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Stability of impulsively perturbed systems

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Until recently most authors have devoted their research to the theory of perturbed systems under continuous perturbations. In this paper, Liapunov's second method is employed to investigate sufficient conditions for integral and integral asymptotic stability of ordinary differential systems with respect to impulsive perturbations.

1.

The study of perturbation problems for ordinary differential equations is an interesting area of research and a great deal of work has been done by many authors in recent years. Historically, there have been two approaches to these problems. The first is to set conditions on the unperturbed system and find out the type of perturbations that preserve the stability behaviour of the unperturbed system ([3], [10], [11], [15]). The second approach is to set the kind of perturbations that will be allowed and find the differential equations whose stability properties are preserved by those perturbations ([2], [4], [13], [14]). Motivated from the works of Vrkoč [13], Okamura [7], and Yoshizawa [15], quite recently Chow and Yorke [1] gave necessary and sufficient conditions for integral and integral asymptotic stability of the unperturbed system.

When a physical system described by an ordinary differential equation is subject to perturbations, the perturbed system is again an ordinary differential equation in which the perturbation functions are assumed to be

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continuous or integrable. Most conspicuously in this case the state of the system changes continuously with respect to time. However in most physical systems, the perturbation functions need not be continuous or integrable (in the usual sense) and thus the state of the system changes discontinuously with respect to time. Indeed, the study of stability properties of solutions of impulsively perturbed systems is indicated by the fact that the description of the physical processes in the language of set functions, distributions, and in particular, generalized functions, gives a more accurate reflection of the real nature of these processes.

The present paper deals with the second approach for impulsively perturbed nonlinear ordinary differential systems.

2.

We shall use the following notation throughout this paper:

R^n = space of n-vectors;

$|x| = \sum_{i=1}^{n} |x_i|$, $x \in R^n$;

$J = [0, \infty)$;

$S_\rho = \{x \in R^n : |x| < \rho, \rho > 0\}$;

K = the class of functions $\phi(r)$, defined and continuous on $[0, \infty)$, $\phi(0) = 0$, and strictly monotone increasing in r ;

$\overline{C}_0(x)$ = the class of functions having uniform Lipschitz constants with respect to x on $J \times S_\rho$.

The object of this investigation is to obtain sufficient conditions for the integral and integral asymptotic stability of the trivial solution $x \equiv 0$ of

$$x' = F(t, x) \tag{2.1}$$

where $F : J \times R^n \to R^n$ is a continuous function.

Now suppose one knows that all the solutions of (2.1) which start near $x \equiv 0$ remain near this solution for all future time, or even approach this solution as time increases. If the differential system (2.1) is acted upon by certain impulsive perturbations, the above property concerning the solutions near $x \equiv 0$ may or may not be true. More precisely, if the trivial solution of (2.1) is asymptotically stable and if the function $p(t)Du$ is small in certain senses, then give sufficient conditions so that the trivial solution $x \equiv 0$ of (2.1) is asymptotically stable with respect to the perturbed system

$$Dx = F(t, x) + p(t)Du \ , \tag{2.2}$$

where $u : J \to R$ is a right continuous function of bounded variation on every compact subinterval of J and $p : J \to R^n$ is integrable with respect to u and the discontinuities $t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ of u tend to ∞ as k tends to ∞. In (2.2), Du denotes the distributional derivative of u and will have the effect of suddenly changing the state of the system at the points of discontinuities of u.

The proofs of the results of this paper crucially depend on almost everywhere differentiability of the function u, and this property is guaranteed because u is a function of bounded variation. In fact, a function of bounded variation has a finite differential coefficient almost everywhere ([12], p. 356). Throughout this paper we denote by v_u, the total variation function of u.

Let $x(t) = x\left(t, t_0, x_0\right)$ be any solution of (2.2) through $\left(t_0, x_0\right)$ existing to the right of $t_0 \geq 0$. Now we give the following definitions which are analogous to the definitions of integral and integral asymptotic stability (*cf.* [1]). Let $V(t, x)$ be a function defined and continuous on $J \times S_\rho$ taking values in J. We define the derivative of V with respect to system (2.1) by

$$V'_{(2.1)}(t, x) \equiv \frac{\partial V}{\partial t} + \nabla V(t, x) . F(t, x) \ ,$$

where $\nabla V = \left(\frac{\partial V}{\partial x_1}, \ \ldots, \ \frac{\partial V}{\partial x_n}\right)$.

DEFINITION 2.1. The trivial solution $x \equiv 0$ of (2.1) is said to be integrally stable, if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such

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that $|x_0| < \delta$ and

$$\int_{t_0}^{\infty} |p(s)|dv_u(s) < \delta$$

imply $|x(t)| < \varepsilon$, for all $|x_0| < \delta$ and $t \geq t_0 \geq 0$.

DEFINITION 2.2. The trivial solution $x \equiv 0$ of (2.1) is said to be integrally attracting, if there exists a $\delta_0 > 0$ and for each $\eta > 0$ there exist $T = T(\eta) > 0$ and $\alpha = \alpha(\eta) > 0$ such that $|x_0| < \delta_0$ and

$$\int_{t_0}^{\infty} |p(s)|dv_u(s) < \alpha$$

imply $|x(t)| < \varepsilon$, for all $t \geq t_0+T$ and $t_0 \geq 0$.

DEFINITION 2.3. The trivial solution $x \equiv 0$ of (2.1) is said to be integrally asymptotically stable if Definitions 2.1 and 2.2 hold simultaneously.

We now prove the following lemmas which will be used in our subsequent discussion.

LEMMA 2.4. *If a solution* $x(t)$ *of* (2.2) *exists and is differentiable for* $t \in [t_{k-1}, t_k)$, $k = 1, 2, 3, \ldots$, *then the inequality*

$$\underset{(2.2)}{V'}\big(t, x(t)\big) \leq \underset{(2.1)}{V'}\big(t, x(t)\big) + L|p(t)||u'(t)| , \quad t \in [t_{k-1}, t_k) ,$$

$k = 1, 2, 3, \ldots$, *where* $|\nabla V(t, x)| \leq L$, *holds.*

Proof. Since u is differentiable on $[t_{k-1}, t_k)$, $k = 1, 2, 3, \ldots$ as long as a solution $x(t)$ of (2.2) exists and is differentiable for $t \in [t_{k-1}, t_k)$, we have

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$$\frac{d}{dt} V(t, x(t)) \underset{(2.2)}{=} V'(t, x(t)) = \frac{\partial}{\partial t} V(t, x(t)) + \nabla V(t, x(t)) . [F(t, x(t))+p(t)u'(t)]$$

$$= \frac{\partial}{\partial t} V(t, x(t)) + \nabla V(t, x(t)).F(t, x(t)) + \nabla V(t, x(t)).p(t)u'(t)$$

$$\le \underset{(2.1)}{V'}(t, x(t)) + L|p(t)||u'(t)| ,$$

and the proof is complete.

LEMMA 2.5. *At the points of discontinuity* t_k , $k = 1, 2, \ldots$ *of* u , *the function* $V(t, x)$ *satisfies the estimate*

$$|V(t_k, x(t_k))-V(t_k, x(\bar{t}_k))| \le L|p(t_k)||u(t_k)-u(\bar{t}_k)|$$

where $x(\bar{t}_k)$ *denotes the left hand limit of* x *at* t_k .

Proof. We know (*cf.* [9]) that $x(t)$ is a solution of (2.2) through (t_0, x_0) if and only if $x(t)$ satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} F(s, x(s))ds + \int_{t_0}^{t} p(s)du(s) , \quad t \ge t_0 .$$

By definition

$$|x(t_k)-x(\bar{t}_k)| = \lim_{h\to 0^+} |x(t_k)-x(t_k-h)| = \lim_{h\to 0^+} \left|\int_{t_k-h}^{t_k} F(s, x(s))ds + \int_{t_k-h}^{t_k} p(s)du(s)\right| . \tag{2.3}$$

Clearly the first limit on the right hand side of (2.3) is zero because of continuity of F ,and we shall prove that

$$\lim_{h\to 0^+} \left|\int_{t_k-h}^{t_k} p(s)du(s)\right| \le |p(t_k)||u(t_k)-u(\bar{t}_k)| . \tag{2.4}$$

Consider the positive set function μ defined by

$$\mu(A) = \left|\int_A p(s)du(s)\right| .$$

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Let $h_1 \geq h_2 \geq h_3 \geq \ldots > 0$ and let $A_n = [t_k - h_n, t_k]$ and $h_n \to 0$ as $n \to \infty$. Then $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ and $\bigcap_{n=1}^{\infty} A_n = A_0$ where $A_0 = \{t_k\}$. Therefore, by [8, Theorem 1.19 (e)], $\mu(A_n) \to \mu(A_0)$. But $\mu(A_0) = |p(t_k)||(u(t_k) - u(\bar{t}_k))|$, by [6, Example S, p. 199]. Thus (2.4) is established and from (2.3) we have

$$|x(t_k) - x(\bar{t}_k)| \leq |p(t_k)||u(t_k) - u(\bar{t}_k)| .$$

Since $V(t, x)$ is uniformly lipschitzian in x, and this property is guaranteed because of $|\nabla V(t, x)| \leq L$, we finally have

$$\begin{aligned} |V(t_k, x(t_k)) - V(t_k, x(\bar{t}_k))| &\leq L|x(t_k) - x(\bar{t}_k)| \\ &\leq L|p(t_k)||u(t_k) - u(\bar{t}_k)| . \end{aligned}$$

This completes the proof.

3.

In this section we give sufficient conditions for integral and integral asymptotic stability of the trivial solution $x \equiv 0$ of (2.1).

THEOREM 3.1. *Let* $F \in \bar{C}_0(x)$ *and* $F(t, 0) = 0$ *for all* $t \geq 0$. *If the trivial solution of* (2.1) *is uniformly stable, then it is also integrally stable.*

Proof. Since the trivial solution of (2.1) is uniformly stable, there exists a Liapunov function $V(t, x)$ on $J \times S_\rho$ satisfying the following hypotheses:

(i) $b(|x|) \leq V(t, x) \leq a(|x|)$;

(ii) $|\nabla V(t, x)| \leq L$;

(iii) $V'_{(2.1)}(t, x) \leq 0$;

where $a, b \in K$ and L is a positive constant. Since $x(t)$ is a solution of (2.2), in view of Lemma 2.4 and hypothesis (iii), it follows that for $t \in [t_{k-1}, t_k)$, $k = 1, 2, \ldots$,

(3.1)
$$V'_{(2.2)}\big(t,\ x(t)\big) \le V'_{(2.1)}\big(t,\ x(t)\big) + L|p(t)|\,|u'(t)|$$
$$\le L|p(t)|\,|u'(t)| \ .$$

Integrating (3.1) for $t \in [t_{k-1},\ t_k)$, we get

(3.2) $$V\big(t,\ x(t)\big) \le V\big(t_{k-1},\ x(t_{k-1})\big) + L\int_{[t_{k-1},t]} |p(s)|\,|u'(s)|ds \ .$$

Since $V(t,\ x)$ is continuous in t for each fixed x , we have

$$V\big(t_k,\ x(t_k)\big) \le \big|V\big(t_k,\ x(t_k)\big)-V\big(t_k,\ x(\bar{t}_k)\big)\big| + \big|V\big(t_k,\ x(\bar{t}_k)\big)\big|$$
$$= \big|V\big(t_k,\ x(t_k)\big)-V\big(t_k,\ x(\bar{t}_k)\big)\big| + \lim_{h\to0^+} V\big(t_k\quad,\ x(t_k\quad)\big) \ .$$

Using Lemma 2.5 and (3.2), we obtain

$$V\big(t_k,\ x(t_k)\big) \le L|p(t_k)|\,|u(t_k)-u(\bar{t}_k)|$$
$$+ \lim_{h\to0^+}\left[V\big(t_{k-1},\ x(t_{k-1})\big)+L\int_{[t_{k-1},t_k\quad]} |p(s)|\,|u'(s)|ds\right] ;$$

that is,

(3.3) $$V\big(t_k,\ x(t_k)\big) \le V\big(t_{k-1},\ x(t_{k-1})\big) + L|p(t_k)|\,|u(t_k)-u(\bar{t}_k)|$$
$$+ L\int_{[t_{k-1},t_k)} |p(s)|\,|u'(s)|ds \ .$$

Thus the inequality (3.2) gives for $t \in [t_0,\ t_1)$,

$$V\big(t,\ x(t)\big) \le V\big(t_0,\ x_0\big) + L\int_{[t_0,t]} |p(s)|\,|u'(s)|ds \ ,$$

and for $t \in [t_1,\ t_2)$,

$$V\big(t,\ x(t)\big) \le V\big(t_1,\ x(t_1)\big) + L\int_{[t_1,t]} |p(s)|\,|u'(s)|ds \ .$$

Hence for $t \in [t_0,\ t_2)$, using (3.3) we get that

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$$V(t,\ x(t)) \le V(t_0,\ x_0) + L|p(t_1)||u(t_1)-u(\bar{t}_1)| + L\int_{[t_0,t_1)} |p(s)||u'(s)|ds + L\int_{[t_1,t]} |p(s)||u'(s)|ds$$
$$\le V(t_0,\ x_0) + L|p(t_1)||u(t_1)-u(\bar{t}_1)| + L\sum_{k=1}^{2}\int_{[t_{k-1},t_k)} |p(s)||u'(s)|ds .$$

Therefore, in general, for $t \ge t_0$, where $t_0 < t_1 < \ldots < t_n = t$, we have

$$V(t,\ x(t)) \le V(t_0,\ x_0) + L\left[\sum_{k=1}^{n-1} |p(t_k)||u(t_k)-u(\bar{t}_k)| + \sum_{k=1}^{n}\int_{[t_{k-1},t_k)} |p(s)||u'(s)|ds\right] ;$$

that is,

$$V(t,\ x(t)) \le V(t_0,\ x_0) + L\int_{t_0}^{t} |p(s)|dv_u(s) . \tag{3.4}$$

Now let $0 < \varepsilon < \rho$ be given. Choose $\delta = \delta(\varepsilon) > 0$, $0 < \delta < \varepsilon$ such that

$$\int_{t_0}^{\infty} |p(s)|dv_u(s) < \delta$$

and $a(\delta) + L\delta < b(\varepsilon)$. For $|x_0| < \delta$ and

$$\int_{t_0}^{\infty} |p(s)|dv_u(s) < \delta ,$$

we have from (i) and (3.4),

$$b(|x(t)|) \le V(t,\ x(t)) \le V(t_0,\ x_0) + L\int_{t_0}^{\infty} |p(s)|dv_u(s)$$
$$\le a(|x_0|) + L\delta$$
$$\le a(\delta) + L\delta < b(\varepsilon) .$$

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This implies that $|x(t)| < \varepsilon$ whenever $|x_0| < \delta$ and

$$\int_{t_0}^{\infty} |p(s)| dv_u(s) < \delta$$

for all $t \geq t_0$, and thus the proof is complete.

In the following theorem we prove the integral asymptotic stability of the trivial solution of (2.1).

THEOREM 3.2. *Let* $F \in \overline{C}_0(x)$ *and* $F(t, 0) = 0$ *for all* $t \geq 0$. *If the trivial solution of* (2.1) *is uniformly asymptotically stable, then it is also integrally asymptotically stable.*

Proof. Since the trivial solution of (2.1) is uniformly asymptotically stable, by a theorem of Massera [5], there exists a Liapunov function $V(t, x)$ on $J \times S_\rho$ with the following properties:

(3.5) $$b(|x|) \leq V(t, x) \leq a(|x|) ,$$

(3.6) $$|\nabla V(t, x)| \leq L ,$$

and

$$V'_{(2.1)}(t, x) \leq -c(|x|) ,$$

where $a, b, c \in K$ and L is a positive constant.

Without loss of generality we can suppose that $L \geq 1$. Let $x(t) = x(t, t_0, x_0)$ be the unique solution of (2.2) existing to the right of $t_0 \geq 0$. By Theorem 3.1 the trivial solution of (2.1) is integrally stable. That is, given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that the inequalities $|x_0| < \delta$ and

$$\int_{t_0}^{\infty} |p(s)| dv_u(s) < \delta$$

imply $|x(t)| < \varepsilon$ for all $t \geq t_0$.

Let $0 < \eta < \varepsilon$ be given. Choose $\delta_0 = \delta(\rho)$, $\alpha(\eta) = \frac{\delta(\eta)}{2L}$, and

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(3.8) $$T(\eta) = \frac{2a(\delta_0)+\delta(\eta)}{2c[\delta(\eta)]} .$$

It is clear that T depends only on η, not on t_0 or x_0. Let $|x_0| < \delta_0$. Now we claim that there exists a $t^* \in [t_0, t_0+T(\eta)]$ such that

(3.9) $$|x(t^*)| < \delta(\eta)$$

whenever $|x_0| < \delta_0$ and

$$\int_{t_0}^{\infty} |p(s)|dv_u(s) < \alpha(\eta) .$$

Suppose not. Then $\delta \le |x(t)| < \rho$ for all $t \in [t_0, t_0+T(\eta)]$.

Proceeding exactly on the same lines as in the proof of Theorem 3.1, we obtain, for all $t \in [t_0, t_0+T(\eta)]$,

$$\begin{aligned} V(t, x(t)) &\le V(t_0, x_0) - c(\delta)T + L\int_{t_0}^{t_0+T} |p(s)|dv_u(s) \\ &\le a(|x_0|) - c(\delta)T + L\alpha(\eta) \\ &< a(\delta_0) - c(\delta)T + \frac{\delta(\eta)}{2} = 0 , \end{aligned}$$

a contradiction, proving (3.9).

Thus from integral stability of (2.1), we have

$$|x(t)| < \eta \text{ for all } t \ge t^*$$

and, in particular,

$$|x(t)| < \eta \text{ for all } t \ge t_0+T(\eta)$$

whenever $|x_0| < \delta_0$ and

$$\int_{t_0}^{\infty} |p(s)|dv_u(s) < \alpha .$$

This completes the proof of Theorem 3.2.

REMARK 3.3. If the perturbation functions in (2.2) are not

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impulsive, that is, when the state of the system changes continuously with respect to time, then our results reduce to some of the results of [1].

REMARK 3.4. Instead of $p(t)Du$ in (2.2), we can as well consider $G(t, x)Du$ with $|G(t, x)| \leq |p(t)|$ for sufficiently small $|x|$ and obtain the corresponding results (with minor changes) with respect to the system

$$Dx = F(t, x) + G(t, x)Du .$$

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