BOOK REVIEWS

SARD, A., *Linear Approximation* (*Mathematical Surveys*, Vol. 9, American Mathematical Society, 1963), 544 pp., $16.80.

This book is divided into two parts. In the first part a basic role is played by Riesz's Theorem and similar theorems. One of these is due to Peano, and gives a convenient representation of functionals which are given as finite sums of Stieltjes integrals of a function of one variable and its derivatives. Many applications of this theorem are given, mainly with the object of finding best approximations of integrals and linear interpolation, both narrow and broad. Equally spaced node and Gaussian quadrature are studied from this point of view, the former in considerable detail, general formulæ being derived.

Spaces of functions of two variables, whose partial derivatives have certain properties, are introduced in order that the above procedure may be applicable to spaces of functions of two variables. Since this topic tends to become rather complicated, it is perhaps helpful that special cases of these spaces are examined first. Again many numerical applications are given, including approximation of double integrals and double linear interpolation.

Fewer applications are given of Riesz's Theorem, which makes a weaker assumption about the given functional, but on the other hand represents it as a Stieltjes integral, in contrast to Peano's Theorem which describes the functional in terms of ordinary integrals. There is considerable repetition of basic ideas in the first part of the book, together with increasing complexity, especially when functions of three variables are discussed. In this connection the diagrams and tables provided are helpful.

The second part of the book, in the reviewer's opinion, contains the more sophisticated topics, and it is perhaps a pity that it is not so well supplied with concrete illustrations as the first part. In order to prove the author's own quotient theorem which enables one to express a given linear continuous operator in a convenient way as the product of two others, factor spaces, Baire's Category theorem, and related theorems for Banach Spaces are included.

The later chapters are mainly concerned with the choice of efficient and minimal operators A from a given set $\mathscr{A}$. In preparation, certain basic topics in Hilbert Space are presented, including the projection theorem and the direct product of two separable Hilbert Spaces. In a typical problem an input x is contaminated by an error δx. $A(x+\delta x)$ is the actual output, Gx the desired output; δx is regarded as an element of $X\psi$ the direct product of a separable Hilbert Space X and separable L^2 space ψ, absolute square integrable relative to probability measure p. So δx is a stochastic process. Theorems are proved using the method of projections which yield conditions for efficiency and strong efficiency of $A \in \mathscr{A}$.

In the later theory, unbiassed operators are considered. These are first studied by the method of projections and then by using the variance V of δx rather than δx itself. For this purpose, certain appropriate theorems regarding operators of finite Schmidt norm, non-negative operators and finite trace are proved. An example is given, using both methods.

The last two chapters, which really form an appendix, give useful formulæ relating to certain discontinuous functions, also an account of Lebesgue-Stieltjes integration including double integrals and functions of bounded variation in two variables.

A number of the results throughout the book are due to the author and have appeared in previous papers by him. The layout of the book is pleasing, and the type clear as indeed is the author's exposition.

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