\begin{pmatrix}6&3\\6&1\end{pmatrix}$	(0,2)	40	$\begin{pmatrix}2&1\\4&3\end{pmatrix}$	(2,2)
4	$\begin{pmatrix}4&5\\5&3\end{pmatrix}$	(0,1)	17	$\begin{pmatrix}3&0\\0&5\end{pmatrix}$	(6,1)	29	$\begin{pmatrix}6&5\\5&1\end{pmatrix}$	(0,2)	41	$\begin{pmatrix}2&2\\2&3\end{pmatrix}$	(2,2)
5	$\begin{pmatrix}4&6\\3&3\end{pmatrix}$	(0,1)	18	$\begin{pmatrix}2&6\\3&6\end{pmatrix}$	(6,1)	30	$\begin{pmatrix}6&6\\3&1\end{pmatrix}$	(0,2)	42	$\begin{pmatrix}2&4\\1&3\end{pmatrix}$	(2,2)
6	$\begin{pmatrix}4&3\\6&3\end{pmatrix}$	(0,1)	19	$\begin{pmatrix}2&3\\6&6\end{pmatrix}$	(6,1)	31	$\begin{pmatrix}0&1\\5&0\end{pmatrix}$	(0,2)	43	$\begin{pmatrix}4&1\\4&5\end{pmatrix}$	(5,2)
7	$\begin{pmatrix}1&3\\6&5\end{pmatrix}$	(1,1)	20	$\begin{pmatrix}2&5\\5&6\end{pmatrix}$	(6,1)	32	$\begin{pmatrix}0&2\\6&0\end{pmatrix}$	(0,2)	44	$\begin{pmatrix}4&2\\2&5\end{pmatrix}$	(5,2)
8	$\begin{pmatrix}1&5\\5&5\end{pmatrix}$	(1,1)	21	$\begin{pmatrix}5&0\\0&3\end{pmatrix}$	(6,1)	33	$\begin{pmatrix}0&3\\4&0\end{pmatrix}$	(0,2)	45	$\begin{pmatrix}4&4\\1&5\end{pmatrix}$	(5,2)
9	$\begin{pmatrix}1&6\\3&5\end{pmatrix}$	(1,1)	22	$\begin{pmatrix}6&4\\1&2\end{pmatrix}$	(6,1)	34	$\begin{pmatrix}0&4\\3&0\end{pmatrix}$	(0,2)	46	$\begin{pmatrix}5&3\\6&4\end{pmatrix}$	(5,2)
10	$\begin{pmatrix}2&0\\0&4\end{pmatrix}$	(1,1)	23	$\begin{pmatrix}6&2\\2&2\end{pmatrix}$	(6,1)	35	$\begin{pmatrix}0&5\\1&0\end{pmatrix}$	(0,2)	47	$\begin{pmatrix}5&5\\5&4\end{pmatrix}$	(5,2)
11	$\begin{pmatrix}4&0\\0&2\end{pmatrix}$	(1,1)	24	$\begin{pmatrix}6&1\\4&2\end{pmatrix}$	(6,1)	36	$\begin{pmatrix}0&6\\2&0\end{pmatrix}$	(0,2)	48	$\begin{pmatrix}5&6\\3&4\end{pmatrix}$	(5,2)
12	$\begin{pmatrix}5&1\\4&1\end{pmatrix}$	(1,1)							49	–	–

a nonsingular linear transformation on $V(4,7)$ induces a collineation of π belonging to the translation complement if it permutes the subspaces V_i, $0 \le i \le 49$ among themselves [9]. From now on we mean by a collineation a collineation from the translation complement. Equivalently it is also known that a nonsingular transformation in the block matrix form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C and D are 2×2 matrices over $GF(7)$ induces a collineation on π if and only if for each $M_i \in \mathcal{C}$, the following conditions are satisfied:

i) $(A + M_i C)^{-1} (B + M_i D) \in \mathcal{C}$, if $(A + M_i C)$ is nonsingular. If $(A + M_i C)$ is singular then $(A + M_i C)$ is the zero matrix and $(B + M_i D)$ is nonsingular.

ii) $C^{-1} D \in \mathcal{C}$, if C is nonsingular. If C is singular then C is the zero matrix and D is nonsingular.

Every matrix of the form $\begin{pmatrix} aI & 0 \\ 0 & aI \end{pmatrix}$ where $a \in GF(7)$, $a \neq 0$, and I is the 2×2 identity matrix trivially satisfies conditions (i) and (ii) and hence induces a collineation of π called a scalar collineation. A scalar collineation fixes the ideal points in all cases and moves the affine points in cases when $a \neq 1$. The group of scalar collineations is of order 6.

3.1. Collineations induced by the left nucleus N_l and the middle nucleus N_m

Since $MA \in \mathcal{C}$ for each $M \in \mathcal{C}$ and $A \in N_l = G$, the mappings $M \longrightarrow MA$ satisfy conditions (i) and (ii) mentioned above and hence induce collineations for all $A \in N_l = G$. These collineations form a group N_λ which acts transitively on the ideal points corresponding to G and $\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} G$ separately. Similarly the mappings $M \longrightarrow BM$ for all $B \in N_m$ induce collineations for all $B \in N_m = G$. These collineations form a group N_μ which acts transitively on the ideal points corresponding to matrices in G and $\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} G$ separately.

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DEFINITION 3.2. *Two ideal points (i) and (j) are said to be companions under a collineation group if whenever a collineation fixes (i) it also fixes (j) and vice versa. In other words any collineation either fixes both (i) and (j) or moves both (i) and (j). The significance of the companions is that any collineation must map companions onto companions only.*

LEMMA 3.3. *There is no collineation of* π *which*

i) *fixes* (0) *and moves* (∞) *onto* (i),

ii) *fixes* (∞) *and moves* (0) *onto* (j),

iii) *moves* (0) *onto* (∞) *and* (∞) *onto* (i), $i \neq 0$ *and*

iv) *moves* (∞) *onto* (0) *and* (0) *onto* j , $(j) \neq (\infty)$.

Proof. An examination of Table 1 reveals that if $M_i \in C$, then $-M_i \in C$. Then the necessary condition for the existence of a collineation satisfying (i) or (ii) is that there is a matrix $M_k \in C$ such that $M + M_k \in C$ for all $M \in C[7]$. This condition is not satisfied by C . Hence the lemma.

It follows from the above lemma that (0) and (∞) are companions.

3.4. Conjugation collineations

A mapping $M \longrightarrow A^{-1}MA$, for $A \in GL\,(2,7)$ such that for each $M \in C$, $A^{-1}MA$ also is in C , satisfies the sufficient conditions (i) and (ii) of Section 3 for the existence of a collineation and hence induces a collineation called a conjugation collineation. The conjugation collineations obviously fix (0) and (∞) . Since $N_l = N_m$, any mapping $M \longrightarrow A^{-1}MA$ is a conjugation collineation if $A \in N_l = N_m = G$. Since any collineation preserves N_l and N_m , the conjugation collineation also must act invariantly on G and hence on $\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} G$ separately. The group G contains exactly 6 matrices, $\left\{ \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 6 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 6 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 5 & 3 \end{pmatrix} \right\}$ with the same characteristic polynomial $\lambda^2 + 1$. We denote the set of these 6 matrices by H . The conjugation collineation must permute the matrices of H among themselves. Let $A = \begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$. Since these matrices are in N_l, the mappings $M \longrightarrow A^{-1}MA$ are collineations of π. From the relations:

$$\begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 6 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 6 & 3 \end{pmatrix};$$

$$\begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 4 & 3 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}; \quad \begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix};$$

$$\begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 4 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 3 & 3 \end{pmatrix}; \quad \begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 4 & 6 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 3 & 3 \end{pmatrix};$$

$$\begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 4 & 5 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 4 \end{pmatrix}; \quad \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 4 \end{pmatrix},$$

We conclude that the group of conjugation collineations acts transitively on the set of ideal points corresponding to the matrices in H.

We now determine all the conjugation collineations which fix the ideal point corresponding to one matrix namely $\begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}$ in H. Since the characteristic polynomial $\lambda^2 + 1$ of the matrix $\begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}$ is irreducible over $GF(7)$, the matrix $\begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}$ belongs to the field $F = \{\begin{pmatrix} a & b \\ 4b & a+b \end{pmatrix} \mid a,b \in GF(7)\}$. By Schur's lemma the normaliser of $\begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}$ consists of nonzero elements of a field contained in $GL(2,7)$ and containing $\begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}$, which is F itself. Thus the normaliser of $\begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}$ is $F - \{0\}$.

In order to show that $A^{-1}MA$ induces a collineation on π we have to verify that $A^{-1}\begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}A$; $A^{-1}\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}A$; $A^{-1}\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}A$ are all in C. This is because $\begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ generate G and $C = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\} \cup G \cup \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} G$. It is easily verified that if $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}$, $\begin{pmatrix} 1 & 5 \\ 6 & 6 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and their scalar multiples, the above mentioned conditions are satisfied. However if $A = \begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix}$, $A^{-1}\begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix} A = \begin{pmatrix} 2 & 0 \\ 2 & 4 \end{pmatrix} \notin C$.

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This implies that $\begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix}$ and its scalar multiples do not induce collineations on π. Further the products of $\begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix}$ with $\begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}$ and their scalar multiples also do not induce collineations on π. Thus the set of all conjugation collineations fixing the ideal point corresponding to $\begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}$ is the group K consisting of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}$ and their scalar multiples. The order of the group K is 24.

Let J be the group of all conjugation collineations of π. Then J is transitive on the 6 ideal points corresponding to the matrices in H. Since the group of all conjugation collineations fixing the ideal point corresponding to $\begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}$ is K, a coset decomposition of J by K gives $J = \cup K\alpha$, where the union extends over some six conjugation collineations α which map $\begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}$ onto each of the elements of H. These collineations α exist since J is transitive on the ideal points corresponding to matrices in H. Clearly the order of J is the product of the size of H and the order of K. Thus $|J| = 6 \times 24 = 144$. Obviously J contains the subgroup of all scalar collineations.

3.5. Collineations fixing (0) and (∞)

It is known that any collineation fixing (0) and (∞) corresponds to the mapping $M \longrightarrow A^{-1}MB$, such that for each $M \in C$, $A^{-1}MB \in C$ where $A, B \in GL(2,7)$. The conjugation collineations are obtained as a special case when $A = B$; they have already been accounted for. Further a collineation $M \longrightarrow A^{-1}MB$ can also be expressed as $M \longrightarrow A^{-1}M_k^{-1}MA$ for some $M_k \in C$. An examination of Table 1 reveals that C has apart from the zero matrix, 24 matrices with determinant 1 and 24 matrices with determinant 2, which forces the choice of M_k to be a matrix with determinant 1. But all the matrices with determinant 1 are in G which is the same as $N_l = N_m$. Thus the mapping $M \longrightarrow A^{-1}M_k^{-1}MA$ is

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a combination of a conjugation collineation and a collineation induced by an element of N_m . Thus the group L of all collineations fixing (0) and (∞) is generated by J and N_μ . Since the subgroup N_μ of L is transitive on the 24 ideal points corresponding to matrices in G, L is transitive on these 24 ideal points. Further all the collineations of L fix (0) and (∞) . The subgroup J consists of all collineations which fix (0), (∞) and the ideal point corresponding to the identity matrix in G . Then a coset decomposition of L by J is given by

$$L = \bigcup_{\alpha \in N_\mu} J\alpha \quad \text{and}$$

$$|L| = 24\ |J| = 24 \times 144 = 3456 .$$

Obviously L contains N_l also.

3.6. Collineations flipping (0) and (∞)

Consider the mapping $\beta : M \longrightarrow \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} M^{-1}$ for $M \in \mathfrak{C}$. If $M \in G$, then $M^{-1} \in G$ and hence $\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} M^{-1} \in \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} G$. If $M \in G\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}$, then $M = P \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}$ for some P in G . Then $\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} M^{-1}$ = $\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}^{-1} P^{-1} = P^{-1} \in G$.

Thus the mapping $M \longrightarrow \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} M^{-1}$ induces a collineation on π interchanging (0) and (∞) and interchanging the ideal points corresponding to matrices in G and the ideal points corresponding to matrices in $\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} G$. Let $G' = \langle L, \beta \rangle$. The group G' divides the ideal points of π into two orbits one containing (0) and (∞) and the other containing the remaining 48 ideal points.

Let $\mathcal{G}$ be the group of all collineations which either fixes both (0) and (∞) or flips (0) and (∞) . Then G' is contained in $\mathcal{G}$ and is therefore transitive on the set of ideal points consisting of (0) and (∞) . Further, since (0) and (∞) are companions, any collineation that fixes (0) must also fix (∞) . Thus the subgroup of $\mathcal{G}$ consisting of all collineations that fix (0) is L itself. A coset decomposition of $\mathcal{G}$ by L is given by

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$$G = L \cup L_\beta \quad \text{which is} \quad G' \quad \text{it self,}$$

then

$$|G| = |G'| = 2 \times 3456 = 6912 .$$

4. Translation complement of π

In this section we prove that G is in fact the translation complement of π .

LEMMA 4.1. *The ideal points corresponding to matrices* I *and* $6I$ *are companions.*

Proof. The mapping $\nu : M \longrightarrow \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}^{-1} M \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ is a collineation belonging to J . This collineation maps an ideal point corresponding to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ onto an ideal point corresponding to the matrix $\begin{pmatrix} a & 2b \\ 4c & d \end{pmatrix}$. This implies that ν fixes ideal points corresponding to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$, $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$ apart from (0) and (∞) and moves all other ideal points. The mapping $\delta : M \longrightarrow \begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}^{-1} M \begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}$ is a collineation belonging to J . This collineation fixes ideal points corresponding to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$ and moves the ideal points corresponding to $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$, $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$. From the actions of ν and δ we conclude that the ideal points corresponding to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$ are companions.

THEOREM 4.2. *There is no collineation of* π *which moves* (0) *and* (∞) *onto* (r) *and* (s) *where* $r, s \neq 0, \infty$.

Proof. Since (0) and (∞) are companions, if a collineation maps (0) onto (r), $r \neq 0, \infty$, then the collineation must map (∞) onto (s) $s \neq 0, \infty$ and (s) the companion of (r) . Since the group G is transitive on the 48 ideal points other than (0) and (∞) , it suffices to consider a collineation η which maps (∞) onto the ideal point corresponding to I and (0) onto the ideal point corresponding to $6I$. Any collineation which sends (∞) onto (s) and (0) onto (r) will be a combination of η and a collineation from G .

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Let $\Gamma_{(r,s)} = \{(M-M_r)^{-1} - (M_s - M_r)^{-1} \mid M \in C\}$,

with the usual understanding that whenever (0) and (∞) occur in the above expression their inverses are to be taken as (∞) and (0). It is known that if a collineation exists which sends (∞) onto (s) and (0) onto (r), then there must exist two matrices $A, B \in GL(2,7)$ such that $A^{-1}\Gamma_{(r,s)} B = C$. Taking $M_r = I$ and $M_s = 6I$, we get

$$\Gamma_{(r,s)} = \{(M + 6I)^{-1} + 4I \mid M \in C\}$$

since C has the property that $M \in C$ implies $-M \in C$, the set $\Gamma_{(r,s)}$ must also inherit this property. Thus for each $M \in C$, there must exist $N \in C$ such that

$$(M + 6I)^{-1} + 4I = -\{(N + 6I)^{-1} + 4I\}.$$

Taking $M = \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}$ and solving the above equation for N, we get $N = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix} \notin C$. Thus (0) and (∞) cannot be moved onto I and $6I$ respectively. Hence the theorem.

Conclusion

The translation complement of π is G itself and it is of order 6912. Further G divides the ideal points into two orbits of lengths 2 and 48. It may be mentioned here that the translation complement of a near-field plane of order 49 also divides the set of ideal points into two orbits of lengths 2 and 48. However the order of the translation complement of the nearfield plane is very much bigger.

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