Double Binary Forms IV.*

(1) Apolarity; (2) Automorphic Transformations.

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§ 1. *Introduction.*

The first part of the following investigation was begun before the discovery that Mr E. KASNER had already touched upon the apolarity theory of double binary forms in an important work on the Inversion Group (*Transactions of the American Mathematical Society*, Vol. I (1900), pp. 471–473). The theory is carried further in what follows, with special reference to the (2, 2) form. The second part answers questions raised by Professor A. R. FORSYTH in the *Quarterly Journal*, 1910, p. 113. It appears that the general (2, 2) form admits of three independent automorphic transformations, but the general (n, n) form admits of none, if n exceeds two.

An automorphic transformation is here a linear transformation of the two independent binary variables of the (n, n) form, which leaves the form unaltered except for a possible non-zero numerical factor.

§ 2. *Apolarity.*

Let x, y be two independent variables and let two (m, n) forms f, ϕ be denoted as follows:

$$f = a_x^m a'^n_y = \sum_{rs} \binom{m}{r} \binom{n}{s} a_{rs}\, x^{m-r} y^{n-s}, \qquad \ldots\ldots\ldots\ldots(1)$$

$$\phi = b_x^m b'^n_y = \sum_{rs} \binom{m}{r} \binom{n}{s} b_{rs}\, x^{m-r} y^{n-s}. \qquad \ldots\ldots\ldots\ldots(2)$$

These forms are said to be *apolar* to each other if their line-linear invariant

$$(f, \phi)^{m, n} = (ab)^m (a'b')^n = \sum_{hk} (-)^{h+k} \binom{m}{h} \binom{n}{k} a_{hk}\, b_{m-h, n-k} \qquad \ldots\ldots(3)$$

vanishes identically.

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* Previous discussions by the author are (1) *Proc. Roy. Soc. Edin.*, XLIII. (1922-3), pp. 43–50; (2) *Proc. Edin. Math. Soc.*, XLI. (1922–3), pp. 116–127, and (3) *Proc. Roy. Soc. Edin.*, XLIV. (1923–4) pp. 23–50.

Thus the apolar relation is symmetrical; also any form of total odd order is apolar to itself.

Since the number of terms in the (m, n) form is $(m+1)(n+1)$, we can always find λ linear independent forms apolar to

$$(m+1)(n+1)-\lambda$$

linearly independent given forms: for the apolar condition is linear in the coefficients of both forms.

Again if ψ is a form of orders $(m+\rho, n+\sigma)$ where $\rho \geqq 0$, $\sigma \geqq 0$, f is apolar to ψ if

$$(f, \psi)^{m, n}=0$$

identically. So f is apolar to all polars of ψ whose orders are not less than m and n respectively.

§ 3. *Connection with quaternary forms.*

Kasner* points out that if $m=n$ this apolarity condition is equivalent to a certain projective apolarity condition of quaternary forms. This is most easily seen as follows:

Let quaternary variables ξ_1, ξ_2, ξ_3, ξ_4, when limited to a certain quadric, be defined by

$$\frac{\xi_1}{xy}=\frac{\xi_2}{x}=\frac{\xi_3}{y}=\frac{\xi_4}{1}=\rho. \quad \text{.......................}(4)$$

Also let symbols r_i, s_i be chosen such that

$$\frac{r_1}{a_1 a_1'}=\frac{r_2}{a_1 a_2'}=\frac{r_3}{a_2 a_1'}=\frac{r_4}{a_2 a_2'}=\frac{r_\xi}{\rho \,.\, a_x a_y'}=\frac{1}{\rho}, \quad \text{..............}(5)$$

where $$r_\xi=r_1\xi_1+r_2\xi_2+r_3\xi_3+r_4\xi_4,$$

and $$a_x=a_1 x+a_2, \quad a_y'=a_1' y+a_2',$$

with a similar set for s in terms of the symbols b, b'. Then the double binary *equi-form* (n, n) corresponds to a unique quaternary form of order n; namely

$$f=a_x^n a_y'^n \text{ corresponds to } F=r_\xi^n,$$

and $$\phi=b_x^n b_y'^n \quad \text{,,} \quad \text{,,} \quad \Phi=s_\xi^n,$$

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* *Loc. cit.*

and conversely, provided that these quaternary forms F, Φ each satisfy the apolar conditions

$$(rqq'q'')^2 r_\xi^{n-2} = 0,\ (sqq'q'')^2 s_\xi^{n-2} = 0\ \{\xi\}^* \quad \ldots\ldots\ldots\ldots\ldots(6)$$

with regard to the quadric

$$2(\xi_1\xi_4 - \xi_2\xi_3) \equiv q_\xi^2 = q_\xi'^2 = q_\xi''^2. \quad \ldots\ldots\ldots\ldots\ldots(7)$$

Geometrically † this correspondence is represented by stereographic projection from a sphere Q to a plane Π, the quadric (7) representing a sphere. If

$$x = X + iY \quad y = X - iY,$$

then X, Y are rectangular Cartesian coordinates in the plane Π; $f = 0$, $\phi = 0$ are the equations of two (n, n) circular curves in this plane; while $F = 0$, $\Phi = 0$ are the equations of two surfaces of order n through the spherical curves answering in this projection to the plane curves. Equations (6) single out, from each singly infinite system of n^{th} order surfaces containing one spherical curve, a unique surface apolar to the sphere.

If now we take the six line coordinates $p_{ij}(i, j = 1, 2, 3, 4;\ i \neq j)$ in space and write the line complex equation of the sphere as

$$0 = (qq'p)^2 = (Qp)^2 = (Q'p)^2 = (Q''p)^2 = \ldots = (Q^{(k)}p)^2 \quad \ldots\ldots\ldots(8)$$

where Q, Q'...are equivalent symbols and

$$(Qp) = \mid q_1 q_2' p_{34} \mid,$$

then the apolarity conditions for double binary forms are readily translated into quaternary symbols. In fact

$$(f, \phi)^{n, n} = (ab)^n (a'b')^n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

becomes a condition which is identically true if n is odd, and becomes

$$I \equiv (rsQ)^2 (rsQ')^2 \ldots (rsQ^{(k)})^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots(10)$$

when n is even, the number of factors $(rsQ)^2$ here being $\frac{1}{2}n$. The degrees of this invariant I are $(1, 1, n)$ in the coefficients of F, Φ, and the sphere respectively.

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* An expressive notation, apparently due to Dr R. Weitzenböck, meaning that the preceding statements are true *for all values of* ξ.

† *Cf. Proc. Roy. Soc. Edin*, XLIV. p. 23.

Conversely it is easy to shew that the only such invariant of degrees $(1,\ 1,\ n)$ in these coefficients is, in its symbolic form,

$$\prod_{i=1}^{n} (rsq^{(i)}q'^{(i)})$$

where the $2n$ symbols $q^{(i)}q'^{(i)}$ are composed of n pairs of equivalent symbols $q,\ q\ ;\ q',\ q'$; .... Also this product reduces to zero if n is odd, and to a numerical multiple of I if n is even, by a simple use of the fundamental symbolic identities, such as

$$2(rsqq')\,(rsqq'')\,(rsq'q''') = (rsqq')^2(rsqq''').$$

To prove the equivalence of (9) and (10), we use the ratios (5) and the values of q_{ij} given by (7), namely

$$q_{14} = -q_{23} = 1, \text{ otherwise } q_{ij} = 0 : \ q = q',$$
$$q_{ij} = q_iq_j = q_jq_i\ ;\ i,\ j, = 1,\ 2,\ 3,\ 4.$$

Also by (5) $r_1r_4 = r_2r_3,\ s_1s_4 = s_2s_3$. Thus

$$(rsQ)^2 = (rsqq')^2 = 2\begin{vmatrix} |\, r_2s_3q_4\,|, & |\, r_3s_1q_4\,| \\ |\, r_2s_4q_1\,|, & |\, r_2s_1q_3\,| \end{vmatrix}$$

on substituting its real values for q',

$$= -2\{(rs)_{14}^2 + (rs)_{23}^2 + 2(rs)_{13}(rs)_{42} + 2(rs)_{12}(rs_{43})\}$$
$$= -2\{r_1s_4 + r_4s_1 - r_2s_3 - r_3s_2\}^2$$

using $r_1r_4 = r_2r_3$ and $s_1s_4 = s_2s_3$,

$$= -\frac{2}{\rho^2}(ab)^2(a'b')^2,$$

whence, if n is even,

$$\rho^n I = (-)^{\frac{n}{2}} 2^{\frac{n}{2}} (ab)^n (a'b')^n \quad \text{.....................(11)}$$

The form (10) of the quaternary equivalent of double binary apolarity is the proper interpretation to take since it is linear and also symmetrical in the coefficients of the quaternary n-ics corresponding to the given double binary forms. It also lends itself to the case of two equi-forms of orders $(n+\lambda,\ n+\lambda)$ and $(n,\ n)$ $\lambda > 0$. The apolarity condition then is

$$I' \equiv I \,.\, r_\xi^\lambda = 0 \ \ (\xi)$$

where $r_\xi^{n+\lambda}$ is the higher of the two corresponding quaternary forms.

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Other interpretations of (10) can however be given. Thus if in (10) r is replaced throughout by u, where u denotes plane coordinates contragredient to ξ, then (10) becomes the tangential equation of the polar reciprocal Ψ of the surface Φ with regard to the sphere. For the polar plane of a point (x) with regard to the sphere has coordinates

$$u_i = q_x q_i,$$

and the result of substituting these values of u_i for r in I vanishes owing to conditions (6).

The same also holds with everything as between r and s interchanged. Hence

When I vanishes, the surface F is apolar to this polar reciprocal of Φ, and Φ is apolar to the polar reciprocal of F: and the spherical curves determined by F and Φ are then apolar to each other in the double binary sense.

§ 4. *The* (2, 2) *form.*

If $m = n = 2$ the form f has four linearly independent (1, 1) polars, and geometrically $f = 0$ represents a bicircular quartic while the (1, 1) polars represent circles, for the particular geometrical interpretation already adopted.

Hence in general this quartic curve has no apolar circles, since this would require all four first polar circles to have a common apolar (*i.e.* orthogonal) circle. This only happens when the fourth degree invariant* $\triangle_4$ vanishes, and conversely.

Canonical form of the apolar condition. We transform to tetracyclic coordinates x_1, x_2, x_3, x_4 which are linear functions of the set (ξ), making equation (6) take the form

$$q_\xi^2 = 2(\xi_1\xi_4 - \xi_2\xi_3) = x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv (x^2). \qquad \text{...........(12)}$$

This admits of a canonical form

$$(ax^2) \equiv a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 \qquad \text{..................(13)}$$

where
$$(a) \equiv a_1 + a_2 + a_3 + a_4 = 0 \qquad \text{.......................(14)}$$

for the general (2, 2) form f.†

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* *Cf. Proc. Roy. Soc. Edin.*, XLIV. p. 30.

† *Cf.* **Kasner,** *loc. cit.* Also *Proc. R. S. E.*, XLIV. p. 25.

In this, the last condition (14) is the equivalent of (6); also, regarded as a quaternary form, the spherical (2, 2) curve given by (12) and (13) is associated with a definite quadric (13) containing the curve, since of all such quadrics

$$(ax^2) + \lambda(x^2) = 0$$

we choose the one when λ vanishes.

Now any other (2, 2) form ϕ will be given by an expression

$$\Sigma b x_i x_j \equiv b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 + b_4 x_4^2 + 2b_{12} x_1 x_2 + \ldots + 2b_{34} x_3 x_4 \ldots\ldots (15)$$

This will be apolar to f if

$$a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

for this is the form taken by the invariant $(rsQ)^2$ when the quadrics are of types (13), (15), (12).

If we replace $\Sigma b x_i x_j$ by $\Sigma b x_i x_j + \lambda(x^2)$ the latter quadric also satisfies this apolar condition (16) in virtue of (14). Hence when two spherical (2, 2) curves are apolar, only one of the corresponding quadrics F need be apolar to the sphere (and satisfy (14)), the other (15) need not be.

§ 5. *Pairs of circles and repeated circles apolar to a* (2, 2) *curve.*

If the four circles, in the plane Π, given by $x_i = 0$, $i = 1, 2, 3, 4$, are called the fundamental circles, then any of the six pairs of fundamental circles is apolar to the (2, 2) curve. For if $\Phi = x_i x_j$ the condition (16) is satisfied.

Hence the ∞^5 (2, 2) forms $\sum\limits_{\nu=1}^{6} p_\nu x_i x_j$ $(i \neq j)$ are apolar to the doubly infinite system of (2, 2) forms which have four fundamental circles in common.

The repeated circle, given by $(lx)^2 \equiv (l_1 x_1 + l_2 x_2 + l_3 x_3 + l_4 x_4)^2 = 0$, is by (16) apolar to (ax^2) if

$$(al^2) \equiv a_1 l_1^2 + a_2 l_2^2 + a_3 l_3^2 + a_4 l_4^2 = 0, \ \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

Hence the corresponding plane $(lx) = 0$ envelopes the quadric

$$(x^2/a) = 0$$

which cuts the sphere in the degree three covariant curve C_3* of $f=(ax^2)$.

In particular if $l_1^2=l_2^2=l_3^2=l_4^2=1$, then there are eight different planes $(lx)=0$ which satisfy (17). Hence the form

$$\sum_{\nu=1}^{8} p_\nu(x_1 \pm x_2 \pm x_3 \pm x_4)^2$$

is apolar to f, Now by § 1 there are eight linearly independent (2, 2) forms apolar to f, so at first sight we seem to have here a possible expression of eight such forms as linear combinations of the same eight squares. This is not so since two linear relations connect these eight forms $(x_1 \pm x_2 \pm x_3 \pm x_4)^2$:—

$$\begin{aligned}(x_1+x_2+x_3+x_4)^2&+(x_1-x_2-x_3+x_4)^2\\&+(x_1-x_2+x_3-x_4)^2+(x_1+x_2-x_3-x_4)^2\\=(x_1+x_2+x_3-x_4)^2&+(x_1+x_2-x_3+x_4)^2\\&+(x_1-x_2+x_3+x_4)^2+(-x_1+x_2+x_3+x_4)^2\\=0.\end{aligned}$$

The first of these conditions is true identically, the second is due to (12). Otherwise the eight forms are independent, as may be verified. So six of them may be taken as linearly independent, giving an alternative expression for the ∞^5 set of (2, 2) curves apolar to the doubly infinite set which possess the same fundamental circles.

§ 6. *Point circles and conjugate points apolar to a* (2, 2) *curve.*

Again two points in the plane of a (2, 2) curve are defined to be conjugate, if the polar circle of one passes through the other.

Thus the points (x, y) and (u, v) are conjugate for $a_x^2 a_y'^2$ if $a_x a_u a'_y a'_v$ vanishes.

It follows that a pair of point circles (of zero radius), situated at such a pair of conjugate points, is apolar to the (2, 2) curve, and that no other pairs of point circles are apolar. For, in the tetracyclic coordinates, $(\gamma x)=0$ represents a point circle only if $(\gamma^2)=0$,

* *Cf. Proc. R. S. E.*, XLIV. p. 29.

and two points, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ and $(\gamma_1', \gamma_2', \gamma_3', \gamma_4')$ are conjugate if

$$(a\gamma\gamma') \equiv a_1\gamma_1\gamma_1' + a_2\gamma_2\gamma_2' + a_3\gamma_3\gamma_3' + a_4\gamma_4\gamma_4' = 0,$$

which is the apolar condition for (ax^2) and $(\gamma x)(\gamma' x)$.

In particular a repeated point circle $(\gamma x)^2$ is only apolar to the (2, 2) curve if its point lies on the curve; and conversely.

Suppose $\delta_i \equiv \alpha_x^{(i)}\beta_y^{(i)} = 0$ to denote a point circle, so that α_x and β_y are real and not merely symbolic factors; then if eight points $(i = 1, 2, \dots 8)$ be chosen in general position on the (2, 2) curve, and if $\delta_i = 0$ denote the corresponding point circles, the expression

$$\sum_i \lambda_i \delta_i$$

is the general (2, 2) form apolar to the given form.

Conversely eight linearly independent (2, 2) forms may each be simultaneously expressed as the sum of squares of the same eight linear degenerate (1, 1) forms, by choosing suitable eight points on the unique apolar (2, 2) curve.

If seven points are taken in general position on the sphere, that is to say, such that no linear relation connects the squares of the forms δ_i $(i = 1, 2, \dots 7)$ determined by their seven-point circles, then a singly infinite system of (2, 2) curves have these for common points, together with an eighth associated point. Hence the seven linearly independent (2, 2) curves all apolar to any two of this system are simultaneously expressible as the sum of the same seven squares.

§7. *Curves of the Argand plane invariantive under a given linear transformation*

$$Z = (aW + b) / (cW + d).$$

The questions raised by Professor A. R. Forsyth* regarding circular curves lend themselves to the foregoing treatment. These questions are two: (1) What circular curves remain unaltered by a given linear transformation as above? (2) How many such transformations leave a given curve unaltered?

* Homographic Transformations. *Quarterly Journal* (1910), p. 113. Also, in the same volume, a note by Steinthal on p. 221.

Geometrically this transformation is equivalent, save for translation, to two successive inversions with regard to two orthogonal circles. Let x_3, x_4, denote such orthogonal circles; so that we may take tetracyclic coordinates x_1, x_2, x_3, x_4 which satisfy the identity

$$(x^2) \equiv x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0. \quad \text{.....................}(18)$$

Then the operation of inversion with regard to the circle x_3 may be denoted by $I\{x_3\}$ and is analytically given by

$$I\{x_3\} f(x_1, x_2, x_3, x_4) = f(x_1, x_2, -x_3, x_4).$$

Similarly

$$I\{x_4\} f(x_1, x_2, x_3, x_4) = f(x_1, x_2, x_3, -x_4).$$

So a transformation $T\{x_3. x_4\}$, equivalent to two such inversions, satisfies the relations

$$T\{x_3, x_4\} = T\{x_4, x_3\} = I\{x_3\} I\{x_4\} = I\{x_4\} I\{x_3\},$$
$$T\{x_3, x_4\} f(x_1, x_2, x_3, x_4) = f(x_1, x_2, -x_3, -x_4). \quad \text{.........}(19)$$

We deal with this modified form of the given transformation, $Z \rightarrow W$, from Z to W given above. The original transformation may be written

$$(Z - p)(W - q) = r$$

where p, q, r are arbitrary complex numbers. If we make $p = q$ the transformation is less general by loss of a translation in one or other of the Z or W planes, but it becomes an exact equivalent of the double inversion transformation $T\{x_3, x_4\}$.

§ 8. Now let $[\xi, \eta]^p$ denote the general binary form in $\xi : \eta$ of order p. With this notation it is easy to write down the general double binary (1, 1), (2, 2), ... equiforms which are left unaltered by the given transformation. Owing to (19) they must be

$$\left.\begin{array}{ll} (1, 1) & ax_1 + bx_2 \text{ and } cx_3 + dx_4, \\ (2, 2) & [x_1, x_2]^2 + [x_3, x_4]^2, \\ (3, 3) & [x_1, x_2]^3 + [x_1, x_2]\,[x_3, x_4]^2 \text{ and} \\ & [x_1, x_2]^2\,[x_3, x_4] + [x_3, x_4]^3, \\ (4, 4) & [x_1, x_2]^4 + [x_1, x_2]^2\,[x_3, x_4]^2 + [x_3, x_4]^4, \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\} \text{......}(20)$$

which are the most general forms unaltered when the signs of x_3, x_4 are simultaneously changed. Hence the corresponding geometrical loci are unaltered by the double inversion.

In the above set the (2, 2) form has six arbitrary coefficients. Since the identical relation, $(x^2) \equiv 0$, exists, these coefficients are effectively five. Hence an ∞^4 of these (2, 2) curves exist.*

This completely answers the first of the two questions cited above. The second is the converse question and requires a special answer for each order of circular curve. It will now be shewn that

Three and only three independent automorphic transformations leave a given general (2, 2) *form invariantive.*

For writing the (2, 2) form symbolically as

$$a_z^2 a_{z'}'^2 = b_z^2 b_{z'}'^2$$

then its branch quartic for the z rays is †

$$(a'b')^2 a_z^2 b_z^2.$$

This is a binary quartic which must remain unaltered since the (2, 2) form remains unaltered by the transformation T. Also the transformation T is equivalent to a linear transformation as far as this quartic is concerned. Since three, and only three, linear transformations leave a given binary quartic unaltered, it follows that not more than three transformations T leave the given (2, 2) form unaltered.

But if we now refer the (2, 2) form to its canonical tetracyclic form

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = 0$$

where
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0,$$

we may account for exactly three independent transformations by selecting

$$(1)\ T\{x_1, x_2\} \qquad (2)\ T\{x_1, x_3\} \qquad (3)\ T\{x_1, x_4\}$$

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* *Cf.* Forsyth, *loc. cit.*, § 12, which agrees with this result, as the (2, 2) curve there treated is restricted by three conditions.

† *Cf. Proc. Roy. Soc. Edin.*, XLIV., p. 36. This geometrical application of double binary algebra was suggested by Mr J. H. Grace, to whom my thanks are due.

or, what is the same thing,

$$(1)\ T\{x_3, x_4\} \qquad (2)\ T\{x_2, x_4\} \qquad (3)\ T\{x_2, x_3\}.$$

Thus *the only automorphic transformations which leave a given* (2, 2) *plane curve unaltered are the three which are expressible by successive inversion in any two of the four fundamental circles.**

It follows at once that for the large class of circular curves, whose equations can by proper choice of coordinates be reduced to the type

$$f(x_1^2, x_2^2, x_3^2, x_4^2) = 0,$$

where f is a rational homogeneous function of its arguments, there are at least these three automorphic transformations which leave the curves unaltered.

It also follows that

If an automorphic transformation leaves a given (2, 2) *curve* f *unaltered, it leaves unaltered all* (2, 2) *curves confocal with* f, *and, more generally, the doubly infinite system of* (2, 2) *curves with the same four fundamental circles.*

For all these curves are expressible in the form

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 = 0.$$

§ 9. It may now be shewn that *for all integral values of* n *greater than* 2 *there is no automorphic transformation which leaves the general* (n, n) *curve unaltered.*

In fact let the general (n, n) form in z, z' be written as

$$\phi = (a_0, a_1, a_2, \ldots a_n)(z', 1)^n$$

a general binary n-ic in z' where the coefficients $a_0 \ldots a_n$ are in turn general binary n-ics in z. Then the discriminant of this binary form in z' is of degree $(2n-2)$ in the coefficients a and therefore is a binary form of order $n(2n-2)$ in the variable z.

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* *Cf.* Forsyth, *loc. cit.*, where the problem is suggested but unanswered. The case when $\Sigma \dfrac{1}{a}$ vanishes, *i.e.* a third degree invariant vanishes, is solved. An algebraic solution of the general problem is given by Steinthal, *loc. cit.* but the method used throws no light on the geometrical theory.

Neglecting terms in the discriminant D containing $a_1, a_2, \ldots a_{n-1}$ it may be written

$$D \equiv a_0^{n-1} a_n^{n-1} \bmod a_1, a_2, \ldots a_{n-1}.$$

Now suppose an automorphic transformation T exists for the form ϕ, then the same transformation must leave D invariantive. It is sufficient to shew that this is impossible even in the special case when all $a_1, a_2, \ldots a_{n-1}$ vanish, and therefore

$$\phi = a_0 z'^n + a_n, \quad D = a_0^{n-1} a_n^{n-1}$$

a_0, a_n being arbitrary binary n ics in the other variable z. In this case D is the $(n-1)^{\text{th}}$ power of an arbitrary $2n$-ic, for which no automorphic transformation exists if $n > 2$ For if the roots of D are $\alpha_1, \alpha_2, \ldots \alpha_i \ldots \alpha_{2n}$, such a transformation implies relations of the type

$$p\alpha_i\alpha_j + q\alpha_i + r\alpha_j + s = 0, \quad i, j = 1, 2, \ldots 2n,$$

p, q, r, s being the same for all the relations. In every case when $n > 2$, these relations imply, when p, q, r, s are eliminated, conditions between the roots α which by hypothesis are independent. This proves the theorem