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A FAMILY OF CRYSTALLOGRAPHIC GROUPS WITH 2-TORSION IN K_0 OF THE RATIONAL GROUP ALGEBRA

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We calculate K_0 of the rational group algebra of a certain crystallographic group, showing that it contains an element of order 2. We show that this element is the Euler class, and use our calculation to produce a whole family of groups with Euler class of order 2.

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1. Let D denote the infinite dihedral group $\langle a,b \mid a^2=b^2=1\rangle$ and $\pi: D\to C_2$ the map onto the cyclic group of order 2 with infinite cyclic kernel. Then π extends to a map on the direct product of n copies of D, $\pi^n: D^n\to C_2$ and for n greater than one we define G_n to be the kernel of π^n (so G_n is the orientable subgroup of D^n).

In this note we show that the Euler class $E(\mathbb{Q}G_n)$ in $K_0(\mathbb{Q}G_n)$ (i.e. the class of the trivial G_n-module $\mathbb{Q}$—see para. 2) of $\mathbb{Q}G_n$ has order two if n is odd, and infinite order if n is even. So in particular no G_n is of type (FL) over $\mathbb{Q}$. The calculation proceeds as follows: in paragraph 2 we show that $2E(\mathbb{Q}G_{\text{odd}})=0$, and in paragraph 3 that $E(\mathbb{Q}G_{\text{even}})$ has infinite order. Paragraph 4 consists of a more detailed calculation which shows that $E(\mathbb{Q}G_3)\neq 0$, and that in fact

$$K_0(\mathbb{Q}G_3)\cong \mathbb{Z}^{13}\oplus \mathbb{Z}/2\mathbb{Z}. \tag{1.1}$$

Finally (para. 5) we prove a lemma about K_0 of direct products $G_m\times G_n$ from which it follows that no G_n is of type (FL), so that $E(\mathbb{Q}G_n)$ has order exactly two for all odd n.

The main calculation (para. 4) can be done using either results due to Quinn [6] or the work of Waldhausen [7]. Quinn's result enables one to calculate K_0 of the rational group algebra of any virtually polycyclic group, while we follow Farrell [2] in using Waldhausen's techniques, which apply to groups which are free products with amalgamation (in the polycyclic case this amounts to saying they have D as a quotient) and are known to be valid over any field k of characteristic zero, so that (1.1) is actually true for $K_0(kG_3)$. In theory, the same methods could be used to calculate $K_0(\mathbb{Q}G_n)$ for all n because we can regard each G_n as the free product with amalgamation of two copies of D^{n-1} over G_{n-1}.

2. Over a field of characteristic zero the group algebra of any polycyclic-by-finite group is Noetherian and of finite global dimension. In particular G_n is of type (FP) over $\mathbb{Q}$ and is in fact an orientable PD^n-group over $\mathbb{Q}$. The trivial G-module $\mathbb{Q}$ admits a finite projective resolution

$$0 \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Q} \to 0, \tag{2.1}$$

and as always, the group has type (FL) over $\mathbb{Q}$ if and only if the Euler class $E = [P_0] - [P_1] + \cdots + (-1)^n [P_n]$ is zero in $K_0(\mathbb{Q}G_n)$. We shall show that for $n = 3$

$$\textit{E can be identified with the element of order 2 in } K_0. \tag{2.2}$$

The fact that $2E = 0$ follows from the following choice of resolution. Let $D = \langle a, b \mid a^2 = b^2 = 1 \rangle$ as before, and let $\mathbb{Q}_0$, $\mathbb{Q}_1$ denote the trivial and non-trivial irreducible $\mathbb{Q}C_2$ modules respectively. Then we define two $\mathbb{Q}D$-modules as follows

$$\begin{aligned} T_0 &= \mathbb{Q}_0 \bigotimes\nolimits_{\mathbb{Q}\langle a \rangle} \mathbb{Q}D \\ T_1 &= \mathbb{Q}_1 \bigotimes\nolimits_{\mathbb{Q}\langle b \rangle} \mathbb{Q}D \end{aligned} \tag{2.3}$$

and note that $\mathbb{Q}$ has a $\mathbb{Q}D$-resolution

$$0 \to T_1 \to T_0 \to \mathbb{Q} \to 0 \tag{2.4}$$

Tensoring together n copies of this we obtain a projective resolution of $\mathbb{Q}$ over $\mathbb{Q}D^n$, and hence by restriction over $\mathbb{Q}G_n$, as in (2.1), with

$$P_i \cong \oplus (T_{\varepsilon(1)} \otimes \cdots \otimes T_{\varepsilon(n)}) \tag{2.5}$$

where the direct sum is taken over all functions $\varepsilon: \{1, \ldots, n\} \to \{0, 1\}$ taking the value 1 i times.

Now D has an outer automorphism ϕ defined by $a\phi = b$, $b\phi = a$, which extends diagonally to an automorphism of D^n and then restricts to an automorphism of G_n, also to be denoted by ϕ. Given any G_n-module M one can form a new module $M\phi$, and one can check that $P_i^\phi \cong P_{n-i}$ as G_n-modules. Twisting the resolution (2.1) by ϕ we therefore obtain a new projective resolution

$$0 \to P_0 \to P_1 \to P_2 \to \cdots \to P_n \to \mathbb{Q} \to 0 \tag{2.6}$$

Applying Schanuel's lemma to (2.1) and (2.3) shows that for $n = 2m + 1$

$$\begin{aligned} (P_0 \oplus P_2 \oplus \cdots \oplus P_{2m}) \oplus (P_0 \oplus P_2 \oplus \cdots \oplus P_{2m}) &\cong (P_1 \oplus P_3 \oplus \cdots \oplus P_{2m+1}) \\ &\oplus (P_1 \oplus P_3 \oplus \cdots \oplus P_{2m+1}) \end{aligned} \tag{2.7}$$

and in particular $2E=0$. The fact that E is non-zero shows that $P_0 \oplus P_2 \oplus \cdots \oplus P_{2m}$ and $P_1 \oplus P_3 \oplus \cdots \oplus P_{2m+1}$ are not even stably isomorphic.

3. The example also shows that Remark 5 of [5] is not quite true as stated. The remark which is true, and which Moody apparently intended, asserts that if P and P' are projective modules (over any polycyclic group algebra $k\Gamma$, k of characteristic zero) with rank functions r and r' then $r=r'$ if and only if $[P]-[P']$ is a torsion element of K_0. The rank functions can be regarded as elements of the free abelian group on the conjugacy classes of Γ, and cannot detect torsion in K_0. Thus, for example, the modules $(P_0 \oplus \cdots \oplus P_{2m})$ and $(P_1 \oplus \cdots \oplus P_{2m+1})$ of paragraph 2 have the same rank functions. In fact this is clear from a result of K. S. Brown. In [1] he proves the following formula:

$$r_\Gamma(E)(s) = \begin{cases} r_{Z(s)}(E)(1) & \textit{if } s \textit{ has finite order} \\ 0 & \textit{if } s \textit{ has infinite order} \end{cases} \tag{3.1}$$

where E denotes the Euler class of Γ on the left, and of $Z(s)=Z_\Gamma(s)$, the centraliser of s in Γ, on the right hand side.

Now for Γ polycyclic-by-finite and non-trivial it is easy to show that $r_\Gamma(E)(1)=0$ if and only if Γ is infinte. So for the 'total Euler characteristic' $r(E)$ to vanish it is necessary and sufficient for every element to have infinite centraliser, and this is satisfied by our groups G_n if and only if n is odd. It follows that $E(\mathbb{Q}G_{\text{even}})$ is of infinite order as claimed.

4. To describe how $K_0(\mathbb{Q}G_3)$ may be calculated we note as above that for $n \geqq 2$ we can regard G_n as the free product with amalgamation of two copies of D^{n-1} over G_{n-1}. Moreover D^{n-1} can in turn be regarded as the amalgamation of two copies of $D^{n-2} \times C_2$ over $D^{n-2} \times 1$, where C_2 denotes a cyclic group of order 2. For the groups D^n this is exactly the point of view taken by Farrell [2], and K_0 can be computed from the general formula

(4.1) *Let $\Gamma = H *_L K$ be an amalgamated free product, with the group algebra $\mathbb{Q}L$ regular coherent. Then the pushout square in the category of groups*

$$\begin{tikzcd} L \arrow[r, "\alpha"'] \arrow[d, "\beta"'] & H \arrow[d] \\ K \arrow[r] \arrow[ur, phantom, "\lrcorner" very near end] & G \end{tikzcd}$$

induces one in the category of albelian groups

$$\begin{tikzcd}
K_0(\mathbb{Q}L) \arrow[r, "\alpha*"'] \arrow[d, "\beta*"'] & K_0(\mathbb{Q}H) \arrow[d] \\
K_0(\mathbb{Q}K) \arrow[r] & K_0(\mathbb{Q}G)
\end{tikzcd}$$

As stated (4.1) *is a very special case of Waldhausen's work* [7].

One also needs the simple observation, already used to similar effect by Farrell [**2**], that for any group Γ, $K_0(\mathbb{Q}[\Gamma\times C_2])=K_0(\mathbb{Q}\Gamma)\oplus K_0(\mathbb{Q}\Gamma)$. Plainly, provided one keeps careful track of the induction maps, it is possible to compute K_0 for the groups D^n, which has effectively been done by Farrell, and the G_n, which is our purpose here. The calculations themselves are not illuminating, and here we simply summarize the results. For each of the groups K_0 has a presentation as the free abelian group on certain idempotents of the group ring modulo certain relations. By including specific idempotents in this way, it is straightforward to follow the induction maps.

To obtain the Euler classes we exploit another simple observation, that an amalgam $H\ast_L K$ acts on a tree with vertex stabilisers H, K and edge stabiliser L. It follows that the augmented simplicial chain complex of the tree has the form

$$0\to\mathbb{Q}\otimes_{\mathbb{Q}L}\mathbb{Q}G\to(\mathbb{Q}\otimes_{\mathbb{Q}H}\mathbb{Q}G)\oplus(\mathbb{Q}\otimes_{\mathbb{Q}K}\mathbb{Q}G)\to\mathbb{Q}\to 0 \tag{4.2}$$

so that

$$E(G)=i_{(H\to G)*}E(H)+i_{(K\to G)*}E(K)-i_{(L\to G)*}E(L). \tag{4.3}$$

Writing $D=C_2*C_2$ and applying (4.1) we obtain

$$K_0(\mathbb{Q}D)=\langle e_1,e_2,e_3,e_4\,|\,e_1+e_2=e_3+e_4\rangle, \tag{4.4}$$

where $e_1=(1+a)/2$, $e_2=(1-a)/2$, $e_3=(1+b)/2$, $e_4=(1-b)/2$. The Euler class $E(D)$ is represented by e_1-e_4.

Writing G_2 as an amalgam of two copies of D over C_∞ we obtain

$$K_0(\mathbb{Q}G_2)=\langle e_1,e_2,\ldots,e_8\,|\,e_1+e_2=e_3+e_4, e_5+e_6=e_7+e_8, e_1+e_2=e_5+e_6\rangle. \tag{4.5}$$

With the presentation $G_2=\langle q_1,q_2,v\,|\,q_1q_2=q_2q_1,\ v^2=1,\ q_i^v=q_i^{-1}\rangle$, the idempotents

$e_1, \dots e_8$ are $(1+v)/2$, $(1-v)/2$, $(1+vq_1)/2$, $(1-vq_1)/2$, $(1+vq_2)/2$, $(1-vq_2)/2$, $(1+vq_2q_1)/2$, $(1-vq_2q_1)/2$. The Euler class $E(G_2)$ is represented by $e_1-e_4+e_5-e_8$.

Now writing D^2 as an amalgam of two copies of $D\times C_2$ over D gives

$$K_0(\mathbb{Q}D^2)=\langle f_1, f_2, \dots, f_{16} \mid f_{1+j}+f_{2+j}=f_{3+j}+f_{4+j},\quad j=0,4,8,12;\quad f_j+f_{4+j}=f_{8+j}+f_{12+j},\ j=1,2,3,4\rangle. \tag{4.6}$$

In fact K_0 is free abelian of rank 9.

Using the presentation

$$D^2=\langle x_1, t_1 \mid t_1^2=1,\ t_1x_1t_1=x_1^{-1}\rangle \times \langle x_2, t_2 \mid t_2^2=1,\ t_2x_2t_2=x_2^{-1}\rangle,$$

the idempotents f_i are given by:

1. $(1+t_2)(1+t_1)/4$	9. $(1+t_2)(1+t_1x_1)/4$
2. $(1-t_2)(1+t_1)/4$	10. $(1-t_2)(1+t_1x_1)/4$
3. $(1+t_2x_2)(1+t_1)/4$	11. $(1+t_2x_2)(1+t_1x_1)/4$
4. $(1-t_2x_2)(1+t_1)/4$	12. $(1-t_2x_2)(1+t_1x_1)/4$
5. $(1+t_2)(1-t_1)/4$	13. $(1+t_2)(1-t_1x_1)/4$
6. $(1-t_2)(1-t_1)/4$	14. $(1-t_2)(1-t_1x_1)/4$
7. $(1+t_2x_2)(1-t_1)/4$	15. $(1+t_2x_2)(1-t_1x_1)/4$
8. $(1-t_2x_2)(1-t_1)/4$	16. $(1-t_2x_2)(1-t_1x_1)/4$

The Euler class $E(\mathbb{Q}D^2)$ is represented by $f_1-f_4-f_{13}+f_{16}$.

Finally regarding G_3 as the pushout $D^2 *_{G_2} D^2$ we obtain the exact sequence

$$0\to\langle e_1,\dots,e_8 \mid \dots\rangle \xrightarrow{\alpha*-\beta*} \langle f_1,\dots,f_{16} \mid \dots\rangle \oplus \langle f_{17},\dots,f_{32} \mid \dots\rangle \to K_0(\mathbb{Q}G_3)\to 0 \tag{4.7}$$

By following the idempotents it is easy to calculate the maps $\alpha*$, $\beta*$. In fact $\alpha*$ takes $e_1,\dots,e_8$ to f_1+f_6, f_2+f_5, f_9+f_{14}, $f_{10}+f_{13}$, f_3+f_8, f_4+f_7, $f_{11}+f_{16}$, $f_{12}+f_{15}$, and $\beta*$ is essentially the same map—just add 16 to the indices of $\alpha*(e_i)$ to get $\beta*(e_i)$.

The Euler class is represented by $(f_1-f_4-f_{13}+f_{16})+(f_{17}-f_{20}-f_{29}+f_{32})-\alpha*(e_1-e_4+e_5-e_8)$.

Now elementary matrix algebra gives the results as stated—$K_0(\mathbb{Q}G_3)$ has rank 13 and an unique torsion element, the Euler class, of order 2.

5. We now use Quinn's work [6] to investigate K_0 of a direct product, and deduce that none of the groups G_n are of type of (FL). Because K_{-1} vanishes on a regular ring, it is a consequence of his Corollary 1.5 that for G polycyclic-by-finite

$$\operatorname{colim} K_0(\mathbb{Q}F)\cong K_0(\mathbb{Q}G) \tag{5.1}$$

where the colimit is taken over the Frobenius category $\Phi(G)$ of finite subgroups of G, with morphisms the inclusions and conjugations. This has not yet been proved over an arbitrary field of characteristic zero, although surjectivity of the map follows from the more general result of Moody [5], as does rational equivalence (see Lorenz [3]).

Now suppose that H, G are polycyclic-by-finite groups such that all their finite subgroups are totally reducible over $\mathbb{Q}$. Then it follows from classical representation theory that

$$\operatorname*{colim}_{\Phi(H)} K_0(\mathbb{Q}F) \otimes \operatorname*{colim}_{\Phi(G)} K_0(\mathbb{Q}F) \cong \operatorname*{colim}_{\Phi(H)\times\Phi(G)} K_0(\mathbb{Q}F) \tag{5.2}$$

where the right hand colimit is taken, as indicated, over the product category $\Phi(H)\times\Phi(G)$. This product is clearly a subcategory of $\Phi(H\times G)$, and it is in fact a final subcategory (see Maclane [4]), so that the right hand colimit is actually isomorphic to the colimit over $\Phi(H\times G)$. To see that it is final just note that if π_H, π_G denote the projections of $H\times G$ onto H, G respectively then every finite subgroup F of $H\times G$ is contained in $\pi_H(F)\times\pi_G(F)$. Putting this together with (5.1) and (5.2) it follows that for H, G as above

$$K_0(\mathbb{Q}H)\otimes K_0(\mathbb{Q}G)\cong K_0(\mathbb{Q}[H\times G]). \tag{5.3}$$

Moreover, since the isomorphism is induced by taking tensor products over $\mathbb{Q}$, and since the tensor product of a $\mathbb{Q}H$-resolution of $\mathbb{Q}$ with a $\mathbb{Q}G$ one gives a $\mathbb{Q}[H\times G]$-resolution of $\mathbb{Q}$, it follows that the map takes the product of Euler classes $E(H)\otimes E(G)$ to $E(H\times G)$.

Finally we note that all the groups we have considered have finite subgroups which are elementary abelian 2-groups, hence completely reducible over $\mathbb{Q}$. We can thus apply (4.3) and the last remark to the results of our calculations above to deduce that none of the groups $G_3\times G_2^m$ is of type (FL) over $\mathbb{Q}$. Since each G_{2m+3} contains one of these as a subgroup of finite index, it follows that no G_{odd} can itself be of type (FL) over $\mathbb{Q}$. In fact the same argument with G_2^m instead of $G_3\times G_2^m$ gives another proof that no G_{even} is of type (FL) over $\mathbb{Q}$.

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