A NOTE ON THE ANGLES IN AN n-DIMENSIONAL SIMPLEX

by A. P. GUINAND

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1. Introduction. Three different sets of equations connecting the sums of angles in an n-dimensional simplex have been given by Sommerville [7], Höhn [5], and Peschl [6].† The equivalence of the first two sets of equations has been proved by Sprott [7].

In the present note it is shown that results are simplified if we consider averages instead of sums, and that the averages form a sequence which is self-reciprocal with respect to the transformation‡

$$q_s = \sum_{r=0}^{s} (-1)^r \binom{s}{r} p_r.$$

The equivalence of the sets of equations is then easily proved by symbolic methods.§

2. Forms of the equations. Given an n-dimensional simplex in spherical or Euclidean space, let s_k denote the sum of the angles at its $\binom{n+1}{k}$ $(n-k)$-cells, each measured as a fraction of the whole angle at the $(n-k)$-flat concerned. Let $s_0 = 1$, and let s_{n+1} be the content of the simplex as a fraction of the whole space.‖ Then the three sets of equations are

$$\sum_{k=r}^{n+1} (-1)^k \binom{k}{r} s_k = \sum_{k=n+1-r}^{n+1} (-1)^{n+1-k} \binom{k}{n+1-r} s_k \quad (0 \leqslant r \leqslant \tfrac{1}{2}n), \qquad \text{(Sommerville)}$$

$$\sum_{k=0}^{p} (-1)^k \binom{n+1-k}{n+1-p} s_k = s_p \quad (1 \leqslant p \leqslant 2[\tfrac{1}{2}n]+2), \qquad \text{(Höhn)}$$

$$\sum_{k=1}^{l+1} \frac{2^{2k}-1}{k} \binom{n-2l+2k-1}{2k-1} B_{2k} s_{2l-2k+2} = s_{2l+1} \quad (0 \leqslant l \leqslant \tfrac{1}{2}n), \qquad \text{(Peschl)}$$

where B_{2k} runs through the Bernoulli numbers $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}, \ldots$.

In each set there are only $[\frac{1}{2}n]+1$ independent equations. An independent set is obtained from Höhn's equations if we take the set of alternate values of p which includes $p = n+1$.

Now put

$$a_k = s_k \Big/ \binom{n+1}{k},$$

so that a_k is the average angle at an $(n-k)$-cell, expressed as a fraction of the whole angle at an $(n-k)$-flat. In terms of the a_k the equations become, respectively,

† See also Coxeter [2].

‡ Cf. Hardy [4].

§ These results were suggested by similarities with equations arising in a problem about Fourier transforms. Cf. Guinand [3].

‖ That is, in the Euclidean case, $s_{n+1} = 0$. Peschl [6] also shows that the same equations hold in hyperbolic space with an appropriate reinterpretation of s_{n+1}.

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$$\sum_{k=r}^{n+1}(-1)^k\binom{n+1-r}{k-r}a_k = \sum_{k=0}^{r}(-1)^k\binom{r}{k}a_{n-k+1} \quad (0 \leqslant r \leqslant \tfrac{1}{2}n),$$

$$\sum_{k=0}^{r}(-1)^k\binom{r}{k}a_k = a_r \quad (1 \leqslant r \leqslant 2[\tfrac{1}{2}n]+2),$$

$$\sum_{k=0}^{r}(2^{r-k}-1)\binom{r}{k}B_{r-k}a_k = 0 \quad (r \text{ even}, 2 \leqslant r \leqslant n+2).$$

In these forms all three sets of equations can readily be expressed in the symbolic or umbral notation. If we put

$$a^k = a_k, \quad B^k = B_k,$$

then the equations can be written, respectively,

$$a^r(1-a)^{n-r+1} = a^{n-r+1}(1-a)^r \quad (0 \leqslant r \leqslant \tfrac{1}{2}n), \tag{S}$$

$$(1-a)^r = a^r \quad (1 \leqslant r \leqslant 2[\tfrac{1}{2}n]+2), \tag{H}$$

$$(B+a)^r = (2B+a)^r \quad (r \text{ even}, 2 \leqslant r \leqslant n+2). \tag{P}$$

3. Equivalence of the sets of equations. Denote the sets of equations by S, H, P as above. Let H_1 denote the set

$$(1-a)^r = a^r \quad (r \text{ odd}, 1 \leqslant r \leqslant n+1), \tag{H_1}$$

and H_2 the set

$$(1-a)^r = a^r \quad (r \text{ even}, 2 \leqslant r \leqslant n+2). \tag{H_2}$$

Then the equivalence of the four sets S, H_1, H_2, P can be proved by the following stages.

(i) $H_1 \supset H$. Suppose that $(1-a)^r = a^r$ for $1 \leqslant r \leqslant 2q-1$. Then any symbolic polynomial in a of degree not greater than $2q-1$ is unchanged in value if a is replaced by $1-a$. The polynomial $(1-a)^{2q}-a^{2q}$ is of degree $2q-1$ only; hence it is equal to $a^{2q}-(1-a)^{2q}$, and therefore

$$(1-a)^{2q} = a^{2q}.$$

Now by H_1 the result $(1-a)^r = a^r$ is true for $r = 1$; hence it is true for $r = 1, 2$. By H_1 it is also true for $r = 3$, and hence for $r = 4$. Continuing this process, we see that it is true for $r = 1, 2, 3, \ldots, 2[\tfrac{1}{2}n]+2$, as required.

(ii) $H_2 \supset H$. Suppose that $(1-a)^r = a^r$ for $0 \leqslant r \leqslant 2q$ and also for $r = 2q+2$. Then the value of any symbolic polynomial in a of degree not greater than $2q$ is unchanged if a is replaced by $1-a$. The polynomial

$$(q+1)\{a^{2q+1}-(1-a)^{2q+1}\}-\{a^{2q+2}-(1-a)^{2q+2}\}$$

is of degree $2q$ only; hence, by an argument as in (i), it is equal to zero. Since $(1-a)^{2q+2} = a^{2q+2}$ by assumption, we have

$$(1-a)^{2q+1} = a^{2q+1}.$$

Now by H_2 the result $(1-a)^r = a^r$ is true for $r = 2$ and it is trivially true for $r = 0$. Hence it is true for $r = 1$. By H_2 it is also true for $r = 4$; so it is true for $r = 0, 1, 2, 4$, and and hence for $r = 3$. Continuing the process, we see that it is true for $r = 0, 1, 2, \ldots, 2[\tfrac{1}{2}n]+2$, as required.

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(iii) H ⊃ S. By H any symbolic polynomial in a of degree not greater than $n+1$ is unchanged in value if a is replaced by $1-a$. Hence

$$a^r(1-a)^{n-r+1} = a^{n-r+1}(1-a)^r$$

for $0 \leqslant r \leqslant n+1$. This includes S.

(iv) S ⊃ H. The equations S run through the same set when $0 \leqslant r \leqslant \frac{1}{2}n$ and when $\frac{1}{2}n+1 \leqslant r \leqslant n+1$. If n is odd, then the remaining equation with $r = \frac{1}{2}n+\frac{1}{2}$ is an identity. Hence S holds for $0 \leqslant r \leqslant n+1$. Thus for $0 \leqslant r \leqslant n$ we have both

$$a^r(1-a)^{n-r+1} = a^{n-r+1}(1-a)^r,$$

and

$$a^{r+1}(1-a)^{n-r} = a^{n-r}(1-a)^{r+1}.$$

Adding these results, we have

$$a^r(1-a)^{n-r}(1-a+a) = a^{n-r}(1-a)^r(a+1-a),$$

or

$$a^r(1-a)^{n-r} = a^{n-r}(1-a)^r,$$

for $0 \leqslant r \leqslant n$. Continuing this process, we get

$$a^p(1-a)^q = a^q(1-a)^p$$

for all p and q in $p \geqslant 0$, $q \geqslant 0$, $p+q \leqslant n+1$. On putting $q = 0$ this gives H, as required.

(v) H ⊃ P. The Bernoulli numbers are determined by the formal expansion

$$e^{Bx} = \sum_{m=0}^{\infty} B_m \frac{x^m}{m!} = \frac{x}{e^x-1}.$$

Hence the function $\phi(x)$, defined by the formal expansion

$$\phi(x) = \sum_{m=0}^{\infty}\{(B+a)^m-(2B+a)^m\}\frac{x^m}{m!}, \qquad (1)$$

is equal to

$$\begin{aligned} e^{(B+a)x}-e^{(2B+a)x} &= e^{ax}(e^{Bx}-e^{2Bx}) \\ &= e^{ax}\left(\frac{x}{e^x-1}-\frac{2x}{e^{2x}-1}\right) \\ &= e^{(a-\frac{1}{2})x}(\tfrac{1}{2}x \operatorname{sech} \tfrac{1}{2}x). \qquad (2) \end{aligned}$$

Now if we replace a by $1-a$ in $(a-\frac{1}{2})^r$, it follows that H implies

$$(a-\tfrac{1}{2})^r = (\tfrac{1}{2}-a)^r$$

for $r = 0, 1, \ldots, n+1$. Hence

$$(a-\tfrac{1}{2})^r = 0$$

for all odd r not greater than $n+1$. Hence the expansion of $\phi(x)$ in the form (2) has no even powers of x lower than x^{n+2}.

By (1) this implies that

$$(B+a)^r-(2B+a)^r = 0$$

for r even and $0 \leqslant r \leqslant n+2$, as required.

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(vi) $P \supset H$. Reversing the argument of (v), we see that P implies that $(a-\frac{1}{2})^r = (\frac{1}{2}-a)^r$ for $0 \leqslant r \leqslant n+1$. Hence we can replace $(a-\frac{1}{2})$ by $(\frac{1}{2}-a)$ in any polynomial in $a-\frac{1}{2}$ of degree not greater than $n+1$. Thus

$$a^r = \{\tfrac{1}{2}+(a-\tfrac{1}{2})\}^r = \{\tfrac{1}{2}+(\tfrac{1}{2}-a)\}^r = (1-a)^r$$

for $0 \leqslant r \leqslant n+1$, as required.

(vii) $H_1 \equiv H_2 \equiv H \equiv S \equiv P$. Since H includes H_1 and H_2, (i) and (ii) give $H_1 \equiv H \equiv H_2$. Then (iii) and (iv) give $H \equiv S$, and (v) and (vi) give $S \equiv P$.

4. Remarks. If $\{p_r\}(r = 0, 1, 2, \ldots)$ is any sequence, and q_s is defined by

$$q_s = \sum_{r=0}^{s} (-1)^r \binom{s}{r} p_r \quad (s = 0, 1, 2, \ldots),$$

then

$$p_s = \sum_{r=0}^{s} (-1)^r \binom{s}{r} q_r.$$

Sequences connected by such a reciprocity may be called " reciprocal sequences ".† With this terminology we can state the equations H thus :

The sequence $\{a_k\}$ $(k = 0, 1, 2, \ldots, n+1)$ of angle averages at $(n-k)$-flats, expressed as fractions of the whole angle at an $(n-k)$-flat, is a self-reciprocal sequence.

A general solution of the equations H is given if we put $c^r = c_r$ where $\{c_r\}$ is any sequence. Then

$$a^r = (\tfrac{1}{2}+c)^r + (\tfrac{1}{2}-c)^r,$$

or

$$a_r = \sum_{k=0}^{[\frac{1}{2}r]} \binom{r}{2k} \left(\frac{1}{2}\right)^{2k-1} c_{r-2k}$$

is a general solution of H.

† Barrucand [1].

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University of Alberta
Edmonton
Canada