C*-ACTIONS OF r-DISCRETE GROUPOIDS AND INVERSE SEMIGROUPS

JOHN QUIGG and NÁNDOR SIEBEN

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Abstract

Groupoid actions on C*-bundles and inverse semigroup actions on C*-algebras are closely related when the groupoid is r-discrete.

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1. Introduction

Many important C*-algebras, such as AF-algebras, Cuntz-Krieger algebras, graph algebras and foliation C*-algebras, are the C*-algebras of r-discrete groupoids. These C*-algebras are often associated with inverse semigroups through the C*-algebra of the inverse semigroup [HR] or through a crossed product construction as in Kumjian's localization [Kum1]. Nica [Nic] connects groupoid C*-algebras with the partial crossed product C*-algebras of Exel [Exe1] and McClanahan [McC]. This gives another connection between groupoid C*-algebras and inverse semigroup C*-algebras since [Sie2] and [Exe2] show that discrete partial crossed products are basically special cases of the inverse semigroup crossed products of [Sie2, Pat, Sie1].

The heart of these connections is Renault's observation in [Ren1] that an r-discrete groupoid can be recovered from the way the inverse semigroup of open G-sets acts on the unit space of the groupoid. In the upcoming monograph [Pat], Paterson further develops this connection by showing that the C*-algebra of an r-discrete groupoid G is the crossed product of $C_0(G^0)$ by the action of the inverse semigroup of open G-sets.

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The purpose of this paper is to explore this connection on the level of C^*-crossed products. Renault [Ren2] defines a C^*-action of a groupoid as a functor to the category of C^*-algebras and homomorphisms, in which the collection of object C^*-algebras are glued together as a C^*-bundle over G^0 and the action is appropriately continuous. We associate to this an action of any sufficiently large inverse semigroup S of open G-sets on the C_0-section algebra of the bundle. Conversely, starting with an action (satisfying certain mild conditions) of S on a C^*-algebra B, we obtain an associated C^*-bundle over G^0 via the realization that $C_0(G^0)$ will act as central multipliers of B. Then we construct the groupoid action using the expected 'germs of local automorphisms' approach that goes back to [Hae] and [Rei]. The C^*-bundles arising this way are typically only upper semicontinuous, rather than continuous. So we use a slight generalization of Renault's theory.

The philosophy is that inverse semigroups and r-discrete groupoids are two sides of the same coin; passing back and forth between groupoid and inverse semigroup constructions may benefit both theories. The theory of groupoid C^*-algebras is more developed, but the inverse semigroup theory is more algebraic. For example, we can show that the C^*-algebra of an r-discrete groupoid is an enveloping C^*-algebra without using Renault's disintegration theorem. In fact one could suspect that for r-discrete groupoids the disintegration theorem follows from the less complicated inverse semigroup disintegration theorem. Other applications could include inverse semigroup versions of Kumjian's [Kum2] groupoid Fell bundles and Renault's imprimitivity theorem [Ren2] (see also [Rae]). This latter may be an important step towards finding a regular representation for inverse semigroup actions, which is very much needed for coactions and crossed product duality.

After some preliminary results, we introduce a slight generalization of Renault's groupoid actions in Section 3. In Section 4 we recall the basic theory of inverse semigroup actions. In Sections 5 and 6 we show how to pass back and forth between groupoid and inverse semigroup actions. In Section 7 we prove our main theorem by showing that the crossed products of the corresponding groupoid and inverse semigroup actions are isomorphic. Finally, as an application, we recover the Hausdorff case of Paterson's theorem connecting groupoid C^*-algebras and inverse semigroup actions. Starting with an inverse semigroup, Paterson builds a universal groupoid [Pat]. This groupoid is not Hausdorff in general. Since we only work with Hausdorff groupoids we did not use this universal groupoid, rather we assumed that our inverse semigroup is always a semigroup of open G-sets of a groupoid. It is likely that our approach works for non-Hausdorff groupoids and the theory generalizes to the level of the universal groupoid.

2. Preliminaries

We will need the following elementary results on representations of C^*-algebras. Since we could not find a reference, we include the proofs for the convenience of the reader. When we refer to a 'representation' of a *-algebra, we mean a *-homomorphism of the algebra into the bounded operators on a Hilbert space.

DEFINITION 2.1. Let D be a *-algebra. We say D *has an enveloping C^*-algebra* if the supremum of the C^*-seminorms on D is finite, and in this case we call the Hausdorff completion of D relative to this largest C^*-seminorm the *enveloping C^*-algebra* of D.

REMARK 2.2. Thus, if D is a *-subalgebra of a C^*-algebra B, and if every representation of D is bounded in the norm inherited from B, then the closure of D in B is the enveloping C^*-algebra of D.

Conversely, if the closure of D in B is the enveloping C^*-algebra of D, then every representation of D is contractive.

Our first elementary result about enveloping C^*-algebras is that ideals have them.

LEMMA 2.3. *Let I be a two-sided, not necessarily closed, *-ideal of a C^*-subalgebra B. Then the closure of I in B is the enveloping C^*-algebra of I.*

PROOF. Let π be a representation of I on H. Since we can replace H by the closure of $\pi(I)H$, we may as well assume $\pi(I)H$ is dense in H. Then of course $\pi(I^2)H$ is also dense in H, so it suffices to show

$$\left\| \sum_1^n \pi(ab_ic_i)\xi_i \right\| \leq \|a\| \left\| \sum_1^n \pi(b_ic_i)\xi_i \right\| \qquad \text{for} \quad \begin{matrix} a, b_1, \ldots, b_n, c_1, \ldots, c_n \in I, \\ \xi_1, \ldots, \xi_n \in H. \end{matrix}$$

We use the Effros-Hahn trick: put

$$d = \left(\|a\|^2 - a^*a \right)^{1/2} \in M(B).$$

We have

$$\left\| \sum_1^n \pi(ab_ic_i)\xi_i \right\|^2 = \sum_{i,j} (\pi(a^*ab_ic_i)\xi_i, \pi(b_jc_j)\xi_j)$$

$$= \sum_{i,j} (\pi(\|a\|^2 b_ic_i - d^2 b_ic_i)\xi_i, \pi(b_jc_j)\xi_j)$$

$$= \sum_{i,j} \|a\|^2 (\pi(b_ic_i)\xi_i, \pi(b_jc_j)\xi_j) - \sum_{i,j} (\pi(d^2 b_ic_i)\xi_i, \pi(b_jc_j)\xi_j),$$

which makes sense since $M(B)I^2 \subset BI \subset I$,

$$
= \|a\|^2 \left\| \sum_1^n \pi(b_i c_i)\xi_i \right\|^2 - \left\| \sum_1^n \pi(db_i c_i)\xi_i \right\|^2
$$
$$
\leq \|a\|^2 \left\| \sum_1^n \pi(b_i c_i)\xi_i \right\|^2,
$$

as desired.   □

REMARK 2.4. As usual with the Effros-Hahn trick, the above argument shows even more: we only need to assume π is a *-homomorphism of I into the *-algebra of *not necessarily bounded* linear operators on H, such that

$$
\overline{\pi(I)H} = H \quad \text{and} \quad (\pi(a)\xi, \eta) = (\xi, \pi(a^*)\eta),
$$

and then the argument shows each $\pi(a)$ is automatically bounded.

DEFINITION 2.5. Say that a family $\{B_e\}_{e \in E}$ of closed ideals of a C^*-algebra B is *upward-directed* if for all $e, f \in E$ there exists $g \in E$ such that $B_e \cup B_f \subset B_g$.

The following elementary result allows us to paste together consistent representations of an upward-directed family of ideals.

LEMMA 2.6. *Let* $\{B_e\}_{e \in E}$ *be an upward-directed family of closed ideals with dense span in a* C^**-algebra* B*, and suppose that for each* $e \in E$ *we have a representation* π_e *of* B_e *on a common Hilbert space* H*, such that*
  (i) $\pi_e = \pi_f | B_e$ *whenever* $B_e \subset B_f$*, and*
  (ii) $\text{span}_{e \in E} \pi_e(B_e)H$ *is dense in* H*.*
Then there is a unique representation of B *on* H *which extends every* π_e*.*

PROOF. By upward-directedness the union $\bigcup_{e \in E} B_e$ is a dense ideal of B. The consistency condition (i) guarantees that the union of the π_e's is a representation π of $\bigcup_{e \in E} B_e$, which is of course contractive since each π_e is. The nondegeneracy condition (ii) shows π extends uniquely to a representation of B.   □

REMARK 2.7. We did not need to appeal to Lemma 2.3 to extend the representation π from $\bigcup_{e \in E} B_e$ to B since the hypotheses already told us π was contractive.

3. Groupoid actions

Throughout, G will be a locally compact Hausdorff groupoid with a Haar system. After the next few paragraphs G will be r-discrete. We should comment a little on the still-evolving definition of this term. Renault [Ren1, Definition 1.2.6] defined a locally compact Hausdorff groupoid G to be r-discrete if the unit space G^0 is open in G. However, this does not quite give all that one wants, in particular a Haar system. In fact, Renault proved that a groupoid which is r-discrete in his sense has a Haar system if and only if the range and domain maps r and d are local homeomorphisms, and in this case counting measures give a Haar system. The current fashion is to build this into the definition of r-discrete. Perhaps one of the most elegant 'modern' definitions is Paterson's [Pat]: for a locally compact (*not necessarily Hausdorff*) groupoid G, Paterson defines G^{op} as the set of open Hausdorff subsets U of G such that $r|U$ and $d|U$ are homeomorphisms onto open subsets of G, and he calls G r-discrete if G^{op} is a base for the topology of G. This is compatible with Renault's definition when G is Hausdorff, and in this case (which is our primary concern) the G^{op} condition just means the range and domain maps r and d are local homeomorphisms. Throughout this paper, when we say G is r-*discrete* we will always mean G is locally compact and Hausdorff, and r and d are local homeomorphisms. In particular, in an r-discrete groupoid G the range and domain maps r and d are open, so the elements of G^{op} are just the open G-sets, also called *bisections* (recall that a G-set is defined as a subset of G on which r and d are injective). It was with some hesitation that we imposed the Hausdorff condition; many r-discrete groupoids occurring in nature are non-Hausdorff, and it would seem reasonable to expect that our results hold for all such groupoids. However, our techniques seem to depend upon Hausdorffness; we intend to explore this in future work.

In [Ren2] Renault developed a notion of actions and crossed products of groupoids in the C^*-category, and we will require a slight generalization of his definition of action. Algebraically, an *action* of G is a functor α from G to the category of C^*-algebras and homomorphisms, with $\alpha_x : A_{d(x)} \to A_{r(x)}$ for $x \in G$. Of course, since G has inverses, by functoriality each α_x will be an isomorphism of $A_{d(x)}$ onto $A_{r(x)}$. Topologically, the collection $\mathscr{A} = \{A_u\}_{u \in G^0}$ of C^*-algebras must be glued together so we can formulate a continuity condition. Renault requires $\mathscr{A}$ to be a continuous C^*-bundle, but we will relax this to *upper* semicontinuity. So, we have a continuous, open surjection of $\mathscr{A}$ onto G^0, and the norm function $\|\cdot\|$ on $\mathscr{A}$ is only upper semicontinuous (see [Bla], [Dix], [DG], [Nil], and [Rie]). Pulling back via the domain map $d : G \to G^0$, we get a Banach bundle $d^* \mathscr{A} = \{(x, a) \in G \times \mathscr{A} : a \in A_{d(x)}\}$ over G. The continuity condition for α is that the map $(x, a) \mapsto \alpha_x(a)$ from $d^* \mathscr{A}$ to $\mathscr{A}$ be continuous. For his main results, Renault also requires separability assumptions, partly because he appeals to direct integral theory. Since we will not need direct integrals, we can dispense here

with separability hypotheses.

We often need to work with sections, rather than the bundle itself. Recall that upper semicontinuity of the C^*-bundle $\mathscr{A}$ can be equivalently expressed as the condition that for every f in the C_0-section algebra $\Gamma_0(\mathscr{A})$ the map $u \mapsto \|f(u)\|$ is upper semicontinuous. We will need the following characterization of continuity for actions in terms of compactly supported sections.

LEMMA 3.1. *Let $\mathscr{A}$ be an upper semicontinuous C^*-bundle over G^0, and let α be a functor from G to the C^*-category with $\alpha_x : A_{d(x)} \to A_{r(x)}$ for each $x \in G$. Then α is continuous (hence an action) if and only if for all $x \in G$, $a \in A_{d(x)}$, and $f, g \in \Gamma_c(\mathscr{A})$ with $f(r(x)) = \alpha_x(a)$ and $g(d(x)) = a$,*

$$\lim_{y \to x} \|f(r(y)) - \alpha_y(g(d(y)))\| = 0.$$

PROOF. First assume α is continuous, and fix x, a, f, g as above. As $y \to x$ we have $r(y) \to r(x)$, so $f(r(y)) \to f(r(x)) = \alpha_x(a)$ by continuity. Similarly, $g(d(y)) \to g(d(x)) = a$, so $\alpha_y(g(d(y))) \to \alpha_x(a)$ by continuity of α. Hence

$$0 = \|f(r(x)) - \alpha_x(a)\| \geq \limsup_{y \to x} \|f(r(y)) - \alpha_y(g(d(y)))\|$$

by upper semicontinuity, giving $\|f(r(y)) - \alpha_y(g(d(y)))\| \to 0$.

Conversely, let $(x, a) \in d^*\mathscr{A}$, and let V be a neighborhood of $\alpha_x(a)$ in $\mathscr{A}$. Pick $f, g \in \Gamma_c(\mathscr{A})$ with $f(r(x)) = \alpha_x(a)$ and $g(d(x)) = a$. By [DG, page 10] we may assume

$$V = \bigcup_{u \in U} \{b \in A_u : \|f(u) - b\| < \epsilon\}$$

for some neighborhood U of $r(x)$ in G^0 and some $\epsilon > 0$. Assuming

$$\lim_{y \to x} \|f(r(y)) - \alpha_y(g(d(y)))\| = 0,$$

we can find a neighborhood N of x in G such that $r(N) \subset U$ and

$$\|f(r(y)) - \alpha_y(g(d(y)))\| < \frac{\epsilon}{2} \qquad \text{for } y \in N.$$

Put

$$W = \bigcup_{u \in d(N)} \left\{b \in A_u : \|g(u) - b\| < \frac{\epsilon}{2}\right\}.$$

Then $N * W := (N \times W) \cap d^* \mathscr{A}$ is a neighborhood of (x, a) in $d^* \mathscr{A}$, and for $(y, b) \in N * W$ we have

$$
\begin{aligned}
&\left\| f(r(y)) - \alpha_y(b) \right\| \\
&\leq \left\| f(r(y)) - \alpha_y(g(d(y))) \right\| + \left\| \alpha_y(g(d(y))) - \alpha_y(b) \right\| \\
&< \frac{\epsilon}{2} + \left\| g(d(y)) - b \right\| < \epsilon,
\end{aligned}
$$

so $\alpha_y(b) \in V$, and we have shown $(y, b) \mapsto \alpha_y(b)$ is continuous at (x, a).   □

Now let α be an action of G on $\mathscr{A}$, and let $r^* \mathscr{A} = \{(a, x) \in \mathscr{A} \times G : a \in A_{r(x)}\}$ be the pull-back via the range map $r \colon G \to G^0$. Renault [Ren2] shows that the vector space $\Gamma_c(r^* \mathscr{A})$ of compactly supported continuous sections becomes a *-algebra with the operations

$$
(fg)(x) = \int f(y) \alpha_y (g(y^{-1}x)) \, d\lambda^{r(x)}(y) \quad \text{and} \quad f^*(x) = \alpha_x (f(x^{-1}))^*.
$$

He then defines (in [Ren2, Section 5]) the *crossed product* $\mathscr{A} \times_\alpha G$ as the Hausdorff completion of $\Gamma_c(r^* \mathscr{A})$ in the supremum of the C^*-seminorms

$$
f \mapsto \| \Pi(f) \|,
$$

where Π runs over all representations of $\Gamma_c(r^* \mathscr{A})$ which are continuous from the inductive limit topology to the weak operator topology. He shows as a consequence of his decomposition theorem [Ren2, Théorème 4.1] that this supremum is finite. We will give an independent proof of this for *r-discrete* groupoids, as an application of our isomorphism (see Theorem 7.1) between $\Gamma_c(r^* \mathscr{A})$ and a *-algebra associated with a corresponding inverse semigroup action. In fact, this will show that for *r*-discrete groupoids the crossed product is the *enveloping C^*-algebra* of $\Gamma_c(r^* \mathscr{A})$, because the inductive limit continuity is automatic, as we show in the following proposition.

PROPOSITION 3.2. *If α is an action of an r-discrete groupoid G on an upper semi-continuous C^*-bundle $\mathscr{A}$, then every representation of $\Gamma_c(r^* \mathscr{A})$ is continuous from the inductive limit topology to the weak operator topology.*

PROOF. Let Π be a representation of $\Gamma_c(r^* \mathscr{A})$ on a Hilbert space H. For each subset T of G define

$$
\Gamma_T(r^* \mathscr{A}) = \{ f \in \Gamma_c(r^* \mathscr{A}) : \operatorname{supp} f \subset T \}.
$$

Claim: it suffices to show that for each $s \in G^{op}$ the restriction $\Pi|_{\Gamma_s(r^* \mathscr{A})}$ is continuous from the (sup) norm topology to the weak operator topology. To see this, let

K be a compact subset of G, and take $\xi, \eta \in H$. Find $s_1, \ldots, s_n \in G^{op}$ such that $K \subset \bigcup_1^n s_i$, and choose a partition of unity $\{\phi_i\}_1^n$ subordinate to the open cover $\{s_i\}_1^n$ of K, so that $\text{supp}\,\phi_i \subset s_i$ and $\sum_1^n \phi_i = 1$ on K. For $f \in \Gamma_K(r^*\mathscr{A})$ we have

$$(\Pi(f)\xi, \eta) = \left(\Pi\left(\sum_1^n f\,\phi_i\right)\xi, \eta\right) = \sum_1^n (\Pi(f\,\phi_i)\xi, \eta).$$

Now just observe that the map $f \mapsto f\,\phi_i \colon \Gamma_K(r^*\mathscr{A}) \to \Gamma_{s_i}(r^*\mathscr{A})$ is norm continuous, and the claim follows.

So, fix $s \in G^{op}$. We will in fact show $\Pi|_{\Gamma_s(r^*\mathscr{A})}$ is continuous for the *norm* topologies of $\Gamma_s(r^*\mathscr{A})$ and $\mathscr{L}(H)$. Note that for $f \in \Gamma_s(r^*\mathscr{A})$ we have

$$f^*f(x) = \sum_{r(y)=r(x)} \alpha_y\big(f(y^{-1})^*f(y^{-1}x)\big),$$

and for any nonzero term in this sum we have $y \in s^*$ and $x \in ys \subset s^*s = d(s)$. Hence, the product f^*f is in $\Gamma_{d(s)}(r^*\mathscr{A})$, and for each $x \in d(s)$ we have $f^*f(x) = \alpha_y(f(y)^*f(y))$, where y is the unique element of s with $d(y) = x$. Now, $\Gamma_{d(s)}(r^*\mathscr{A})$ is an ideal in the C_0-section algebra $\Gamma_0((r^*\mathscr{A})|_{G^0})$, which in turn is a C^*-subalgebra of $\mathscr{A} \times_\alpha G$. Consequently, Lemma 2.3 tells us the restriction $\Pi|_{\Gamma_{d(s)}(r^*\mathscr{A})}$ is contractive, so

$$\|\Pi(f)\|^2 \le \|f^*f\|_\infty = \|f\|_\infty^2,$$

since

$$\sup_{x\in d(s)} \|f^*f(x)\| = \sup_{y\in s} \|f(y)^*f(y)\| = \sup_{y\in s} \|f(y)\|^2.$$

$\square$

4. Inverse semigroup actions

Here we mainly follow the conventions of [Pat, Sie2] and [Sie1]. Let B be a C^*-algebra. A *partial automorphism* [Exe1] of B is an isomorphism between two (closed) ideals of B. The set of all partial automorphisms of B forms an inverse semigroup PAut B under composition. An *action* of an inverse semigroup S on B is a homomorphism $\beta\colon S \to$ PAut B such that the domain ideals of the partial automorphisms $\{\beta_s\}_{s\in S}$ are upward-directed and have dense span in B. Of course, for $s \in S$ the domain of the partial automorphism β_s will also be the range of β_{s^*}, and furthermore will only depend upon the domain idempotent $d(s) := s^*s$. Thus, for each idempotent $e \in E_S$ we have an ideal B_e of B, and each β_s is an isomorphism

of $B_{d(s)}$ onto $B_{r(s)}$. Recall from Section 2 that the upward-directedness of the ideals $\{B_e\}_{e \in E_S}$ means that any two of them are contained in a third, and is automatic if E_S itself is upward-directed in the sense that for all $e, f \in E_S$ there exists $g \in E_S$ such that $e, f \leq g$.

A *representation* of S on a Hilbert space H is a $*$-homomorphism U of S to $\mathcal{L}(H)$ such that the span of the ranges of the operators $\{U_s\}_{s \in S}$ is dense in H. Of course, each U_s will be a partial isometry, since

$$U_s U_s^* U_s = U_s U_{s^*} U_s = U_{ss^*s} = U_s.$$

A *covariant representation* of an inverse semigroup action (B, S, β) is a pair (π, U) consisting of a nondegenerate representation π of B and a representation U of S, both on the same Hilbert space H, such that

$$U_e H = \pi(B_e) H \qquad \text{for } e \in E_S$$

and

$$U_s \pi(b) U_s^* = \pi(\beta_s(b)) \qquad \text{for } b \in B_{d(s)}.$$

Note that when we are checking whether a pair (π, U) is covariant, we do not need to verify the span of the ranges of the U_s is dense, since this follows from nondegeneracy of π and the covariance condition.

The disjoint union of $\{B_e\}_{e \in E_S}$ forms a Banach bundle $\mathcal{B}$ over the discrete space E_S. Pulling back via the range map $r : S \to E_S$, we get a Banach bundle $r^* \mathcal{B} = \{(b, s) \in \mathcal{B} \times S : b \in B_{r(s)}\}$ over S. The set $\Gamma_c(r^* \mathcal{B})$ of finitely supported sections becomes a $*$-algebra with operations defined on the generators by

$$(b, s)(c, t) = (\beta_s(\beta_{s^*}(b)c), st) \quad \text{and} \quad (b, s)^* = (\beta_{s^*}(b^*), s^*),$$

and then extended additively. For every covariant representation (π, U) of (B, S, β) the *integrated form* of (π, U) is the representation Π of $\Gamma_c(r^* \mathcal{B})$ defined by

$$\Pi\left(\sum_1^n (b_i, s_i)\right) = \sum_1^n \pi(b_i) U_{s_i}.$$

The integrated form Π is nondegenerate, and we have

$$\Pi(\Gamma_c(r^* \mathcal{B})) = \operatorname*{span}_{s \in S} \pi(B_{r(s)}) U_s \quad \text{and} \quad \left\| \Pi\left(\sum_1^n (b_i, s_i)\right) \right\| \leq \sum_1^n \|b_i\|.$$

The *crossed product* of (B, S, β) is the Hausdorff completion $B \times_\beta S$ of $\Gamma_c(r^* \mathcal{B})$ in the supremum of the C^*-seminorms

$$f \mapsto \|\Pi(f)\|,$$

where Π runs over the integrated forms of all covariant representations. Warning: there is some collapsing when $\Gamma_c(r^*\mathcal{B})$ is mapped into $B \times_\beta S$, since whenever Π is the integrated form of a covariant representation we have

$$\Pi(b, s) = \Pi(b, t) \qquad \text{for } b \in B_{r(s)}, s \leq t$$

(see [Pat, Proposition 3.3.2], [Sie1, Lemma 3.4.4], [Sie2, Lemma 4.5]). For $b \in B_{r(s)}$ let $[b, s]$ denote the image of (b, s) in $B \times_\beta S$, so that

$$B \times_\beta S = \overline{\text{span}}\{[b, s] : b \in B_{r(s)}, s \in S\}.$$

For every covariant representation (π, U) there is a unique representation $\pi \times U$ of $B \times_\beta S$ such that

$$(\pi \times U)[b, s] = \pi(b)U_s \qquad \text{for } b \in B_{r(s)}, s \in S,$$

and in fact this gives a bijection between the covariant representations of the action (B, S, β) and the nondegenerate representations of the C^*-algebra $B \times_\beta S$ [Pat, Corollary 3.3.1], [Sie1, Proposition 3.4.7].

Caution: the crossed product $B \times_\beta S$ is (usually) *not* the enveloping C^*-algebra of $\Gamma_c(r^*\mathcal{B})$; although it is true (and not hard to show) that $\Gamma_c(r^*\mathcal{B})$ does in fact have an enveloping C^*-algebra, there can in general be representations of $\Gamma_c(r^*\mathcal{B})$ which are not integrated forms of covariant representations [Sie2, Example 4.9]. Which representations of $\Gamma_c(r^*\mathcal{B})$ are integrated forms? Paterson [Pat] has found an answer:

DEFINITION 4.1. A representation Π of $\Gamma_c(r^*\mathcal{B})$ is called *coherent* if

$$\Pi(b, e) = \Pi(b, f) \qquad \text{whenever } b \in B_e \text{ and } e \leq f.$$

Actually, Paterson requires $\Pi(b, e) = \Pi(b, f)$ whenever $b \in B_e B_f$, which is clearly equivalent to the above logically weaker condition, and he builds this condition right into his definition of a representation of $\Gamma_c(r^*\mathcal{B})$. He also requires the map Π to be bounded in the L^1-norm

$$\left\| \sum(b_s, s) \right\|_1 := \sum \|b_s\| \left(= \sum \|(b_s, s)\| \right),$$

although this follows automatically from Proposition 4.3 below. Clearly, Π is coherent if and only if its kernel contains the ideal generated by the subspace

$$\text{span}\{(b, e) - (b, f) : b \in B_e, e, f \in E_S, e \leq f\}.$$

We need to identify this ideal explicitly. Recall that the partial order in an inverse semigroup is given by

$$s \leq t \quad \text{if and only if} \quad s = ss^*t,$$

which should be regarded as saying s is a 'restriction' of t.

LEMMA 4.2. *The ideal of $\Gamma_c(r^*\mathcal{B})$ generated by $\{(b, e) - (b, f) : b \in B_e, e, f \in E_S, e \leq f\}$ coincides with the subspace*

$$I_\beta := \mathrm{span}\{(b, s) - (b, t) : b \in B_s, s \leq t\}.$$

PROOF. It is easy to check that I_β is a self-adjoint left ideal. Hence, it suffices to observe that for $b \in B_{r(s)}$ and $s \leq t$ we can factor $b = cd$ for some $c, d \in B_{r(s)}$, and then

$$(b, s) - (b, t) = (cd, ss^*t) - (cd, tt^*t) = (c, ss^*)(d, t) - (c, tt^*)(d, t)$$
$$= \big((c, ss^*) - (c, tt^*)\big)(d, t).$$

$\square$

Is there a coherent representation whose kernel is I_β? We do not know in general, but it follows from Theorem 7.1 below that the answer is yes for representations related to groupoids.

The following proposition, which appears in various forms in [Pat, Corollary 3.3.1], and [Sie1, Proposition 3.4.7], establishes a bijective correspondence between coherent, nondegenerate representations of $\Gamma_c(r^*\mathcal{B})$ and covariant representations of (B, S, β). We include the outline of the argument for the convenience of the reader; in particular, this is the first time the automatic continuity of representations of $\Gamma_c(r^*\mathcal{B})$ has been adequately handled.

PROPOSITION 4.3 ([Pat, Sie1]). *Every representation of $\Gamma_c(r^*\mathcal{B})$ is contractive on each fiber $(B_{r(s)}, s)$. A nondegenerate representation Π of $\Gamma_c(r^*\mathcal{B})$ is the integrated form of a covariant representation of (B, S, β) if and only if Π is coherent.*

PROOF. Let Π be a representation of $\Gamma_c(r^*\mathcal{B})$. For each $e \in E_S$ define a representation π_e of B_e by

$$\pi_e(b) = \Pi(b, e).$$

The first statement follows from the estimate

$$\|\Pi(b, s)\|^2 = \|\Pi(b, s)^*\Pi(b, s)\| = \|\Pi(\beta_{s^*}(b^*), s^*)\Pi(b, s)\|$$
$$= \big\|\Pi\big((\beta_{s^*}(b^*), s^*)(b, s)\big)\big\| = \big\|\Pi\big(\beta_{s^*}(\beta_s\beta_{s^*}(b^*)b), s^*s\big)\big\|$$
$$= \|\Pi(\beta_{s^*}(b^*b), d(s))\| = \big\|\pi_{d(s)}(\beta_{s^*}(b^*b))\big\|$$
$$\leq \|\beta_{s^*}(b^*b)\| = \|b^*b\| = \|b\|^2 = \|(b, s)\|^2.$$

For the other part, first assume Π is the integrated form of a covariant representation (π, U). Then for $b \in B_e$ and $e \leq f$ in E_S,

$$\Pi(b, e) = \pi(b)U_e = \pi(b)U_{ef} = \pi(b)U_eU_f = \pi(b)U_f = \Pi(b, f),$$

since $U_e H = \pi(B_e)H$, and so Π is coherent.

Conversely, assume Π is a nondegenerate and coherent representation of $\Gamma_c(r^*\mathscr{B})$ on H, and recall the above definition of the π_e. We have $\pi_e = \pi_f | B_e$ whenever $B_e \subset B_f$, by coherence, and $\text{span}_e\, \pi_e(B_e)H$ is dense in H, so by Lemma 2.6 there is a unique representation π of B on H which extends every π_e.

To get the other half of our covariant representation we use the construction of McClanahan [McC, Proposition 2.8] (where the context was partial actions of groups), which was adapted to inverse semigroup actions by the second author [Sie2, proof of Proposition 4.7]. Fix $s \in S$ and a bounded approximate identity $\{b_i\}$ for $B_{r(s)}$. Claim: the net $\{\Pi(b_i, s)\}$ is Cauchy in the strong operator topology. To see this, take $\xi \in H$. Then

$$
\begin{aligned}
\left\| (\Pi(b_i, s) - \Pi(b_j, s))\xi \right\|^2 &= \left((\Pi(b_i, s) - \Pi(b_j, s))^* (\Pi(b_i, s) - \Pi(b_j, s))\xi, \xi \right) \\
&= \left(\Pi\Big(((b_i, s) - (b_j, s))^* ((b_i, s) - (b_j, s)) \Big)\xi, \xi \right) \\
&= \Big(\Pi\big((\beta_{s^*}(b_i), s^*)(b_i, s) + (\beta_{s^*}(b_j), s^*)(b_j, s) \\
&\quad - (\beta_{s^*}(b_i), s^*)(b_j, s) - (\beta_{s^*}(b_j), s^*)(b_i, s) \big)\xi, \xi \Big) \\
&= \left(\pi_{d(s)}\big(\beta_{s^*}(b_i^2 + b_j^2 - b_i b_j - b_j b_i) \big)\xi, \xi \right) \\
&\to 0,
\end{aligned}
$$

since $\{b_i^2\}$, $\{b_j^2\}$, $\{b_i b_j\}$, and $\{b_j b_i\}$ are all bounded approximate identities (the latter two for the product direction) for $B_{r(s)}$, hence their images under β_{s^*} are approximate identities for $B_{d(s)}$. This verifies the claim, so we can let U_s be the strong operator limit of $\{\Pi(b_i, s)\}$.

Since multiplication and involution are jointly strong operator continuous on bounded sets,

$$
\begin{aligned}
U_s^* U_s &= \lim \Pi(b_i, s)^* \Pi(b_i, s) = \lim \Pi(\beta_{s^*}(b_i^2), d(s)) \\
&= \lim \pi_{d(s)}(\beta_{s^*}(b_i^2)) \\
&= \text{projection on } \pi_{d(s)}(B_{d(s)})H.
\end{aligned}
$$
(4.1)

Hence, U_s is a partial isometry with initial subspace $\pi_{d(s)}(B_{d(s)})H$. In particular, the following computation shows U_s is independent of the choice of the bounded approximate identity $\{b_i\}$: for $c \in B_{d(s)}, \xi \in H$

$$
\begin{aligned}
U_s \pi_{d(s)}(c)\xi &= \lim \Pi(b_i, s)\Pi(c, d(s))\xi = \lim \Pi(\beta_s(\beta_{s^*}(b_i)c), s)\xi \\
&= \Pi(\beta_s(c), s)\xi,
\end{aligned}
$$
(4.2)

since $\{\beta_{s^*}(b_i)\}$ is a bounded approximate identity for $B_{d(s)}$ and Π is continuous on the fiber $(B_{r(s)}, s)$ of $r^*\mathscr{B}$.

To see that U is multiplicative, take bounded approximate identities $\{b_i\}$ for $B_{r(s)}$ and $\{c_j\}$ for $B_{r(t)}$:

$$U_s U_t = \lim \Pi(b_i, s)\Pi(c_j, t) = \lim \Pi(\beta_s(\beta_{s^*}(b_i)c_j), st) = U_{st},$$

since $\{\beta_s(\beta_{s^*}(b_i)c_j)\}$ is a bounded approximate identity for $\beta_s(B_{d(s)}B_{r(t)}) = B_{r(st)}$. A similar computation shows $U_s^* = U_{s^*}$.

For covariance, the computation (4.1) implies $U_e H = \pi_e(B_e)H$ for all $e \in E_S$, and for $b \in B_{d(s)}$ the computation (4.2) shows

$$
(4.3) \qquad
\begin{aligned}
U_s \pi(b) &= \Pi(\beta_s(b), s) = \Pi(\beta_{s^*}(\beta_s(b^*)), s^*)^* \\
&= \big(U_{s^*}\pi(\beta_s(b^*))\big)^* = \pi(\beta_s(b^*))^* U_{s^*}^* = \pi(\beta_s(b)) U_s.
\end{aligned}
$$

π is nondegenerate, since if $\pi(B)\xi = 0$ then for all $s \in S$, $b \in B_{d(s)}$ we have

$$0 = U_s \pi_{d(s)}(b)\xi = \Pi(\beta_s(b), s)\xi,$$

so $\xi = 0$ by nondegeneracy of Π. Finally, the computation (4.3) also implies $\Pi(b, s) = \pi(b) U_s$ for $b \in B_{r(s)}$, so Π is the integrated form of (π, U). $\square$

The above proposition allows us to express the crossed product as an enveloping C^*-algebra:

COROLLARY 4.4. *The crossed product $B \times_\beta S$ is the enveloping C^*-algebra of the $*$-algebra $\Gamma_c(r^*\mathscr{B})/I_\beta$.*

We will actually need a technical generalization of the above result. Suppose that for each $e \in E_S$ we have a dense ideal B_e' of B_e, such that

$$\beta_s(B_{d(s)}') = B_{r(s)}' \qquad \text{for all } s \in S.$$

Write $\Gamma_c(r^*\mathscr{B}')$ for the linear span of $\{(b, s) \in \mathscr{B} \times S : b \in B_{r(s)}'\}$ in $\Gamma_c(r^*\mathscr{B})$. Note that $\Gamma_c(r^*\mathscr{B}')$ is a $*$-subalgebra (in fact, an ideal) of $\Gamma_c(r^*\mathscr{B})$ which is dense for the pointwise convergence topology. We need to know that the $*$-algebra $\Gamma_c(r^*\mathscr{B}')$ determines the C^*-algebra $B \times_\beta S$.

LEMMA 4.5. *Every representation Π of $\Gamma_c(r^*\mathscr{B}')$ is continuous from the pointwise convergence topology to the norm topology of operators, and hence has a unique extension to a representation $\overline{\Pi}$ of $\Gamma_c(r^*\mathscr{B})$. Moreover, Π is coherent if and only if $\overline{\Pi}$ is.*

PROOF. For the first part it suffices to show Π is norm continuous on each fiber $(B'_{r(s)}, s)$ of $r^*\mathscr{B}'$. Each fiber $(B_{r(s)}, s)$ of $r^*\mathscr{B}$ is given the Banach space structure of $B_{r(s)}$. For every $e \in E_S$ define a representation π_e of B'_e by $\pi_e(b) = \Pi(b, e)$ for $b \in B'_e$. For $b \in B'_{r(s)}$ we have

$$
\begin{aligned}
\|\Pi(b, s)\|^2 = \|\Pi(b, s)^*\Pi(b, s)\| &= \left\|\Pi\big((b, s)^*(b, s)\big)\right\| \\
&= \left\|\Pi\big((\beta_{s^*}(b^*), s^*)(b, s)\big)\right\| = \|\Pi(\beta_{s^*}(b^*b), s^*s)\| \\
&= \|\pi_{s^*s}(\beta_{s^*}(b^*b))\| \le \|\beta_{s^*}(b^*b)\|,
\end{aligned}
$$

by Lemma 2.3, since $B'_{d(s)}$ is an ideal of $B_{d(s)}$. So

$$
\|\Pi(b, s)\|^2 \le \|\beta_{s^*}(b^*b)\| = \|b^*b\| = \|b\|^2.
$$

For the coherence, obviously Π is coherent if $\overline{\Pi}$ is, so assume Π is coherent. Take $b \in B_{r(s)}$ and $s \le t$, and choose a bounded approximate identity $\{c_i\}$ for $B_{r(s)}$ which is contained in the dense ideal $B'_{r(s)}$. Then

$$
\overline{\Pi}(b, s) = \lim_i \Pi(c_i b, s) = \lim_i \Pi(c_i b, t) = \overline{\Pi}(b, t).
$$

$\square$

COROLLARY 4.6. *With the above notation, $B \times_\beta S$ is the enveloping C^*-algebra of the quotient*

$$
\Gamma_c(r^*\mathscr{B}') \big/ \big(I_\beta \cap \Gamma_c(r^*\mathscr{B}')\big).
$$

PROOF. This follows from the above lemma and Proposition 4.3, since a representation of $\Gamma_c(r^*\mathscr{B}')$ is coherent if and only if it kills $I_\beta \cap \Gamma_c(r^*\mathscr{B}')$. $\square$

5. From groupoids to inverse semigroups

Let α be an action of an r-discrete groupoid G on an upper semicontinuous C^*-bundle $\mathscr{A}$. Recall that since G is r-discrete, the family G^{op} of open G-sets is a base for the topology of G. Further, G^{op} is an inverse semigroup with operations

$$
st = \{xy : (x, y) \in (s \times t) \cap G^2\} \quad \text{and} \quad s^* = \{x^{-1} : x \in s\}.
$$

Note that an element s of G^{op} has domain idempotent

$$
\begin{aligned}
d(s) = s^*s &= \{x^{-1}y : (x^{-1}, y) \in (s^{-1} \times s) \cap G^2\} \\
&= \{x^{-1}y : x, y \in s, r(x) = r(y)\} \\
&= \{x^{-1}x : x \in s\} = \{d(x) : x \in s\},
\end{aligned}
$$

and similarly s has range idempotent

$$r(s) = ss^* = \{r(x) : x \in s\}.$$

We want to associate to the groupoid action $(\mathscr{A}, G, \alpha)$ an inverse semigroup action (B, S, β). For B we take $\Gamma_0(\mathscr{A})$. To construct partial automorphisms of B, we will need the following elementary lemma, which is an easy consequence of, for example, [DG, Proposition 1.4].

LEMMA 5.1. *Let C and D be upper semicontinuous C^*-bundles over locally compact Hausdorff spaces X and Y, respectively. Let ϕ be a homeomorphism of X onto Y, and for each $x \in X$ let γ_x be an isomorphism of C_x onto $D_{\phi(x)}$. For $f \in \Gamma_c(C)$ and $y \in Y$ define*

$$\gamma(f)(y) = \gamma_{\phi^{-1}(y)}(f(\phi^{-1}(y))) \in D_y.$$

If $\gamma(\Gamma_c(C)) \subset \Gamma_0(D)$, then γ extends uniquely to an isomorphism of $\Gamma_0(C)$ onto $\Gamma_0(D)$. Moreover, the extension is given by the above formula for $f \in \Gamma_0(C)$.

For S we want to allow some flexibility; roughly speaking, we can take any sufficiently large inverse subsemigroup of G^{op}.

DEFINITION 5.2. We call an inverse subsemigroup S of G^{op} *full* if S is a base for the topology of G and E_S is upward-directed in the sense that every two elements of E_S have a common upper bound.

Take S to be any inverse subsemigroup of G^{op} which is full in the above sense. It might be useful to mention that to determine whether S is a base it is enough to check the idempotents; more precisely, an inverse subsemigroup S of G^{op} is a base for the topology of G if and only if S covers G and the idempotent semilattice E_S is a base for the topology of the unit space G^0.

Note that G^{op} is a full inverse subsemigroup of itself since it is a base for the topology of G, and the open G-sets in G^0 are just the open sets in G^0. For a more interesting example, consider a transformation groupoid $G = X \times H$ where H is a discrete group with identity e. If $\mathscr{B}$ is an upward-directed base for the topology of the locally compact space X then $S = \{U \times \{h\} : U \in \mathscr{B}, h \in H\}$ is full since it covers G and $E_S = \{U \times \{e\} : U \in \mathscr{B}\}$ is an upward-directed base for the topology of $G^0 = X \times \{e\}$.

The ideals of the semigroup action β will be given by

$$B_e = \{f \in \Gamma_0(\mathscr{A}) : f = 0 \quad \text{off} \quad e\} \qquad \text{for } e \in E_S.$$

The reader can immediately verify that each B_e is a closed ideal of $\Gamma_0(\mathscr{A})$, the span of these ideals is dense in $\Gamma_0(\mathscr{A})$, and the family $\{B_e\}_{e \in E_S}$ is upward-directed.

THEOREM 5.3. *Let α be an action of an r-discrete groupoid G on an upper semi-continuous C^*-bundle $\mathscr{A}$, and let S be a full inverse semigroup of open G-sets. Then there is a unique action β of S on $\Gamma_0(\mathscr{A})$ such that*

$$\beta_s(f)(u) = \begin{cases} \alpha_{us}(f(s^*us)) & \text{if } u \in r(s), \\ 0 & \text{else}, \end{cases}$$

for $s \in S$, $f \in B_{d(s)}$, and $u \in G^0$.

PROOF. We first show that the above formula defines an isomorphism $\beta_s : B_{d(s)} \xrightarrow{\cong} B_{r(s)}$, equivalently, an isomorphism of the C_0-section algebra $\Gamma_0(\mathscr{A}|_{d(s)})$ of the restricted bundle $\mathscr{A}|_{d(s)}$ onto $\Gamma_0(\mathscr{A}|_{r(s)})$, and for this we aim to apply the above lemma. The map $u \mapsto sus^*$ gives a homeomorphism of $d(s)$ onto $r(s)$, with inverse $u \mapsto s^*us$. Moreover, $u \mapsto us = s(s^*us)$ is a homeomorphism of $r(s)$ onto s, and $d(us) = s^*us$, and similarly for $u \mapsto su: d(s) \to s$. For each $u \in d(s)$, α_{su} is an isomorphism of A_u onto A_{sus^*}. Thus, the proposition follows from the above lemma once we verify that if $f \in \Gamma_c(\mathscr{A}|_{d(s)})$ then $\beta_s(f) \in \Gamma_0(\mathscr{A}|_{r(s)})$. The continuity properties of f and α ensure that $\beta_s(f)$ is a continuous section. Also, if $u \notin s(\text{supp } f)s^*$ then $\beta_s(f)(u) = 0$, so $\beta_s(f)$ has compact support.

It remains to check that β is a homomorphism. For $s, t \in S$ the domain of $\beta_s\beta_t$ is

$$\beta_t^{-1}(B_{d(s)} \cap B_{r(t)}) = \beta_{t^*}(B_{d(s) \cap r(t)}) = B_{t^*d(s)r(t)t} = B_{t^*s^*st} = B_{d(st)},$$

which is the domain of β_{st}. For $f \in B_{d(st)}$ and $u \in r(st)$ we have $u \in r(s)$ and $s^*us \in r(t)$, so

$$\beta_s\beta_t(f)(u) = \alpha_{us}(\beta_t(f)(s^*us)) = \alpha_{us}(\alpha_{s^*ust}(f(t^*s^*ust)))$$
$$= \alpha_{uss^*ust}(f(t^*s^*ust)) = \beta_{st}(f)(u),$$

since

$$uss^*ust = uust = ust,$$

and this is enough to show $\beta_s\beta_t = \beta_{st}$. ☐

6. From inverse semigroups to groupoids

As in the preceding section, let G be an r-discrete groupoid, and let S be a full inverse semigroup of open G-sets. Suppose we are given an action β of S on a C^*-algebra B. We want to construct an action of G from which (B, S, β) arises as in the

construction of the preceding section. We first need to find an upper semicontinuous C^*-bundle $\mathscr{A}$ over G^0 such that $B \cong \Gamma_0(\mathscr{A})$. We know from [Bla, DG, Nil] and [Rie] (for example) that this is equivalent to B being a $C_0(G^0)$-algebra, that is, to the existence of a faithful, nondegenerate homomorphism of $C_0(G^0)$ into the central multipliers $ZM(B)$. So, assume we have an injective, nondegenerate homomorphism $\phi\colon C_0(G^0) \to ZM(B)$. For $u \in G^0$ put

$$I_u = \{f \in C_0(G^0) : f(u) = 0\}$$
$$K_u = \phi(I_u)B \qquad \text{(a closed ideal of } B)$$
$$A_u = B/K_u.$$

Then put $\mathscr{A} = \bigcup_{u \in G^0} A_u$, and define $\Phi\colon B \to \prod_{u \in G^0} A_u$ by

$$\Phi(b)(u) = b + K_u.$$

Then there is a unique topology on $\mathscr{A}$ making $\mathscr{A}$ an upper semicontinuous C^*-bundle and each $\Phi(b)$ a continuous section, and moreover Φ is an isomorphism of B onto $\Gamma_0(\mathscr{A})$.

We need to relate the homomorphism ϕ to the action (B, S, β). For $e \in E_S$ define the ideal

$$C_e = \{f \in C_0(G^0) : f = 0 \text{ off } e\}$$

of $C_0(G^0)$. For our purposes, the appropriate connection between ϕ and β is

(6.1) $$\phi(C_e)B = B_e \qquad \text{for } e \in E_S,$$

so we assume this henceforth. Although we do not need it, we point out that (6.1) implies ϕ is equivariant for β and an obvious action of S on $C_0(G^0)$.

To simplify the writing, we use the isomorphism $\Phi\colon B \to \Gamma_0(\mathscr{A})$ to *replace B by* $\Gamma_0(\mathscr{A})$. Then β is an action of S on $\Gamma_0(\mathscr{A})$, and the homomorphism $\phi\colon C_0(G^0) \to ZM(B)$ becomes the canonical embedding of $C_0(G^0)$ in $ZM(\Gamma_0(\mathscr{A}))$. Since

$$C_e\Gamma_0(\mathscr{A}) = \{f \in \Gamma_0(\mathscr{A}) : f = 0 \text{ off } e\} \qquad \text{for } e \in E_S,$$

our hypothesis (6.1) tells us the ideals associated with the inverse semigroup action β are $B_e = \{f \in \Gamma_0(\mathscr{A}) : f = 0 \text{ off } e\}$.

We want to construct an action α of the groupoid G on the C^*-bundle $\mathscr{A}$. For a start, if $x \in G$ we need an isomorphism α_x of $A_{d(x)}$ onto $A_{r(x)}$. Take any $s \in S$ such that $x \in s$ (and note that such s form a neighborhood base at x in G). For $u \in G^0$ we have

$$K_u = \{f \in \Gamma_0(\mathscr{A}) : f(u) = 0\}.$$

Furthermore, if $u \in e \in E_S$ we have

$$B = B_e + K_u.$$

Therefore,

$$A_u = B/K_u = (B_e + K_u)/K_u \cong B_e/(B_e K_u).$$

LEMMA 6.1. *With the above notation, there is a unique homomorphism α_x from $A_{d(x)}$ to $A_{r(x)}$ such that*

$$\alpha_x(f\,(d(x))) = \beta_s(f\,)(r(x)) \qquad for\ x \in s \in S, f \in B_{d(s)}.$$

PROOF. Identifying $A_{d(x)}$ with $B_{d(s)}/(B_{d(s)}K_{d(x)})$, and similarly for $A_{r(x)}$, the conclusion of the lemma is equivalent to the assertion that there is a homomorphism α_x making the diagram

$$
\begin{array}{ccc}
B_{d(s)} & \xrightarrow{\ \ \beta_s\ \ } & B_{r(s)} \\
\downarrow & & \downarrow \\
B_{d(s)}\big/\big(B_{d(s)}K_{d(x)}\big) & \xrightarrow{\ \ \alpha_x\ \ } & B_{r(s)}\big/\big(B_{r(s)}K_{r(x)}\big)
\end{array}
$$

commute. For this we must show

$$\beta_s\big(B_{d(s)}K_{d(x)}\big) \subset B_{r(s)}K_{r(x)}.$$

Take $f \in B_{d(s)}K_{d(x)}$. By density and continuity we can assume $\operatorname{supp} f \subset e$ for some $e \in E_S$ with $e \subset d(s)$ and $d(x) \notin e$. Then

$$\beta_s(f\,)(r(x)) = \beta_s\beta_e(f\,)(r(x)) = \beta_{se}(f\,)(r(x)) = 0,$$

since

$$\beta_{se}(f\,) \in B_{r(se)} = B_{ses^*} \quad \text{and} \quad r(x) = sd(x)s^* \notin ses^*.$$

$\square$

THEOREM 6.2. *Let G be an r-discrete groupoid, let S be a full inverse semigroup of open G-sets, and let β be an action of S on a C^*-algebra B. Assume that there is an injective, nondegenerate homomorphism ϕ of $C_0(G^0)$ into $ZM(B)$ such that $\phi(C_e)B = B_e$ for every $e \in E_S$. Then B is isomorphic to $\Gamma_0(\mathscr{A})$ for an upper semicontinuous C^*-bundle $\mathscr{A}$, and the map α defined in the above lemma is an action of G on $\mathscr{A}$.*

PROOF. By the above discussion, the only thing left to check is that α is an action. We must check functoriality and continuity. Let $(x, y) \in G^2$ and $a \in A_{d(y)}$. Choose $s, t \in S$ such that $x \in s$ and $y \in t$, and then choose $f \in B_{d(st)}$ such that $f(d(y)) = a$. Then $xy \in st$, so by the above lemma we have

$$
\begin{aligned}
\alpha_x \alpha_y (f(d(y))) = \alpha_x \big(\beta_t(f)(r(y))\big) &= \alpha_x \big(\beta_t(f)(d(x))\big) \\
&= \beta_s \beta_t(f)(r(x)) = \beta_{st}(f)(r(xy)) \\
&= \alpha_{xy} \big(f(d(xy))\big) = \alpha_{xy} \big(f(d(y))\big).
\end{aligned}
$$

Thus, α preserves compositions. To see that it preserves identities, that is, $\alpha_u = \mathrm{id}_{A_u}$ for $u \in G^0$, just note that α_u will be an idempotent surjection from A_u to itself.

For the continuity, we appeal to Lemma 3.1. Take $x \in G$, $a \in A_{d(x)}$, and $f, g \in \Gamma_c(\mathscr{A})$ with $f(r(x)) = \alpha_x(a)$ and $g(d(x)) = a$. Cutting down f and g, we can assume that $f \in B_{r(s)}$ and $g \in B_{d(s)}$ for some $s \in S$ with $x \in s$. Then for $y \in s$ we have

$$
\begin{aligned}
\big\| f(r(y)) - \alpha_y(g(d(y))) \big\| &= \big\| f(r(y)) - \beta_s(g)(r(y)) \big\| \\
&= \big\| (f - \beta_s(g))(r(y)) \big\|,
\end{aligned}
$$

which goes to 0 as $y \to x$ since the norm is upper semicontinuous and $f - \beta_s(g)$ is a continuous section which is 0 at $r(x)$. $\square$

7. The isomorphism

Let G be an r-discrete groupoid, and let S be a full inverse semigroup of open G-sets. In the preceding two sections we established a correspondence between the actions of G and certain actions of S. Our main result will be that the associated crossed products are isomorphic. To be specific, let α be an action of G on an upper semicontinuous C^*-bundle $\mathscr{A}$, and let β be the corresponding action of S on the C_0-section algebra $B := \Gamma_0(\mathscr{A})$. Recall that the ideals of β are given by

$$
B_e := \{ f \in \Gamma_0(\mathscr{A}) : f = 0 \text{ off } e \} \qquad \text{for } e \in E_S,
$$

and the partial automorphisms are given by

$$
\beta_s(f)(r(x)) = \alpha_x(f(d(x))) \qquad \text{for } x \in s \in S, f \in B_{d(s)}.
$$

We will show in Theorem 7.2 that $B \times_\beta S \cong \mathscr{A} \times_\alpha G$. At the same time, we will fulfill our promise from Section 3 by giving an independent proof that $\mathscr{A} \times_\alpha G$ exists, that is, the $*$-algebra $\Gamma_c(r^*\mathscr{A})$ has an enveloping C^*-algebra. We emphasize that our proof of

this is completely independent of Renault's (or anyone else's) decomposition theorem for representations of G. There is no measure theory (other than the hypothesis that counting measure is a Haar system on G); rather, the techniques are topological. As a (minor) byproduct, we have no separability requirements.

The crossed product $B \times_\beta S$ is the enveloping C^*-algebra of a quotient of the finitely supported section algebra of the pull-back bundle $r^*\mathscr{B}$. However, for our proof we will need to work with a subbundle having incomplete fibers. For $e \in E_S$ put $\Gamma_e(\mathscr{A}) = \{f \in \Gamma_c(\mathscr{A}) : \operatorname{supp} f \subset e\}$, and for $s \in S$ put

$$C_s = (\Gamma_{r(s)}(\mathscr{A}), s),$$

giving a subbundle $\mathscr{C} = \bigcup_{s \in S} C_s$ of $r^*\mathscr{B}$.

THEOREM 7.1. *With the above notation, the map* $\Psi: \Gamma_c(\mathscr{C}) \to \Gamma_c(r^*\mathscr{A})$ *defined on the generators by*

$$\Psi(b, s)(x) = \begin{cases} b(r(x)) & \text{if } x \in s, \\ 0 & \text{else} \end{cases}$$

(and extended additively) is a surjective *-*homomorphism with kernel* $I_\beta \cap \Gamma_c(\mathscr{C})$.

PROOF. Since the range map r takes each G-set $s \in S$ homeomorphically onto $r(s)$, Ψ takes each fiber C_s of $\mathscr{C}$ isometrically and isomorphically onto the linear subspace

$$\Gamma_s(r^*\mathscr{A}) := \{f \in \Gamma_c(r^*\mathscr{A}) : \operatorname{supp} f \subset s\}$$

of $\Gamma_c(r^*\mathscr{A})$. Since the elements of S cover the groupoid G, a standard partition of unity argument shows Ψ maps $\Gamma_c(\mathscr{C})$ onto $\Gamma_c(r^*\mathscr{A})$.

Fix $(b, s), (c, t) \in \mathscr{C}$. We have

$$\Psi\big((b, s)(c, t)\big)(x) = \Psi(\beta_s(\beta_{s^*}(b)c), st)(x)$$
$$= \begin{cases} \beta_s(\beta_{s^*}(b)c)(r(x)) & \text{if } x \in st, \\ 0 & \text{else.} \end{cases}$$

Now, if $x \in st$ then x factors *uniquely* as $x = yz$ with $y \in s$ and $z \in t$, and then

$$\beta_s(\beta_{s^*}(b)c)(r(x)) = \beta_s(\beta_{s^*}(b)c)(r(y)) = \alpha_y\big((\beta_{s^*}(b)c)(d(y))\big)$$
$$= \alpha_y\big(\beta_{s^*}(b)(d(y))c(d(y))\big)$$
$$= \alpha_y\big(\alpha_{y^{-1}}(b(r(y)))\big)\alpha_y\big(c(d(y))\big)$$
$$= b(r(y))\alpha_y\big(c(r(y^{-1}x))\big)$$
$$= \Psi(b, s)(y)\alpha_y\big(\Psi(c, t)(y^{-1}x)\big)$$
$$= \sum_{r(w)=r(x)} \Psi(b, s)(w)\alpha_w\big(\Psi(c, t)(w^{-1}x)\big)$$
$$= \big(\Psi(b, s)\Psi(c, t)\big)(x).$$

On the other hand,

$$\mathrm{supp}\big(\Psi(b, s)\Psi(c, t)\big) \subset \big(\mathrm{supp}\,\Psi(b, s)\big)\big(\mathrm{supp}\,\Psi(c, t)\big) \subset st,$$

so if $x \notin st$ then

$$0 = \big(\Psi(b, s)\Psi(c, t)\big)(x).$$

Hence, Ψ is multiplicative. A similar computation shows Ψ preserves adjoints, so Ψ is a *-homomorphism.

It remains to show the kernel of the map $\Psi \colon \Gamma_c(\mathscr{C}) \to \Gamma_c(r^*\mathscr{A})$ is the ideal $I_\beta \cap \Gamma_c(\mathscr{C})$. This will involve a couple of mildly fussy partition-of-unity arguments, so we have made an attempt to isolate the hard bit by factoring the map Ψ through an auxiliary bundle: let $\mathscr{X}$ be the bundle over S with (incomplete) fibers $\{(\Gamma_s(r^*\mathscr{A}), s)\}_{s \in S}$. Then define $\Theta \colon \Gamma_c(\mathscr{C}) \to \Gamma_c(\mathscr{X})$ and $\Lambda \colon \Gamma_c(\mathscr{X}) \to \Gamma_c(r^*\mathscr{A})$ by

$$\Theta(b, s) = (\Psi(b), s) \quad \text{and} \quad \Lambda(f, s) = f,$$

so that $\Psi = \Lambda \circ \Theta$. The map Θ is a linear isomorphism of $\Gamma_c(\mathscr{C})$ onto $\Gamma_c(\mathscr{X})$, since it is induced by the identity map on the base space S of the bundles $\mathscr{C}$ and $\mathscr{X}$, and by linear isomorphisms between the fibers $(\Gamma_{r(s)}(\mathscr{A}), s)$ and $(\Gamma_s(r^*\mathscr{A}), s)$. Moreover, Θ takes $I_\beta \cap \Gamma_c(\mathscr{C})$ onto the span of $\{(f, s) - (f, t) : f \in \Gamma_s(r^*\mathscr{A}), s \leq t\}$. Thus, it remains to show the kernel of Λ coincides with this span.

We first show

$$(7.1) \qquad \ker \Lambda = \mathrm{span}\{(f, s) - (f, t) : f \in \Gamma_{s \cap t}(r^*\mathscr{A})\}.$$

Let I denote the right hand side. Certainly $I \subset \ker \Lambda$. For the opposite inclusion, suppose $\Lambda\big(\sum_1^n (f_i, s_i)\big) = 0$. We need to show $\sum_1^n (f_i, s_i) \in I$. We have $\sum_1^n f_i = 0$, so if $n = 1$ then $f_1 = 0$ and so $\sum_1^n (f_i, s_i) = (f_1, s_1) = 0$. Hence, we can assume $n > 1$. Let Ω denote the family of subsets of $\{1, \dots, n\}$ with cardinality at least 2, and for $\omega \in \Omega$ put

$$V_\omega = \left(\bigcap_{i \in \omega} s_i\right) \setminus \left(\bigcup_{i \notin \omega} \mathrm{supp}\, f_i\right).$$

Then $\{V_\omega\}_{\omega \in \Omega}$ forms an open cover of $\bigcup_1^n \mathrm{supp}\, f_i$, since if $f_i(x) \neq 0$ then also $f_j(x) \neq 0$ for at least one $j \neq i$, and upon taking limits we get $\mathrm{supp}\, f_i \subset \bigcup_{j \neq i} \mathrm{supp}\, f_j$, so if $x \in \mathrm{supp}\, f_i$ and $\omega = \{j : x \in s_j\}$, then $\omega \in \Omega$ and $x \in V_\omega$. Choose a partition of unity $\{\phi_\omega\}_{\omega \in \Omega}$ subordinate to the open cover $\{V_\omega\}_{\omega \in \Omega}$ of $\bigcup_1^n \mathrm{supp}\, f_i$. Note that whenever $i \notin \omega$ we have $f_i \phi_\omega = 0$ since $\mathrm{supp}\, f_i \cap V_\omega = \emptyset$. Hence,

$$\sum_{i \in \omega} f_i \phi_\omega = \sum_{i=1}^n f_i \phi_\omega = (0)\phi_\omega = 0 \qquad \text{for all } \omega \in \Omega.$$

We have

$$\sum_1^n (f_i, s_i) = \sum_1^n \left(\sum_{\omega \in \Omega} f_i \phi_\omega, s_i \right) = \sum_\omega \sum_1^n (f_i \phi_\omega, s_i)$$
$$= \sum_\omega \sum_{i \in \omega} (f_i \phi_\omega, s_i).$$

Fix $\omega \in \Omega$, and pick any two distinct elements j, k of ω. Then

$$\sum_{i \in \omega} (f_i \phi_\omega, s_i) = (f_j \phi_\omega, s_j) + (f_k \phi_\omega, s_k) + \sum_{i \in \omega \setminus \{j,k\}} (f_i \phi_\omega, s_i)$$
$$= (f_j \phi_\omega, s_j) + (f_k \phi_\omega, s_k) + \sum_{i \in \omega \setminus \{j,k\}} (f_i \phi_\omega, s_i)$$
$$+ \sum_{i \in \omega \setminus \{j,k\}} (f_i \phi_\omega, s_k) - \sum_{i \in \omega \setminus \{j,k\}} (f_i \phi_\omega, s_k)$$
$$= (f_j \phi_\omega, s_j) + \sum_{i \in \omega \setminus \{j\}} (f_i \phi_\omega, s_k)$$
$$+ \sum_{i \in \omega \setminus \{j,k\}} ((f_i \phi_\omega, s_i) - (f_i \phi_\omega, s_k))$$
$$= (f_j \phi_\omega, s_j) - (f_j \phi_\omega, s_k)$$
$$+ \sum_{i \in \omega \setminus \{j,k\}} ((f_i \phi_\omega, s_i) - (f_i \phi_\omega, s_k)),$$

because $f_j \phi_\omega + \sum_{i \in \omega \setminus \{j\}} f_i \phi_\omega = 0$. Since $\operatorname{supp} f_i \phi_\omega \subset s_l$ for every $i, l \in \omega$, we conclude that $\sum_{i \in \omega} (f_i \phi_\omega, s_i)$ is an element of I, so we have shown (7.1).

Now put

$$J = \operatorname{span}\{(f, s) - (f, t) : f \in \Gamma_s(r^* \mathscr{A}), s \le t\}.$$

Clearly $J \subset \ker \Lambda$. For the opposite containment, by the above argument it suffices to show that if $f \in \Gamma_{s \cap t}(r^* \mathscr{A})$ then $(f, s) - (f, t) \in J$. Since S is a base for the topology of G, we can find $s_1, \ldots, s_n \in S$ such that

$$\operatorname{supp} f \subset \bigcup_1^n s_i \subset s \cap t.$$

choose a partition of unity $\{\phi_i\}_1^n$ subordinate to the open cover $\{s_i\}_1^n$ of $\operatorname{supp} f$. We

have

$$(f, s) - (f, t) = \sum_1^n (f\,\phi_i, s) - \sum_1^n (f\,\phi_i, t)$$

$$= \sum_1^n (f\,\phi_i, s) - \sum_1^n (f\,\phi_i, s_i) + \sum_1^n (f\,\phi_i, s_i) - \sum_1^n (f\,\phi_i, t)$$

$$= \sum_1^n \big((f\,\phi_i, s) - (f\,\phi_i, s_i)\big) + \sum_1^n \big((f\,\phi_i, s_i) - (f\,\phi_i, t)\big).$$

Since each ϕ_i has support contained in $s_i \cap s \cap t$, the latter sums are elements of J, and we are done.   □

THEOREM 7.2. *Let α be an action of an r-discrete groupoid G on an upper semicontinuous C^*-bundle $\mathscr{A}$, S a full inverse semigroup of open G-sets (as in Definition 5.2), and β the associated action of S on $B := \Gamma_0(\mathscr{A})$ (as in Theorem 5.3). Then the *-algebra $\Gamma_c(r^*\mathscr{A})$ has an enveloping C^*-algebra $\mathscr{A} \times_\alpha G$. Moreover, the map Ψ of Theorem 7.1 extends uniquely to an isomorphism of $B \times_\beta S$ onto $\mathscr{A} \times_\alpha G$.*

PROOF. By Theorem 7.1, the map Ψ factors through an isomorphism Ψ' of the quotient $\Gamma_c(\mathscr{C})/(I_\beta \cap \Gamma_c(\mathscr{C}))$ onto $\Gamma_c(r^*\mathscr{A})$. Since each $\Gamma_e(\mathscr{A})$ is a dense ideal of B_e and $\beta_s(\Gamma_{d(s)}(\mathscr{A})) = \Gamma_{r(s)}(\mathscr{A})$ for every $s \in S$, Corollary 4.6 tells us $B \times_\beta S$ is the enveloping C^*-algebra of $\Gamma_c(\mathscr{C})/(I_\beta \cap \Gamma_c(\mathscr{C}))$. The result follows.   □

8. Application

We show how Theorem 7.2 allows us to recover Paterson's representation [Pat] of the C^*-algebra of an r-discrete groupoid as a semigroup crossed product. Since our groupoids are Hausdorff and Paterson requires only the unit space of the groupoids to be Hausdorff, we cannot get his theorem in full generality. We believe the connection between groupoid and inverse semigroup crossed products should also work for non-Hausdorff groupoids.

If G is a not necessarily Hausdorff r-discrete groupoid then Paterson [Pat] calls an inverse subsemigroup S of G^{op} *additive* if S is a base for the topology of G and s, $t \in S$ with $s \cup t \in G^{op}$ implies $s \cup t \in S$. Note that additivity is a strictly stronger condition than fullness in the sense of Definition 5.2. Paterson shows [Pat, Theorem 3.3.1] that if S is an additive inverse subsemigroup of G^{op} then $C^*(G)$ is isomorphic to $C_0(G^0) \times_\beta S$ if S acts on $C_0(G^0)$ canonically (see below). We can deduce the same result if we assume that G is also Hausdorff, and in fact we can get away with slightly less than additivity:

THEOREM 8.1. *Let G be an r-discrete (Hausdorff) groupoid and let S be a full inverse semigroup of open G-sets. Then S has an action β on $B := C_0(G_0)$ defined by*

$$\beta_s(f)(u) = \begin{cases} f(s^*us) & \text{if } u \in r(s), \\ 0 & \text{else}, \end{cases}$$

for $f \in B_{d(s)} := \{f \in C_0(G^0) : f = 0 \text{ off } d(s)\}$, and $C^*(G)$ is isomorphic to $B \times_\beta S$.

PROOF. G has an action α on the trivial C^*-bundle $\mathscr{A} = \mathbb{C} \times G^0$, where $\alpha_x : A_{d(x)} \to A_{r(x)}$ is the identity map between two copies of $\mathbb{C}$. It is clear that $C^*(G)$ is isomorphic to $\mathscr{A} \times_\alpha G$. By Theorem 7.2 $\mathscr{A} \times_\alpha G$ is isomorphic to $B \times_\beta S$. Since $\Gamma_0(\mathscr{A})$ is isomorphic to $C_0(G^0)$ and α_x is the identity map for all $x \in G$, β is exactly the canonical action of S on $C_0(G^0)$ used by Paterson. □

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Department of Mathematics
Arizona State University
Tempe, Arizona 85287
e-mail: quigg@math.la.asu.edu, nandor.sieben@asu.edu