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ON GOING DOWN IN POLYNOMIAL RINGS

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1. Introduction. Our main purpose is to enlarge upon the studies of McAdam [**9**; **10**] on the property of going down (GD) for prime ideals in extensions of (commutative integral) domains. Unlike the investigations of McAdam and the earlier work of Krull [**8**] and Cohen-Seidenberg [**4**] on GD and the related property of going up (GU), this paper is not primarily concerned with integral extensions. Consideration of more general extensions of domains $A \subset B$ is facilitated by the following basic definitions. A prime ideal P of A is *unibranched in* B if there exists exactly one prime ideal Q of B satisfying $Q \cap A = P$. The extension $A \subset B$ is *unibranched* (respectively, *mated*) if every prime ideal of A (respectively, every prime ideal P of A such that $PB \neq B$) is unibranched in B.

Throughout the paper, $R \subset T$ is an extension of domains, K is the quotient field of R, and x is an indeterminate commuting with the appropriate coefficient rings. Any unexplained terminology is standard, as in [**6**].

We begin by recalling the following result [**10**].

THEOREM (McAdam). *Let T be contained in the integral closure of R (in K). Then $R[x] \subset T[x]$ satisfies* GD *if and only if $R[x] \subset T[x]$ is unibranched.*

In (2.1) we modify McAdam's argument and obtain the following generalization.

THEOREM. *Let $T \subset K$. If $R[x] \subset T[x]$ satisfies* GD, *then $R \subset T$ satisfies* GD *and $R[x] \subset T[x]$ is mated.*

In Propositions 3.1 and 3.2, we use the data in [**9**] on prime ideals of polynomial rings to find necessary and sufficient conditions that $R[x] \subset T[x]$ be mated (respectively, satisfy GD). Together with a recent result of Kaplansky characterizing integrality in terms of GU in polynomial rings (see (3.8)), these lead in (3.9) to an example where GD is not inherited by polynomial rings.

Section 4 is a detailed study of the case R pseudo-Bézout (GCD) and $T = R[u]$ for some u in K. Then GD or matedness of $R \subset T$ or $R[x] \subset T[x]$ amounts to T being a localization of R. Several characterizations of Bézout domains are thus obtained.

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2. An implication of GD. In this brief section, we consider the effect of removing the integrality assumption in the result of McAdam quoted above. The result obtained below generalizes McAdam's, as the properties of GU and lying over show any integral mated extension is unibranched and satisfies GD. (2.1) also extends [**9**, Theorem A], itself a generalization of an argument of Kaplansky, and part of [**9**, Theorem C].

THEOREM 2.1. *Let $T \subset K$. If $R[x] \subset T[x]$ satisfies* GD, *then $R \subset T$ satisfies* GD *and $R[x] \subset T[x]$ is mated.*

Proof. That $R \subset T$ inherits GD from $R[x] \subset T[x]$ was shown in [**9**, Lemma 2]. To prove $R[x] \subset T[x]$ is mated, we ape the proof in [**10**] as much as possible. Let N be a prime ideal of $R[x]$ such that $NT[x] \neq T[x]$; set $P = N \cap R$. Since $PT \neq T$ and we are assuming $R[x] \subset T[x]$ satisfies GD, [**9**, Theorem A] provides a unique prime Q of T lying over P. By [**6**, 1-6, Exercise 38] or [**9**, Lemma 1], there also exists a prime of $T[x]$ lying over N.

If N is not unibranched, let M_1 and M_2 be distinct primes of $T[x]$ lying over N. Since $M_i \cap R = N \cap R = P$, unibranchedness of P implies $M_i \cap T = Q$, whence $QT[x] \subset (M_1 \cap M_2)$. If $f(x)$ is chosen of minimal degree in $(M_1 \backslash M_2) \cup (M_2 \backslash M_1)$, it follows that the leading coefficient α of $f(x)$ is not in Q. Hence α is not in $NT[x]$.

Next, localize at $S = \{1, \alpha, \alpha^2, \ldots\}$. From the description of primes in localizations [**6**, Theorem 34], we see easily that $R[x] \subset T_S[x] = T[x]_S$ inherits GD from $R[x] \subset T[x]$. Since α is not in $NT[x]$, it follows that $NT_S[x] \neq T_S[x]$. Then $M_iT_S[x] (i = 1, 2)$ are distinct primes which lie over N and whose intersection does not contain $f(x)$. Replacing T by T_S, we may assume α is a unit in T and, hence, that $f(x)$ is monic. As $M_i \cap T = Q$, $f(x)$ is non-constant. We may now apply the argument in [**10**] concerning splitting fields and integrality to obtain a contradiction and complete the proof.

3. Matedness and GD. Throughout this section, we use the following data from [**9**]. If Q is a prime of T and $Q \cap R = P$, let F_P (respectively, K_Q) denote the quotient field of R/P (respectively, T/Q). View $F_P \subset K_Q$ in the usual way. The primes of $R[x]$ lying over P are $P^* = PR[x]$ and uppers $\langle P, \alpha(x) \rangle$ arising from monic irreducible $\alpha(x)$ in $F_P[x]$. By definition, $\langle P, \alpha(x) \rangle$ is the set of all $f(x)$ in $R[x]$ whose canonical images in $(R/P)[x] \subset F_P[x]$ are divisible by $\alpha(x)$. Define Q^* and uppers $\langle Q, \beta(x) \rangle$ in $T[x]$ similarly. The basic facts about uppers are summarized in [**9**, Theorems 1 and 2].

PROPOSITION 3.1. *$R[x] \subset T[x]$ is mated if and only if the following two conditions hold:*

(a) *$R \subset T$ is mated,*

(b) *If Q is a prime of T and $P = Q \cap R$, then $F_P \subset K_Q$ is purely inseparable.*

Proof. The proof of [**9**, Theorem 3] applies with minor changes. For example, in the "only if" half, use of [**9**, Proposition 1] to establish (a) is replaced by the observation that $PT \neq T$ implies $P^*T[x] \neq T[x]$. For the "if" half, note $P^*T[x] \neq T[x]$ implies $PT \neq T$.

PROPOSITION 3.2. *$R[x] \subset T[x]$ satisfies* GD *if and only if the following three conditions hold:*

(a) *$R \subset T$ satisfies* GD.

(b) *If $\langle P, \alpha(x) \rangle$ and $\langle Q, \beta(x) \rangle$ are uppers in $R[x]$ and $T[x]$ respectively, such that $\langle Q, \beta(x) \rangle$ is minimal among primes of $T[x]$ containing $\langle P, \alpha(x) \rangle T[x]$, then $P = Q \cap R$.*

(c) *There do not exist a prime Q of T and an upper $\langle P, \alpha(x) \rangle$ in $R[x]$ such that Q^* is minimal among primes of $T[x]$ containing $\langle P, \alpha(x) \rangle T[x]$.*

Proof. Assume (a), (b) and (c). To show $R[x] \subset T[x]$ satisfies GD, we check equivalently (see [**6**, 1-6, Exercise 37]) the criterion that if $\mathfrak{g}$ is a prime of $R[x]$ and $\mathfrak{q}$ minimal among primes of $T[x]$ containing $\mathfrak{g}T[x]$, then $\mathfrak{g} = \mathfrak{q} \cap R[x]$.

Case 1. $\mathfrak{g} = P^*$: Subcase (i) $\mathfrak{q} = Q^*$. If suffices to show $P = Q \cap R$. By (a) and the cited GD criterion, we need only show Q is minimal among primes of T containing PT. If $Q \supset Q_1 \supset PT$ for some prime Q_1 of T, then $Q^* \supset (Q_1)^* \supset PT[x]$. Minimality of $\mathfrak{q}$ implies $Q^* = (Q_1)^*$, whence $Q = Q_1$, as required.

Subcase (ii) $\mathfrak{q} = \langle Q, \beta(x) \rangle$. Then

$$Q = \mathfrak{q} \cap T[x] \supset \mathfrak{g} \cap R[x] = P \text{ and } Q^* \supset PT[x],$$

contradicting minimality of $\mathfrak{q}$.

Case 2. $\mathfrak{g} = \langle P, \alpha(x) \rangle$: Subcase (i) $\mathfrak{q} = Q^*$. This is explicitly ruled out by (c).

Subcase (ii) $\mathfrak{q} = \langle Q, \beta(x) \rangle$. By (b), $P = Q \cap R$. According to [**9**, Theorem 2(i)], we need only show $\beta(x) | \alpha(x)$ in $K_Q[x]$. Clearing denominators yields r in $R \backslash P$ such that $\bar{r} = r + P$ in R/P satisfies: $\bar{r}\alpha(x)$ is in $(R/P)[x]$. Let $d(x)$ in $R[x]$ reduce to $\bar{r}\alpha(x)$ modulo P. By definition of uppers, $d(x)$ is in $\langle P, \alpha(x) \rangle \subset \langle Q, \beta(x) \rangle$. Hence, $\beta(x) | \bar{r}\alpha(x)$. Since $\bar{r} \neq 0$, $\beta(x) | \alpha(x)$.

Conversely, assume $R[x] \subset T[x]$ satisfies GD. Then (a) was proved in [**9**, Lemma 2]. As for (b), the above GD criterion implies $\langle P, \alpha(x) \rangle = \langle Q, \beta(x) \rangle \cap R[x]$; intersecting with R shows $P = Q \cap R$. Finally, for (c), GD implies $Q^* \cap R[x] = \langle P, \alpha(x) \rangle$; intersect with R to get $Q \cap R = P$, whence $Q^* \cap R[x] = P^*$, contradicting [**9**, Theorem 1].

We note that Proposition 3.2 supplies an amusing proof that $R[x] \subset T[x]$ satisfies GD whenever T is a field.

COROLLARY 3.3. *$R[x] \subset T[x]$ is unibranched if and only if the following two conditions hold:*

(a) *$R[x] \subset T[x]$ is mated.*

(b) *For every prime P of R, there exists a prime Q of T such that $P = Q \cap R$.*

Proof. Apply [**9**, Theorem 3] and Proposition 3.1.

COROLLARY 3.4. *Let R be Noetherian of* (*Krull*) *dimension* 1. *Assume either T*

is integral over R or $T \subset K$. Then $R[x] \subset T[x]$ satisfies GD *if and only if the following condition holds:*

If $\langle 0, \alpha(x)\rangle$ and $\langle Q, \beta(x)\rangle$ are uppers in $R[x]$ and $T[x]$ respectively, such that $\langle Q, \beta(x)\rangle$ is minimal among primes of $T[x]$ containing $\langle 0, \alpha(x)\rangle T[x]$, then $Q = 0$.

Proof. If Q is a nonzero, not necessarily prime, ideal of T, then $Q \cap R \neq 0$. (In the integral case, consider the constant coefficient of an integrality equation of minimal degree for a nonzero element of Q. In case $T \subset K$, consider the numerator of a nonzero fraction in Q.) Hence, the last condition in the statement of the corollary is precisely the case $P = 0$ of Proposition 3.2(b). It, therefore, suffices to verify conditions (a) and (c) and the case $P \neq 0$ of (b) in Proposition 3.2.

Condition (a) is immediate since R has dimension 1. For (b) in case $P \neq 0$, note

$$Q \cap R = \langle Q, \beta(x)\rangle \cap R \supset \langle P, \alpha(x)\rangle T[x] \cap R \supset P \neq 0,$$

whence $Q \cap R = P$ by one-dimensionality. Finally, for (c), assume Q^* is minimal among primes of $T[x]$ containing $\langle P, \alpha(x)\rangle T[x]$. Then $Q \neq 0$ and, as noted above, $P_1 = Q \cap R \neq 0$. As $(P_1)^* = Q^* \cap R[x] \supset \langle P, \alpha(x)\rangle$, $P^* \subsetneq \langle P, \alpha(x)\rangle$, we have $P \subsetneq P_1$, and $P = 0$ by one dimensionality. Since R is Noetherian, [**6**, Theorem 149] shows that $(P_1)^*$ and $\langle 0, \alpha(x)\rangle$ each have height 1 in $R[x]$. Thus, $(P_1)^* = \langle 0, \alpha(x)\rangle$ and $P_1 = (P_1)^* \cap R = \langle 0, \alpha(x)\rangle \cap R = 0$, a contradiction, to complete the proof.

Remark 3.5. The following result gives some information in the direction of the converse of Theorem 2.1. Assume $R \subset T$ satisfies GD at P (obvious definition) and P^* is unibranched in $T[x]$. Then $R[x] \subset T[x]$ satisfies GD at P^*. (The key is to observe that the radical $\sqrt{P^*T[x]}$ is prime in $T[x]$.) A similar result for uppers, one prime at a time, would be of interest.

We shall see in Example 3.9 that the condition in Corollary 3.4 does not always hold. The case of Dedekind R, included in the next result, is however much simpler. First, recall that if R is Prüfer then T is R-flat [**3**, VII, Proposition 4.2] and hence $R \subset T$ satisfies GD [**11**, (5.D), p. 33].

Proposition 3.6. *If R is Prüfer, then $R[x] \subset T[x]$ satisfies* GD.

Proof. As noted above, T is R-flat. Hence, $T[x] \cong T \otimes_R R[x]$ is $R[x]$-flat and an application of [**11**, (5.D), p. 33) completes the proof.

We next give an example where GD and unibranchedness imply integrality.

Proposition 3.7. *Let R be Prüfer, $R \subset T$ be unibranched and $T \subset K$. Then $R = T$.*

Proof. Let M be a maximal ideal of R and N the (necessarily maximal) prime of T lying over M. It is clear from the behavior of primes in localizations that $R_M \subset T_N$ is unibranched. However, R_M is a valuation ring [**6**, Theorem

64], hence maximal with respect to dominance inside K [**1**, Théorème 1, p. 89]. Thus, $R_M = T_N$ and $R = \bigcap R_M = \bigcap T_N = T$.

We pause to record the principal result of [**7**].

THEOREM 3.8 (Kaplansky). *If $R[x] \subset T[x]$ satisfies* GU, *then T is integral over R.*

Example 3.9. We now give an example where R is Noetherian quasi-local of (Krull) dimension 1, $R \subset T$ is unibranched, $T \subset K$, T is not integral over R, $R[x] \subset T[x]$ does not satisfy GD. (Thus, GD is not necessarily inherited by polynomial rings even if, as the Krull-Akizuki Theorem [**6**, Theorem 93] shows is the case here, the coefficient rings are one-dimensional Noetherian.)

For the example, let R be a Noetherian quasi-local one-dimensional domain whose integral closure D contains distinct maximal ideals Q_1 and Q_2. (An example of such R may be fashioned from [**13**, Example 2, p. 102].) Let T be the discrete valuation ring D_{Q_1}. By considering a numerator of a nonzero fraction in T, we see that the maximal ideal of T meets R nontrivially and, hence, that $R \subset T$ is unibranched. Of course, T is not integral over R, since $D \subsetneq T$.

If $R[x] \subset T[x]$ satisfies GD, then $R[x] \subset T[x]$ is mated by Theorem 2.1. As $R \subset T$ is unibranched, Corollary 3.3 shows $R[x] \subset T[x]$ is unibranched. Hence $R[x] \subset T[x]$ satisfies GU. Theorem 3.8 implies T is integral over R, a contradiction, showing that $R[x] \subset T[x]$ does not satisfy GD.

As an immediate corollary of the preceding proof, we have

COROLLARY 3.10. *Assume $T \subset K$ and $R[x] \subset T[x]$ satisfies* GD. *Then the following conditions are equivalent.*

(a) *T is integral over R.*

(b) *For every prime P of R, there exists a prime Q of T such that $P = Q \cap R$.*

(c) *$R[x]$ is unibranched in $T[x]$.*

Remark 3.11. Because of (3.7) and (3.9), it would be interesting to know under what additional conditions one can infer $R = T$, given $T \subset K$, R integrally closed, $R \subset T$ unibranched and GD. In this regard, recall that the (u, u^{-1}) lemma implies the following result [**6**, 1-6, Exercise 19]. If R is integrally closed quasi-local of dimension 1, $T = R[u]$ for some u in K and $R \subset T$ is unibranched, then $R = T$.

4. GCD domains. In this final section we consider simple extensions of GCD (or pseudo-Bézout) domains, i.e. domains in which every pair of nonzero elements has a greatest common divisor. The localization example $Z \subset Z[\frac{1}{2}] = Z_2$ will be seen to be typical of such extensions which are mated or satisfy GD.

PROPOSITION 4.1. *Let R be a* GCD *domain, a and b nonzero relatively prime elements of R such that $u = ab^{-1}$ is not in R, and $T = R[u]$.*

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(i) *There exists a prime of* R *containing* b *but not containing* a. *For any such prime* P, *there exists no ideal* Q *of* T *such that* $P = Q \cap R$.

(ii) *Let* P *be a prime of* R *not containing* b. *Then* $PR_b \cap T$ *is minimum among primes of* T *lying over* P.

(iii) *Let* P *be a prime of* R *containing* b. *If there exists a prime of* T *lying over* P, *then* P *contains* a. *Conversely, if* P *contains* a, *then* PT *is prime.*

Proof. (i) Assume that a lies in every prime of R containing b. Then a is in the radical $\sqrt{Rb}$, whence $a^n = rb$ for some r in R and $n \geqq 1$. Since a and b are relatively prime, [**6**, 1-6, Exercise 7] may be applied repeatedly to show b is a unit. Hence, u is in R, contradicting our standing hypotheses.

Now, let P be any prime of R containing b but not containing a. If $P = Q \cap R$ for an ideal Q of T, then a is not in Q. However, $bu = a$ is in Q, thus proving that no such Q exists.

(ii) As b is not in P, PR_b is a prime of R_b; since $T \subset R_b$, it follows that $I = PR_b \cap T$ is a prime of T. Moreover, $I \cap R = PR_b \cap (R \cap T) = PR_b \cap R = P$; i.e., I lies over P. Note $R_b = T_b$. Hence, if Q is any prime of T lying over P, then $I = PT_b \cap T \subset QT_b \cap T = Q$.

(iii) Assume some prime Q of T lies over P. If a is not in P then a is not in Q and $QT_a \neq T_a$, whence $PT_a \neq T_a$. Since $a^{-1}u = b^{-1}$ is in T_a and b is in P, we have a contradiction. Thus a is in P.

Conversely, assume that a is in P. Let $f : R[x] \to T$ be the surjective R-algebra homomorphism sending x to u. As a and b are relatively prime, a (gcd-) content argument [**2**, Exercice 23 (a), p. 87] implies $\ker(f) = (bx - a)R[x]$. In particular, $\ker(f) \subset PR[x]$, and so $f(PR[x]) = PT$ is a prime of T.

THEOREM 4.2. *Let* R *be a* GCD *domain,* a *and* b *nonzero relatively prime elements of* R *such that* $u = ab^{-1}$ *is not in* R, *and* $T = R[u]$. *Then the following conditions are equivalent.*

(1) $R[x] \subset T[x]$ *satisfies* GD.

(2) $R[x] \subset T[x]$ *is mated.*

(3) $R \subset T$ *is mated.*

(4) $Ra + Rb = R$.

(5) $T = R_b$.

(6) T *is* R*-flat.*

(7) $R \subset T$ *satisfies* GD.

Proof. Some of the implications are immediate: (1) $\Rightarrow$ (2) by Theorem 2.1; (2) $\Rightarrow$ (3) by Proposition 3.1; since $R_b[x] = R[x]_b$, (5) $\Rightarrow$ (1) by the usual description of primes in localizations. If $ra + sb = 1$ for some r and s in R, then $b^{-1} = ru + s$ is in $R[u] = T$, whence $R_b \subset T$. Since $T \subset R_b$, this proves (4) $\Rightarrow$ (5).

To show (3) $\Rightarrow$ (4), assume $R \subset T$ is mated. If (4) fails, choose a prime P containing a and b. Choose an upper $\langle P, \alpha(x)\rangle$ and let $f : R[x] \to T$ be the map considered in the proof of Proposition 4.1 (iii). Then $\mathfrak{q} = f(\langle P, \alpha(x)\rangle)$ is a prime of T such that $P_1 = \mathfrak{q} \cap R$ contains a and b. Since matedness implies P_1 is unibranched in T, Proposition 4.1 (iii) shows $\mathfrak{q} = P_1T$. Considering inverse images under f yields $\langle P, \alpha(x) \rangle = P_1R[x]$. Intersect with R to get $P = P_1$ and $\langle P, \alpha(x) \rangle = PR[x]$, contradicting [**9**, Theorem 1]. Hence (3) $\Rightarrow$ (4).

As we saw in the proof of Proposition 4.1 (iii), $\ker(f) = (bx - a)R[x]$. Hence, $T \cong R[x]/(bx - a)R[x]$. Note that the R-submodule of R generated by the coefficients of elements of $(bx - a)R[x]$ is $Ra + Rb$. Since $(bx - a)R[x]$ is an invertible ideal of $R[x]$, we may apply a result of Ohm-Rush [**12**, Corollary 1.3] to infer (4) $\Leftrightarrow$ (6).

Hence, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6). Since (1) $\Rightarrow$ (7) by [**9**, Lemma 2], it suffices to prove (7) $\Rightarrow$ (6). Assume $R \subset T$ satisfies GD. We need only prove T is locally flat, i.e., that $T \otimes_R R_M$ is R_M-flat for every maximal ideal M of R. However, R_M is GCD [**2**, Exercice 21, p. 86],

$$T \otimes_R R_M \cong T_{R \setminus M} = R_M[u],$$

and $R_M \subset T_{R\setminus M}$ inherits GD from $R \subset T$. Thus, we may assume R is quasi-local with maximal ideal M. Since (4) $\Rightarrow$ (6), it suffices to prove $Ra + Rb = R$. By Proposition 4.1 (i), choose a prime P of R containing b but not a. Since no prime of T lies over P and $R \subset T$ satisfies GD, we see that no prime of T lies over M. Thus Proposition 4.1 (iii) implies that a is not in M; i.e., a is a unit and $Ra + Rb = R$.

COROLLARY 4.3. *The following conditions are equivalent.*

(1) *R is Bézout.*

(2) *R is* GCD *and S is R-flat for all R-submodules S of K.*

(3) *R is* GCD *and $R \subset S$ satisfies* GD *for all rings $R \subset S \subset K$.*

(4) *R is* GCD *and $R \subset R[u]$ satisfies* GD *for all u in K.*

Proof. Since Bézout domains are GCD and Prüfer, the remarks preceding Proposition 3.6 give (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Note (3) $\Rightarrow$ (4) is trivial.

Next, assume (4). By Theorem 4.2, $R[u]$ is R-flat for all u in K. Moreover, [**14**, Proposition 2] or [**6**, Theorem 50] shows R is integrally closed. Then [**14**, Proposition 3] and [**5**, Theorem 13, (10) $\Rightarrow$ (9)] imply R is Prüfer. As R is also GCD, [**6**, 1-6, Exercise 15] then shows that every finitely generated ideal of R is principal, i.e., R is Bézout. Hence, (4) $\Rightarrow$ (1).

Finally, we give an alternate, possibly simpler, proof that (4) $\Rightarrow$ (1). Assume (4). As noted in [**6**, p. 32], we need only show that, if a and b are nonzero elements of R with greatest common divisor d, then $Ra + Rb = Rd$. Without loss of generality, suppose ab^{-1} is not in R. Since ad^{-1} and bd^{-1} are relatively prime [**6**, Theorem 49(b)] and $R \subset R[ab^{-1}] = R[(ad^{-1})(bd^{-1})^{-1}]$ satisfies GD, Theorem 4.2 shows $Rad^{-1} + Rbd^{-1} = R$. Thus, $Ra + Rb = Rd$, to complete the proof.

Applying Theorem 4.2 to Corollary 4.3(4) produces six more characterizations of Bézout domains, giving nine in all. Since valuation domains are characterized as quasi-local Bézout [**6**, Theorem 63], these remarks also yield nine characterizations of valuation domains.

Our final result follows immediately from the preceding corollary.

COROLLARY 4.4. *Let R have (Krull) dimension* 1. *Then R is Bézout if and only if R is* GCD.

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