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PROJECTIONS OF HYPERSURFACES IN $\mathbb{R}^4$ WITH BOUNDARY TO PLANES

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Abstract. We study orthogonal projections of generic embedded hypersurfaces in $\mathbb{R}^4$ with boundary to 2-spaces. Therefore, we classify simple map germs from $\mathbb{R}^3$ to the plane of codimension less than or equal to 4 with the source containing a distinguished plane which is preserved by coordinate changes. We also go into some detail on their geometrical properties in order to recognize the cases of codimension less than or equal to 1.

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1. Introduction. In this paper we study the singularities of orthogonal projections of a generic embedded hypersurface M in $\mathbb{R}^4$ with boundary to a two-dimensional plane. The singularities occurring at interior points have already been classified in [**13**] (see also [**17**]), so we shall be concerned solely with the nature of the projections at boundary points. We consequently need to classify map germs from $\mathbb{R}^3$ to the plane with the source containing a distinguished plane which is preserved by coordinate changes. The singularities of such maps measure, for instance, the contact of M with two-dimensional planes. Bruce and Giblin [**4**] investigated the singularities of projections of generic surfaces in $\mathbb{R}^3$ with boundary to a plane, i.e. the authors classify map germs from plane to plane with the source containing a line which is preserved by coordinate changes. Tari [**18**] generalized such problem, considering projections of more general piecewise smooth surfaces in $\mathbb{R}^3$. If the space considered is $\mathbb{R}^4$, then studies on surfaces and their projections can be found, for example, in [**6**, **12**, **15**], and on hypersurfaces in [**13**, **14**].

Given a germ of an immersion at $0 \in \mathbb{R}^3$ of the set $V = \{(x, y, z); z \geq 0\}$ into $\mathbb{R}^4$, we can regard the image as a small piece of a smooth hypersurface M in $\mathbb{R}^4$ with boundary. Projections of M to planes are parametrized by the Grassmanian of 2-planes in $\mathbb{R}^4$, $G(2, 4)$. Then an orthogonal projection of M to plane can be represented locally by a germ of a map $(\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$, as we explain in Section 4. If a and b are orthonormal bases of the plane of projection $u \in G(2, 4)$, then the family of orthogonal projections to 2-spaces is given by

$$\begin{array}{rcl}\Pi : M \times G(2,4) & \to & \mathbb{R}^2 \\ (p,u) & \mapsto & (\langle p,a\rangle, \langle p,b\rangle).\end{array}$$

Given $u \in G(2,4)$, the map Π_u measures the contact between M and the plane orthogonal to u, the kernel of Π_u. (Note that Π_u is of corank at most 1). If p is a corank 1 singular point of Π_u, then the orthogonal plane to u is a subset of T_pM.

Let $\mathcal{E}_n$ be the local ring of germs of functions $(\mathbb{R}^n, 0) \to \mathbb{R}$ and m_n be its maximal ideal (which is the subset of germs that vanish at the origin). Denote by $\mathcal{E}(n,p)$ the p-tuples of elements in $\mathcal{E}_n$. Let $\mathcal{A} = \mathcal{R} \times \mathcal{L} = Diff(\mathbb{R}^n, 0) \times Diff(\mathbb{R}^p, 0)$ denote the group of right–left equivalences, which acts smoothly on $m_n.\mathcal{E}(n,p)$ by $(h,k).f = k \circ f \circ h^{-1}$. We shall consider the subgroup $\mathcal{B}(n)$ of $\mathcal{A}$ ($\mathcal{B}(n) = \mathcal{A}_V$ in notation of [**7**], with $V = \{(x_1, \ldots, x_n); x_n \geq 0\}$) which consists of pairs of germs of diffeomorphisms (h,k), where h preserves the n-dimensional manifold as well as its boundary $(\mathbb{R}^{n-1}, 0)$ (that is, h takes the variety V into itself) and k is any diffeomorphism in the target. When the context is clear, we write $\mathcal{B}$ for $\mathcal{B}(n)$. Since the $\mathcal{A}$-equivalence classes of Π_u do not depend on the choice of orthonormal basis a, b of u (see [**13**]), then we can expect the generic $\mathcal{B}$-equivalence classes of Π_u to be those of $\mathcal{B}_e$-codimension less than or equal to 4. We classify simple germs of maps $(\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ of codimension less than or equal to 4 up to smooth origin preserving changes of coordinates in the source, which preserve the manifold as well as its boundary, and any smooth origin preserving changes of coordinates in the target. This yields an action of $\mathcal{B}$ on $m_3 . \mathcal{E}(3,2)$. The group $\mathcal{B}$ is a geometric subgroup of $\mathcal{A}$ in Damon's terminology [**7**]. The list of orbits of simple germs of corank at most 1 and $\mathcal{B}_e$-codimension less than or equal to 4 of this action are given in Theorem 1.1. Note that the germs (x, y) and $(x, z + g(x, y))$ are all submersions.

THEOREM 1.1. *The $\mathcal{B}$-simple map-germs $(\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ of corank at most 1 and $\mathcal{B}_e$-codimension ≤ 4 are given in Table 1.*

In an analogous way to the work [**4**] (see also Theorem 4.1 in [**8**]), the classification in Theorem 1.1 can be thought as a classification of invariant map germs. Let $T : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be given by $T(x, y, z) = (x, y, -z)$; T yields a $\mathbb{Z}_2$-action on $(\mathbb{R}^3, 0)$. One can classify invariant map-germs $(\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ up to equivariant changes of coordinates in the source and any changes of coordinates in the target. The list of simple invariant germs of corank at most 1 and of codimension ≤ 4 can be obtained by replacing z by z^2 in Table 1.

The classification in Theorem 1.1 also has application in the study of germs of pairs of codimension one regular foliations in $\mathbb{R}^3$, which can be assumed to be given by germs of differential 1-forms ω and $\eta = dz$. An important feature of the pair (ω, η) is its discriminant $D(\omega, \eta)$, that is, the locus of points where the foliations are tangents. This is generically a germ of a space curve. In local coordinates, the discriminant is given by the fibre of a map-germ $F : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$, which we call the discriminant map-germ. In [**10**] the authors show that the discriminant $D(\omega, \eta)$ determines the local topological type of the pair (ω, η) and obtain a complete list of discrete topological models (Theorem 4.1, p. 108). Theorem 1.1 models codimension less than or equal to four simple singularities of the discriminant map-germ up to subgroup $\mathcal{B}$. The $\mathcal{B}$-class of a discriminant map-germ determines if the associated pair of foliations is topologically equivalent to one of the discrete topological models given in [**10**]. Indeed, suppose

Table 1. Normal forms of $\mathcal{B}_e$-codimension ≤ 4 $(\epsilon, \epsilon_1 = \pm 1)$

Normal form	$\mathcal{B}_e$-codimension
(x, y)	0
$(x, z + \epsilon y^2)$	0
$(x, z + xy + y^3)$	0
$(x, z + y^3 + \epsilon^{k-1} x^k y),\ k \geq 2$	$k - 1$
$(x, z + xy + \epsilon y^4)$	1
$(x, z + xy + y^5 + \epsilon y^7)$	2
$(x, z + xy + y^5)$	3
$(x, z + xy + \epsilon y^6 + y^9)$	4
$(x, z + xy^2 + \epsilon y^4 + y^{2k+1}),\ k \geq 2$	k
$(x, z + xy^2 + y^5 + \epsilon y^6)$	3
$(x, z + xy^2 + y^5 + \epsilon y^9)$	4
$(x, z + x^2 y + \epsilon y^4 + \epsilon_1 y^5)$	3
$(x, z + x^2 y + \epsilon y^4)$	4
$(x, y^2 + \epsilon z^2 + {\epsilon_1}^k x^{k-1} z),\ k \geq 2$	$k - 2$
$(x, y^2 + xz + \epsilon z^3)$	1
$(x, y^2 + xz + \epsilon z^4 + \epsilon_1 z^6)$,	2
$(x, y^2 + xz + \epsilon z^4)$	3
$(x, yz + xy + \epsilon y^3)$	1
$(x, yz + xy + y^4 + \epsilon y^6)$	2
$(x, yz + xy + y^4)$	3
$(x, xy + z^2 + y^3 + \epsilon y^k z),\ k \geq 2$	k

that the discriminant map-germ of (ω, η) has a $\mathcal{B}$-simple singularity of codimension less than or equal to four and that it is transverse, away from the origin, to the pair of foliations. Then the topological type of the pair is determined by the number of branches of $D(\omega, \eta)$ in each half region delimited by the leaf of ω (or η) through the origin, provided this number does not exceed two. We can calculate the number of branches of the zero-fibre of a normal form F given in Table 1, in the semi-spaces $z > 0$ and $z < 0$, as follows. Take, for example, the normal form $F(x, y, z) = (x, z + y^2)$. Then $F^{-1}(0, 0)$ is the parabola $z + y^2 = 0$ in the plane $x = 0$. So there are two branches of $F^{-1}(0, 0)$ in the semi-space $z \leq 0$ and none in $z > 0$.

The paper is organized as follows. In Section 2 we give some preliminary concepts from singularity theory. In Section 3 we give the classification, which is carried out inductively on the jet level until a sufficient jet is found. We deal with the geometry of $\mathcal{B}$-simple map germs given in Theorem 1.1 of codimension ≤ 1 in Section 4. We observe that all $\mathcal{B}$-germs of codimension ≤ 1 are simple (Theorem 3.7). This section also contains pictures which can be useful to recognize various cases. Our notation and terminology will follow closely [**4**].

2. Preliminaries.

In this paper we use a method of classification of map germs that is similar to the well known method for group $\mathcal{A}$, and also works for group $\mathcal{B}$.

Given a map-germ $f \in m_n.\mathcal{E}(n, p)$, θ_f denotes the set of germs of vector fields along f (these are sections of the pull-back of the tangent bundle of the target manifold). We set $\theta_n = \theta_{id_{\mathbb{R}^n,0}}$ and $\theta_p = \theta_{id_{\mathbb{R}^p,0}}$, where $id_{\mathbb{R}^n,0}$ and $id_{\mathbb{R}^p,0}$ denote the germs of identity maps on $(\mathbb{R}^n, 0)$ and $(\mathbb{R}^p, 0)$ respectively. One can define the homomorphisms $tf : \theta_n \to \theta_f$ by $tf(\psi) = Df.\psi$, and $wf : \theta_p \to \theta_f$ by $wf(\phi) = \phi \circ f$.

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The tangent space to the $\mathcal{A}$-orbit of f at the germ f is given by

$$\begin{aligned} L\mathcal{A}(f) &= tf(m_n.\theta_n) + wf(m_p.\theta_p) \\ &= m_n.\{f_{x_1}, \ldots, f_{x_n}\} + f^*(m_p).\{e_1, \ldots, e_p\}, \end{aligned}$$

where f_{x_i} denotes the partial derivative with respect to x_i ($i = 1, \ldots, n$), $\{e_1, \ldots, e_p\}$ is the standard basis vectors of $\mathbb{R}^p$ considered as elements of $\mathcal{E}(n, p)$ and $f^*(m_p)$ is the pull-back of the maximal ideal in $\mathcal{E}_p$.

The extended tangent space to the $\mathcal{A}$-orbit of f at the germ f is given by

$$\begin{aligned} L_e\mathcal{A}(f) &= tf(\theta_n) + wf(\theta_p) \\ &= \mathcal{E}_n.\{f_{x_1}, \ldots, f_{x_n}\} + f^*(\mathcal{E}_p).\{e_1, \ldots, e_p\}. \end{aligned}$$

The codimension of the orbit of f is given by

$$\dim_{\mathbb{R}}(m_n.\mathcal{E}(n, p)/L\mathcal{A}(f)),$$

and the codimension of the extended orbit ($\mathcal{A}_e$-codimension) is given by

$$\dim_{\mathbb{R}}(\mathcal{E}(n, p)/L_e\mathcal{A}(f)).$$

Let $k \geq 1$ be an integer. We denote by $J^k(n, p)$ the space of kth order Taylor expansions without constant terms of elements of $\mathcal{E}(n, p)$ and write j^kf for the k-jet of the map f. A germ f is said to be $k - \mathcal{A}$-*determined* if every map g with $j^kg = j^kf$ is $\mathcal{A}$-equivalent to f (notation: $g \sim f$). The k-jet of f is then called a sufficient jet. (See for example [**2**, **3**, **19**] for finite determinacy criteria.)

The method used here is that of complete transversal [**5**] together with Mather's Lemma [**11**], given below, where $\mathcal{A}_1$ denotes the normal subgroup of $\mathcal{A}$ whose elements have 1-jets at 0 equal to the identity. The classification (i.e. the listing of representatives of the orbits) of germs of codimension less than or equal to four is carried out inductively on the jet level. The first below result checks whether f is k-determined and therefore can we stop the induction method. Otherwise, by using the subgroup $\mathcal{A}_1$, the second result says what monomials of degree $k+1$ we need to add to j^kf. Finally, Mather's lemma says if the monomials are really necessary for the $\mathcal{A}$-classification. After this we return to the first theorem to check whether f is finitely determined.

THEOREM 2.1. (Theorem 6.10 in [**2**]).
(1) *A germ* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ *is finitely* $\mathcal{A}$*-determined if and only if for some* N *we have* $m_n^N.\mathcal{E}(n, p) \subset L\mathcal{A}.f$.

(2) *A germ* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ *is* $(2r+1)$-$\mathcal{A}$*-determined if*

$$m_n^{r+1}.\mathcal{E}(n, p) \subset L\mathcal{A}.f + m_n^{2r+2}.\mathcal{E}(n, p).$$

(3) *A germ* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ *is* r-$\mathcal{A}_1$*-determined if and only if*

$$m_n^{r+1}.\mathcal{E}(n, p) \subset L\mathcal{A}_1.f.$$

(4) *A germ* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ *is* r-$\mathcal{A}_1$*-determined if and only if*

$$m_n^{r+1}.\mathcal{E}(n, p) \subset L\mathcal{A}_1.f + m_n^{r+1}.(f^*m_p.\mathcal{E}_n + m_n^{r+1}).\mathcal{E}(n, p).$$

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PROPOSITION 2.2. (Complete transversal, Proposition 2.2 in [**5**]). *Let g be a k-jet in $J^k(n,p)$, and let T be a vector subspace of the set $H^{k+1}(n,p)$ of homogeneous jets of degree $k+1$, such that*

$$H^{k+1}(n,p) \subset T + L(J^{k+1}\mathcal{A}_1)(g)\,.$$

Then any $(k+1)$-jet $j^{k+1}f$ with $j^k g = j^k f$ is $J^{k+1}\mathcal{A}_1$-equivalent to $g+t$ for some $t \in T$. (The vector subspace T is called the complete $(k+1)$-transversal of g.)

LEMMA 2.3. (Mather's Lemma, Lemma 3.1 in [**11**]). *Let $\mathcal{G}$ be a Lie group acting smoothly on a finite dimensional manifold X. Let V be a connected submanifold of X. Then V is contained in a single orbit of $\mathcal{G}$ if and only if*

(1) *for each $x \in V$, $T_xV \subset T_x\mathcal{G}(x) = L\mathcal{G}(x)$;*

(2) $\dim T_x\mathcal{G}(x)$ *is constant for all $x \in V$.*

The notion of simple germs is defined in [**1**] as follows.

DEFINITION 2.4. ([**1**]). *Let X be a manifold and $\mathcal{G}$ a Lie group acting on X. The modality of a point $x \in X$ under the action of $\mathcal{G}$ on X is the least number m such that a sufficiently small neighbourhood of x may be covered by a finite number of m-parameter families of orbits. The point x is said to be simple if its modality is 0, that is, a sufficiently small neighbourhood intersects only a finite number of orbits. The modality of a finitely determined map-germ is the modality of a sufficient jet in the jet-space under the action of the jet-group.*

The results on finite determinacy and complete transversal are stated above for the group $\mathcal{A}$ and were initially proved for the groups $\mathcal{L}, \mathcal{R}, \mathcal{C}, \mathcal{K}$ and $\mathcal{A}$ (see [**5**, **3**]). However, Damon[**7**] showed that these results are also valid for a larger class of subgroups of $\mathcal{K}$ and $\mathcal{A}$, which he called *geometric subgroups* of $\mathcal{K}$ and $\mathcal{A}$. These are subgroups that satisfy some algebraic properties that ensure that all the results on finite determinacy and versal unfoldings are valid for the action of such subgroups on $m_n.\mathcal{E}(n,p)$. As we said before, the group $\mathcal{B}$ is a geometric subgroup of $\mathcal{A}$.

3. Classification. The singularities occurring at interior points of the surface were classified by using the group $\mathcal{A}$ in [**13**]. For boundary points we need to use the group $\mathcal{B}$ as described below.

We shall use (x,y,z) coordinates on $\mathbb{R}^3$, and $\mathbb{R}^2 \times \{0\}$ is, naturally, the xy-plane. This corresponds to the boundary of our manifold with boundary, whose interior is taken to be that part of $\mathbb{R}^3$ with $z > 0$. Then our aim is to classify simple map germs $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ of corank at most 1 and $\mathcal{B}_e$-codimension ≤ 4.

The group $\mathcal{B}$ in the Introduction is the subgroup of Mather's group $\mathcal{A}$ consisting of pairs of germs of diffeomorphisms (h,k) in $\mathit{Diff}(\mathbb{R}^3) \times \mathit{Diff}(\mathbb{R}^2)$ with h preserving the manifold as well as its boundary $\mathbb{R}^2 \times \{0\}$ (that is, h takes the variety $V = \{(x,y,z) : z \geq 0\}$ into itself). Then $\mathcal{B}$ acts on the set of map germs $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ and we wish to classify $\mathcal{B}$-orbits of low codimension. Therefore, if $(h,k) \in \mathcal{B}$, we can write $h(x,y,z) = (h_1(x,y,z), h_2(x,y,z), zh_3(x,y,z))$ with $h_3(0,0,0) > 0$ for germs of smooth functions $h_i,\ i = 1,2,3$.

The group $\mathcal{B}$ inherits the action of the group $\mathcal{A}$ on $m(x,y,z).\mathcal{E}(3,2)$. As it is a Damon geometric subgroup (see [**7**]), the determinacy results in Section 2 (or [**3**]) apply here. As we explained in the Introduction, we are interested in obtaining the list of the orbits of simple germs of codimension less than or equal to four of this action.

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The $\mathcal{B}$ (resp. $\mathcal{B}_1$, i.e. the subgroup of $\mathcal{B}$ whose elements have with 1-jets at 0 the identity) tangent space of $f \in m(x, y, z).\mathcal{E}(3, 2)$ is given by

$$L\mathcal{B}.f = m(x, y, z).\{f_x, f_y\} + \mathcal{E}_3\{zf_z\} + f^*m(u, v).\{e_1, e_2\},$$
$$L\mathcal{B}_1.f = m^2(x, y, z).\{f_x, f_y\} + m(x, y, z).\{zf_z\} + f^*m^2(u, v).\{e_1, e_2\}.$$

The situation here is very similar to that considered in [**4**], where a classification of codimension ≤ 2 singularities of map-germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ up to diffeomorphisms in the source that preserve the variety $\{(x, y) : y \geq 0\}$ and any diffeomorphism in the target is given. The results on finite determinacy in [**4**] can be adapted to our situation.

THEOREM 3.1. ([**7**]). *A map-germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ is finitely $\mathcal{B}$-determined if and only if $L\mathcal{B}.f$ contains $m^N(x, y, z).\mathcal{E}(3, 2)$ for some N.*

THEOREM 3.2. (Compare with Theorem 1.2 in [**4**]). *Let $U \subset \mathcal{B}$ be a subgroup with $\mathcal{B}_1 \subset U$ and J^1U be a unipotent group, and let $f \in m(x, y, z).\mathcal{E}(3, 2)$. If*

$$m^{r+1}(x, y, z).\mathcal{E}(3, 2) \subset LU.f$$

then f is r-$\mathcal{B}$-determined. (Taking $U = \mathcal{B}_1$, one can deduce that f is r-$\mathcal{B}_1$-determined.)

We use the following result, which is an adaptation of Corollary 1.3 in [**4**], to check the inclusion in Theorem 3.2.

COROLLARY 3.3. *If f satisfies*

$$m^l(x, y, z).\mathcal{E}(3, 2) \subset \mathcal{E}_3\{f_x, f_y, zf_z\} + f^*m(u, v).\mathcal{E}(3, 2) + m^{l+1}(x, y, z).\mathcal{E}(3, 2)$$

and

$$m^{r+1}(x, y, z).\mathcal{E}(3, 2) \subset L\mathcal{B}_1.f + m^{r+l+1}(x, y, z).\mathcal{E}(3, 2)$$

then f is r-$\mathcal{B}_1$-determined.

Of course, if a germ is r-$\mathcal{B}_1$-determined then it is k-$\mathcal{B}$-determined for some $k \leq r$. We can use Corollary 3.3 to obtain the degree of $\mathcal{B}_1$-determinacy and then a combination of the complete transversal method and Mather's lemma to find the degree of $\mathcal{B}$-determinacy.

The classification is carried out inductively on the jet level as we mentioned in the Introduction. When working in $J^k(3, 2)$, the symbol $\sim$ means here $J^k\mathcal{B}$-equivalence.

The 1*-jets*

Write $j^1f = (a_1x + a_2y + a_3z, b_1x + b_2y + b_3z)$. If $a_1b_2 - a_2b_1 \neq 0$, then $j^1f \sim (x, y)$. If $a_1b_2 - a_2b_1 = 0$ but one of the coefficients a_i or b_i, $i = 1, 2$, is not zero, then $j^1f \sim (x, z)$ or $j^1f \sim (x, 0)$. If $a_1 = a_2 = b_1 = b_2 = 0$, then $j^1f \sim (z, 0)$ or $j^1f \sim (0, 0)$. So the orbits in $J^1(3, 2)$ are

$$(x, y),\ (x, z),\ (x, 0),\ (z, 0),\ (0, 0).$$

Note that the 1-jet $(0, 0)$ leads to germs of corank 2 and so cannot arise as the projection of a hypersurface in $\mathbb{R}^4$ to plane. It is not hard to show that the 1-jet (x, y) is 1-$\mathcal{B}$-determined and is stable (that is $\mathcal{B}_e$-codimension is zero).

We now follow these germs and carry out the classification inductively on the jet level, using the complete transversal method [**5**] and the 'Transversal' package [**9**]. We observe that we adapted such package to work in the case of hypersurface with boundary. The tangent space calculated by this original package is for the usual groups of Mather, so we changed it for the group $\mathcal{B}$.

Higher jets

Case 1. Suppose $j^1 f \sim (x, z)$. In this case f is $\mathcal{B}$-equivalent to $(x, z + g(x, y))$, for some germ g of a smooth function of degree greater or equal to 2 in (x, y). Consider $h(x, y) = (x, g(x, y))$ as a map-germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$.

PROPOSITION 3.4. *The map germ $f(x, y, z) = (x, z \pm g(x, y))$ is r-$\mathcal{B}$-determined (resp. simple) if and only if the map-germ $h(x, y) = (x, g(x, y))$ is r-$\mathcal{A}$-determined (resp. simple). We also have $\mathcal{B}_e$-cod$(f) = \mathcal{A}_e$-cod(h).*

Proof. We observe that

$$\begin{aligned} L\mathcal{B}.f &= \mathcal{E}_3.\{(z, 0), (0, z)\} + m(x, y).\{(1, g_x), (0, g_y)\} + h^* m(u, v).\{e_1, e_2\} \\ &= \mathcal{E}_3.\{(z, 0), (0, z)\} + L\mathcal{A}.h. \end{aligned}$$

So by Theorems 2.1 and 3.1, f is $\mathcal{B}$-determined if and only if h is $\mathcal{A}$-determined. Now using the complete transversal method and Mather's Lemma, one can show that both germs have the same degree of determinacy.

For the result on simplicity, we observe that any one-parameter family f_a map-germ with $f_0(x, y, z) = (x, z + g_0(x, y))$ can be written in a suitable coordinate system in the form $(x, z + g_a(x, y))$, for some one-parameter family of functions $g_a(x, y)$. Then f_a is not equivalent to $f_{a'}$, for $a \neq a'$, if and only if $(x, g_a(x, y))$ is not $\mathcal{B}$-equivalent to $(x, g_{a'}(x, y))$. □

The germ $h(x, y) = (x, g(x, y))$ in Proposition 3.4 is of corank ≤ 1 (see [**17**] for the corank 2 simple germs). The $\mathcal{A}$-simple germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ of corank at most 1 and $\mathcal{A}_e$-codimension ≤ 4 are given in [**16**]. In view of Proposition 3.4, we can take $(x, g(x, y))$ as in Table 2 to obtain the germs $(x, z \pm g(x, y))$. We make changes of scale in the source and target to obtain the classification in Table 1. Recall that, since the $\mathcal{B}$-group preserves the set $\{(x, y, z); z \geq 0\}$ as well as its boundary, we can use only scalar change of coordinate for z of the form $z = kZ$, $k > 0$. Therefore, there is a sign $\epsilon = \pm 1$ in Table 1 in front of even powers of y in the second components of the germs in Table 2.

Observe that $(x, z + y)$ is equivalent to (x, y).

Case 2. Suppose that $j^1 f \sim (x, 0)$. A complete two-transversal is given by

$$(x, b_1 xy + b_2 xz + b_3 yz + b_4 y^2 + b_5 z^2)$$

for some $b_1, b_2, b_3, b_4, b_5 \in \mathbb{R}$. We can now make linear changes of coordinates and obtain the following orbits in $J^2(3, 2)$ (the superscript (ns) is used to mean that the germs are non-simple and will not be followed):

$b_4 \neq 0 \Rightarrow (x, y^2 + xz \pm z^2)$, $(x, y^2 + xz)$, $(x, y^2 \pm z^2)$, $(x, y^2)^{(ns)}$. Case (2.1).
$b_4 = 0, b_1 \neq 0 \Rightarrow (x, xy + yz)$, $(x, xy + z^2)$, $(x, xy)^{(ns)}$. Case (2.2).
$b_4 = 0, b_1 = 0 \Rightarrow (x, xz + z^2)^{(ns)}$, $(x, xz)^{(ns)}$, $(x, z^2)^{(ns)}$, $(x, yz)^{(ns)}$, $(x, 0)^{(ns)}$. Case (2.3).

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Table 2. $\mathcal{A}$-simple germs of map-germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, $\epsilon = \pm 1$ ([**16**])

Type	Normal form	$\mathcal{A}_e$-codimension
1	(x, y)	0
2	(x, y^2)	0
3	$(x, xy + y^3)$	0
4_k	$(x, y^3 + \epsilon^{k-1}x^k y)$, $k \geq 2$	$k - 1$
5	$(x, xy + y^4)$	1
6	$(x, xy + y^5 + \epsilon y^7)$	2
7	$(x, xy + y^5)$	3
9	$(x, xy + y^6 + y^9)$	4
11_{2k+1}	$(x, xy^2 + y^4 + y^{2k+1})$, $k \geq 2$	k
12	$(x, xy^2 + y^5 + y^6)$	3
13	$(x, xy^2 + y^5 + \epsilon y^9)$	4
16	$(x, x^2y + y^4 + \epsilon y^5)$	3
17	$(x, x^2y + y^4)$	4

Case (2.1).

Suppose $j^2 f = (x, y^2) + j^2(0, g(x, y, z))$. All orbits from Case (2.1) can be considered as subcases of this one, and we call them by (2.1.1), (2.1.2), (2.1.3) and (2.1.4) respectively. Any complete k-transversal of $(0, y^2)$ can be written in the form $(x, y^2 + g(x, z))$ for some germ g of a polynomial in (x, z) with a zero 2-jet. We take here g to have a zero 1-jet to include the cases $(x, y^2 + xz \pm z^2)$, $(x, y^2 + xz)$ and $(x, y^2 \pm z^2)$.

Let $h(x, z) = (x, g(x, z))$ and consider the action of the subgroup $\mathcal{B}(2)$ of $\mathcal{A}$ with diffeomorphisms in $(\mathbb{R}^2, 0)$ preserving the line $z = 0$ and the set $\{(x, z) : z > 0\}$, and any diffeomorphism in the target, on the set of map-germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ (see [**4**]).

Proposition 3.5. *The map-germ* $f(x, y, z) = (x, y^2 \pm g(x, z))$ *is r-*$\mathcal{B}(3)$*-determined (resp. simple) if and only if the map-germ* $h(x, z) = (x, g(x, z))$ *is r-*$\mathcal{B}(2)$*-determined (resp. simple). We have* $\mathcal{B}_e(3)$*-cod*$(f) = \mathcal{B}_e(2)$*-cod*(h).

Proof. We can write

$$\begin{aligned} L\mathcal{B}(3).f &= \mathcal{E}_3.\{(y, 0), (0, xy), (0, y^2), (0, yz)\} \\ &\quad + \mathcal{E}_2.\{x(1, g_x), z(1, g_x), z(0, g_z)\} + h^* m(u, v).\{e_1, e_2\} \\ &= \mathcal{E}_3.\{(y, 0), (0, xy), (0, y^2), (0, yz)\} + L\mathcal{B}(2).h. \end{aligned}$$

So by Theorem 3.1 and its version for the $\mathcal{B}(2)$-singularities of map-germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ [**4**], f is $\mathcal{B}(3)$-determined if and only if h is $\mathcal{B}(2)$-determined. Using the complete transversal method and Mather's Lemma, one can show that both germs have the same degree of determinacy. The simplicity follows the same argument in the proof of the Proposition 3.4. □

Map-germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ of $\mathcal{B}_e$-codimension ≤ 2 and some other consequences of these are given in [**4**]. Also, all the $\mathcal{B}$-simple germs can be obtained from the calculations in [**4**]. The corank 1 cases are listed in Table 3. (There is also a corank 2 series given by $(z + x^{2k+1}, x^2)$.) Then by Proposition 3.5, $\mathcal{B}$-simple germs $(\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ in the form $(x, y^2 \pm g(x, z))$ from subcases (2.1.1), (2.1.2) and (2.1.3) can be obtained by using Table 3. As we explained in Case 1, we make changes of coordinates in the source and target to obtain the classification in Table 1, but we can not use $z = kZ$ with $k < 0$. Therefore, there is a sign $\epsilon = \pm 1$ in Table 1 in front of z^k, $k = 2, 3, 4$ in the second components of the germs in Table 3.

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Table 3. Corank 1 $\mathcal{B}(2)$-simple singularities of map-germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ [**4**]

Normal form	$\mathcal{B}_e$-codimension
$(x, z^2 + \epsilon^k x^{k-1} z),\ k \geq 2$	$k-2$
$(x, xz + z^3)$	1
$(x, xz + z^4 + \epsilon z^6)$,	2
$(x, xz + z^4)$	3

The cases of germs with 2-jet equivalent to $(x, y^2 \pm z^2)$ or $(x, y^2 + xz \pm z^2)$ give us only the family $(x, y^2 \pm z^2 + \epsilon^k x^{k-1} z)$, $k \geq 2$, which comes from Table 3. We now follow other cases that do not come from Table 3, that is, other cases with $2 < cod \leq 4$. Then we need to analyse the germs with 4-jet equivalent to $(x, y^2 + xz)$ or 2-jet equivalent to (x, y^2). We will prove that they generate non-simple germs. Therefore, the next step also justifies the fact that Table 3 does not have simple germs of codimension 4.

Others cases for (2.1.2): $j^4 f \sim (x, y^2 + xz)$

A complete five-transversal (that is the 5-CT) is $j^5 f \sim (x, y^2 + xz + b_1 z^5)$ with codimension ≥ 3, and by Mather's lemma we can take $b_1 = \epsilon = \pm 1$ or $b_1 = 0$. If $b_1 = \epsilon$, then 6-CT has $(0, z^6)$ but by Mather's Lemma this vector is on the tangent space. Therefore, $j^7 f \sim (x, y^2 + xz + \epsilon z^5 + b_3 z^7)$ with $b_3 = \epsilon_1 = \pm 1$ or $b_3 = 0$. If $b_3 = \epsilon_1$ then we have

(1) $j^8 f \sim (x, y^2 + xz + \epsilon z^5 + \epsilon_1 z^7 + b_4 z^8)$,

where b_4 is a modulus. So they are non-simple germs. If $b_3 = 0$, then $j^8 f \sim (x, y^2 + xz + \epsilon z^5 + b_4 z^8)$ with $b_4 = \epsilon_1$ or $b_4 = 0$, these are also non-simple germs because they have non-simple germs (1) next to them.

If $b_1 = 0$, then $j^8 f \sim (x, y^2 + xz + b_2 z^6 + b_4 z^8)$. By Mather we have the following cases:

$b_2 = \epsilon$ and $b_4 = \epsilon_1$. Then we have

(2) $j^9 f \sim (x, y^2 + xz + \epsilon z^6 + \epsilon_1 z^8 + b_5 z^9)$,

where b_5 is modulus. So these are non-simple germs;

$b_4 = 0$ and $b_2 \neq 0$. Then $j^9 f \sim (x, y^2 + xz + \epsilon z^6 + b_5 z^9)$, $b_5 = \epsilon_1$ or $b_5 = 0$, are also non-simple because they have non-simple germs (2) next to them;

$b_2 = 0$. Then $j^7 f \sim (x, y^2 + xz + b_3 z^7)$, $b_3 = \epsilon_1$ or $b_3 = 0$, that are not sufficient jets but have codimension ≥ 5 and also are non-simple germs. In fact, $j^k f \sim (x, y^2 + xz + \epsilon z^{k-3} + \epsilon_1 z^{k-1} + a z^k)$ with a modulus.

(2.1.4) $j^2 f \sim (x, y^2)$

A 3-CT is $j^3 f \sim (x, y^2 + b_1 x^2 z + b_2 x z^2 + b_3 z^3)$. The codimension is at least 2. If $b_1 b_2 \neq 0$ then

(3) $j^3 f \sim (x, y^2 + \epsilon x^2 z + x z^2 + b_3 z^3)$,

where b_3 is a modulus, so we will not consider it. Consequently, we have the next cases:

$b_1 = 0$, $b_2 \neq 0$ and $b_3 \neq 0$ then $j^3 f \sim (x, y^2 + xz^2 + \epsilon z^3)$;

$b_2 = 0$, $b_1 \neq 0$ and $b_3 \neq 0$ then $j^3 f \sim (x, y^2 + \epsilon x^2 z + \epsilon_1 z^3)$;

$b_1 = 0$, $b_2 = 0$ and $b_3 \neq 0$ then $j^3 f \sim (x, y^2 + \epsilon z^3)$;

$b_2 = 0$, $b_3 = 0$ and $b_1 \neq 0$ then $j^3 f \sim (x, y^2 + \epsilon x^2 z)$;

$b_1 = 0$, $b_3 = 0$ and $b_2 \neq 0$ then $j^3 f \sim (x, y^2 + xz^2)$;

$b_1 = 0$, $b_2 = 0$ and $b_3 = 0$ then $j^3 f = (x, y^2)$.

So we do not have simple germs for this case because they have non-simple germ (3) next to them.

Case (2.2).
(2.2.1) $j^2f \sim (x, xy + yz)$

If $j^2f \sim (x, xy + yz)$ then $j^3f \sim (x, xy + yz + b_1y^3)$ is three-determined and have codimension 1 if $b_1 \neq 0$, in which case we can take $b_1 = \epsilon$ by Mather's lemma. If $b_1 = 0$, we have the cases $j^6f \sim (x, xy + yz + y^4 + b_3y^6)$, six-determined with $b_3 = \epsilon$ and codimension 2 or if $b_3 = 0$ then $j^6f \sim (x, xy + yz + y^4)$, six-determined with codimension 3.

If $j^4f \sim (x, xy + yz)$ then we have non-simple germs. In fact, $j^8f \sim (x, xy + yz + b_4y^5 + b_5y^7 + b_6y^8)$, where b_4 and b_5 are ϵ or zero, and codimension is ≥ 3. If $b_4b_5 \neq 0$, then b_6 is a modulus. Then for all other subcases, with $b_4b_5b_6 = 0$, we have non-simple germs.

(2.2.2) $j^2f \sim (x, xy + z^2)$

A complete three-transversal is given by $(x, xy + z^2 + b_1y^3 + b_2y^2z)$. Then the orbits in $J^3(3, 2)$ over the above 2-jet are

$$(x, xy + z^2 + y^3 + \epsilon y^2z),\ (x, xy + z^2 + y^3),\ (x, xy + z^2 + \epsilon y^2z)^{(ns)},\ (x, xy + z^2)^{(ns)}.$$

Higher jets of $(x, xy + z^2)$

- $j^3f \sim (x, xy + z^2 + y^3 + \epsilon y^2z)$

This germ is 3-$\mathcal{B}$-determined and has $\mathcal{B}_e$-codimension 2. This germ appears in Table 1 by using the same formula determined in the next case with $k = 2$.

• $j^3f \sim (x, xy + z^2 + y^3)$

Any $(k + 1)$-jet $(k \geq 3)$ with k-jet equal to $(x, xy + z^2 + y^3)$ is equivalent to $(x, xy + z^2 + y^3 + b_1y^kz)$. When $b_1 \neq 0$, a change of scale reduces to $(x, xy + z^2 + y^3 + \epsilon y^kz)$. The germ $(x, xy + z^2 + y^3 + \epsilon y^kz)$ is $(k + 1)$-$\mathcal{B}$-determined and has $\mathcal{B}_e$-codimension k.

• $j^3f \sim (x, xy + z^2 + \epsilon y^2z)$

A complete four-transversal is given by $(x, xy + z^2 + \epsilon y^2z + b_1y^4)$ with codimension ≥ 3. One can show by using Mather's lemma that b_1 is a parameter modulus.

• $j^3f \sim (x, xy + z^2)$

The 4-CT has $(0, y^4)$ and $(0, y^3z)$, then we have the following cases:
$j^5f \sim (x, xy + z^2 + \epsilon y^4 + y^3z + b_1y^4z)$ with b_1 modulus, codimension ≥ 4 for any b_1;
$j^6f \sim (x, xy + z^2 + b_0y^3z + b_1y^6 + b_2y^5z)$, $b_0 = \epsilon$ or $b_0 = 0$ with at least one modulus, codimension ≥ 5;
$j^{k+1}f \sim (x, xy + z^2 + \epsilon y^4 \pm y^kz + b_1y^{k+1}z)$ with b_1 modulus, codimension $\geq k$.

The 5-CT of $j^4f \sim (x, xy + z^2)$ has $(0, y^5)$ and $(0, y^4z)$ and generates non-simple germs as in the former cases. The codimension is at least 5.

(2.2.3) $j^2f \sim (x, xy)$

The complete three-transversal gives us
(1) $j^3f \sim (x, xy + b_1y^3 + b_2y^2z + b_3yz^2 + b_4z^3)$
with codimension of at least 3. If $b_3b_4 \neq 0$, then $j^3f \sim (x, xy + b_1y^3 + b_2y^2z + yz^2 + \epsilon z^3)$ with two moduli. For all other subcases of (1), with $b_1b_2b_3b_4 = 0$, we have non-simple germ.

Table 4. Corank 1 $\mathcal{B}$-simple singularities of map-germs $f:(\mathbb{R}^3,0)\to(\mathbb{R}^2,0)$ of $\mathcal{B}_e$-codimension ≤ 4 and with $j^1f\sim(x,0)$ of case(2.2), $\epsilon=\pm1$

Normal form	$\mathcal{B}_e$-codimension
$(x, yz+xy+\epsilon y^3)$	1
$(x, yz+xy+y^4+\epsilon y^6)$	2
$(x, yz+xy+y^4)$	3
$(x, xy+z^2+y^3+\epsilon y^k z),\ k\geq 2$	k

We summarise the above classification in Table 4.

By using f from Table 4, we get some of the normal forms given in Theorem 1.1.

Case (2.3). All subcases are non-simple germs.

(2.3.1) $j^2f\sim(x, xz+z^2)$

The germs in $J^3(3,2)$ with this 2-jet are equivalent to $(x, xz+z^2+b_1x^2y+b_2xy^2+b_3y^3+b_4y^2z+b_5yz^2)$ for some $b_i\in\mathbb{R}$, $i=1,\ldots,5$, with codimension ≥ 3. By using Mather in $J^3(3,2)$ we confirm that these orbits always are non-simple. We have many cases and we will not describe everything here. For example, y^3 is not at the tangent space if $b_3\neq 0$, $-b_4^2+3b_5b_3+4b_4b_2\neq 0$, $-b_4^2b_1+4b_1b_2b_4+b_5b_2^2\neq 0$, so in these connected components the germs are non-simple. Also, if $b_3=0$ ($-b_4^2+3b_5b_3+4b_4b_2=0$ or $-b_4^2b_1+4b_1b_2b_4+b_5b_2^2=0$), they are non-simple germs because near this case there is non-simple germ.

(2.3.2) $j^2f\sim(x, xz)$

$j^3f\sim(x, xz+b_1x^2y+b_2xy^2+b_3y^3+b_4y^2z+b_5yz^2+b_6z^3)$ with codimension $\geq$ 4. By using Mather's lemma, y^3 is not at the tangent space if $b_5\neq 0$, $b_2\neq 0$, $b_4b_5^2+3b_6b_3b_5-4b_6b_4^2\neq 0$, $-b_2^2+3b_3b_1\neq 0$. So for any case these are non-simple germs.

(2.3.3) $j^2f\sim(x, z^2)$

$j^3f\sim(x, z^2+b_1x^2y+b_2xy^2+b_3y^3+b_4y^2z+b_5xyz+b_6z^3+b_7x^2z)$ with codimension ≥ 4. By using Mather x^2y is not at the tangent space if $b_3\neq 0$, $b_5\neq 0$, $-3b_3b_5+2b_4b_2\neq 0$ and $3b_3b_7+b_4b_1-b_2b_5\neq 0$. So for any case these are non-simple germs.

(2.3.4) $j^2f\sim(x, yz)$

The orbits in $J^3(3,2)$ are given by $(x, yz+b_1x^2y+b_2xy^2+b_3y^3)$. If $b_2b_3\neq 0$ then the germs with this 3-jet are equivalent to $(x, yz+b_1x^2y+xy^2+\epsilon y^3)$ with one modulus. Then all the next cases are non-simple germs because they have non-simple germs next to them. If $b_3=0$: $b_1b_2\neq 0$ then $j^3f\sim(x, yz+\epsilon x^2y+xy^2)$; $b_1=0$ and $b_2\neq 0$ then $j^3f\sim(x, yz+xy^2)$. If $b_2=0$: $b_1b_3\neq 0$ then $j^3f\sim(x, yz+x^2y+\epsilon y^3)$; $b_1=0$ and $b_3\neq 0$ then $j^3f\sim(x, yz+\epsilon y^3)$; $b_3=0$ and $b_1\neq 0$ then $j^3f\sim(x, yz+\epsilon x^2y)$; $b_1=b_3=0$ then $j^3f\sim(x, yz)$.

(2.3.5) $j^2f\sim(x, 0)$

The 3-CT give us $j^3f\sim(x, b_1x^2y+b_2xy^2+b_3y^3+b_4y^2z+b_5xyz+b_6z^3+b_7x^2z+b_8xz^2+b_9yz^2)$ with codimension ≥ 5. By using Mather x^2y is not at the tangent if $b_6\neq 0$, $b_9\neq 0$, $b_9^2b_4+3b_3b_6b_9-4b_4^2b_6\neq 0$ and $2b_4b_8-b_9b_5\neq 0$. So for any case these are also non-simple germs.

Case 3. Suppose $j^1f\sim(z,0)$. All the cases are non-simple germs.

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A complete two-transversal is given by

$$(z + a_1x^2 + a_2xy + a_3y^2, b_1x^2 + b_2xy + b_3y^2 + b_4xz + b_5yz).$$

If $4b_3b_1 - b_2^2 \neq 0$, then by using Mather and scalar changes of coordinates $j^2f \sim (z + \epsilon_1x^2 + a_2xy + a_3y^2, x^2 + \epsilon\, y^2)$ with new coefficients a_i, $\epsilon = \pm 1$, $\epsilon_1 = \pm 1$. First we deal with the case $\epsilon_1 = 1$. If $\epsilon = 1$, the conditions of Mather are true if $a_2^2 + (a_3 - 1)^2 \neq 0$. Then we need to consider two cases (one representant in each connected component) $(a_2, a_3) = (0, 1)$ and $(a_2, a_3) = (0, 0)$ for which the normal forms are $(z, x^2 + y^2)$ and $(z + x^2, x^2 + y^2)$ respectively. If $\epsilon = -1$, the conditions of Mather are true if $(a_2 + 1 + a_3)(a_2 - 1 - a_3) \neq 0$ (lines that divide the plane (a_2, a_3) in four regions). Then we need to consider the cases $(a_2, a_3) = (0, 0)$, $(a_2, a_3) = (0, -2)$ (which are equivalents). These are in the region $(a_2 + 1 + a_3)(a_2 - 1 - a_3) < 0$ and we will use the representative form $(z + x^2, x^2 - y^2)$. When $(a_2 + 1 + a_3)(a_2 - 1 - a_3) > 0$, we can take $(a_2, a_3) = (-2, 0)$, $(a_2, a_3) = (2, 0)$ (which are equivalents), then by using $(2, 0)$ we have $(z + x^2 + 2xy, x^2 - y^2)$. On the lines given by $(a_2 + 1 + a_3)(a_2 - 1 - a_3) = 0$, that is $a_2 = \pm(1 + a_3)$, the 2-jets are equivalent to $(z \pm (1 + a_3)xy + (1 + a_3)y^2, x^2 - y^2)$. The intersection of these lines, $(a_2, a_3) = (0, -1)$, gives us $(z, x^2 - y^2)$.

Now we consider the case $\epsilon_1 = -1$. If $\epsilon = 1$, the conditions of Mather are true if $a_2^2 + (a_3 + 1)^2 \neq 0$. Then we need to consider two cases $(a_2, a_3) = (0, -1)$ and $(a_2, a_3) = (0, 0)$ for which the normal forms are $(z, x^2 + y^2)$ (that appears above) and $(z - x^2, x^2 + y^2)$ respectively. If $\epsilon = -1$, the conditions of Mather are true if $(a_2 - 1 + a_3)(a_2 + 1 - a_3) \neq 0$ (lines that divide the plane (a_2, a_3) in four regions). Then we need to consider the cases $(a_2, a_3) = (0, 0)$, $(a_2, a_3) = (0, 2)$ for which the normal forms are respectively equivalent to $(z - x^2, x^2 - y^2)$ and $(z + x^2, x^2 - y^2)$ (this last one appears before). These are in the region $(a_2 - 1 + a_3)(a_2 + 1 - a_3) < 0$. When $(a_2 - 1 + a_3)(a_2 + 1 - a_3) > 0$ then we can take $(a_2, a_3) = (-2, 0)$, $(a_2, a_3) = (2, 0)$ (which are equivalents). So by using $(2, 0)$ we have $(z - x^2 + 2xy, x^2 - y^2)$. On the lines given by $(a_2 - 1 + a_3)(a_2 + 1 - a_3) = 0$, that is $a_2 = \pm(a_3 - 1)$, the 2-jets are $(z \pm (a_3 - 1)xy + (a_3 - 1)y^2, x^2 - y^2)$. The intersection of these lines, $(a_2, a_3) = (0, 1)$, gives us a form equivalent to $(z, x^2 - y^2)$.

We analyse all these cases below.

(3.1) $j^2f \sim (z + \epsilon_1x^2, x^2 + \epsilon y^2)$:

The orbits in $J^3(3, 2)$ are given by a modular form $(z + \epsilon_1x^2 + a_4x^3 + a_5y^3, x^2 + \epsilon y^2)$, so all the subcases are non-simple germs. The codimension is at least 2.

(3.2) $j^2f \sim (z, x^2 + \epsilon y^2)$:

The 3-jet is equivalent to $(z + a_1y^3 + a_2xy^2 + a_3x^2y + a_4x^3, x^2 + \epsilon y^2)$ that are non-simple germs because by Mather $(x^3, 0)$ is not at the tangent space if $a_1 \neq 0$ and $3a_1^2 - 2\epsilon a_3a_1 + \epsilon a_2^2 \neq 0$. Therefore, all cases are non-simple. The codimension is at least 4.

(3.3) $j^2f \sim (z + \epsilon_1x^2 + 2xy, x^2 - y^2)$

The 3-jet is equivalent to $(z + \epsilon_1x^2 + 2xy + a_4y^3, x^2 - y^2 + b_6y^3)$ for some $a_4, b_6 \in \mathbb{R}$. By Mather $(0, y^3)$ is not in the tangent space if $a_4 \neq 0$. So we always have non-simple germs. The codimension is at least 2.

(3.4) $j^2f \sim (z \pm (1 + a_3)xy + (1 + a_3)y^2, x^2 - y^2)$ or $j^2f \sim (z \pm (a_3 - 1)xy + (a_3 - 1)y^2, x^2 - y^2)$:

These 2-jets are non-simple, a_3 is a modulus.

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Table 5. Map germs $(\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ of codimension ≤ 1, $\epsilon, \epsilon_1 \pm 1$

No.	Normal form	$\mathcal{B}_e$-codimension of orbit
I	(x, y)	0
II	$(x, z + \epsilon y^2)$	0
III	$(x, z + xy + y^3)$	0
IV	$(x, z + \epsilon x^2 y + y^3)$	1
V	$(x, z + xy + \epsilon y^4)$	1
VI	$(x, y^2 + \epsilon z^2 + xz)$	0
VII	$(x, y^2 + \epsilon z^2 + \epsilon_1 x^2 z)$	1
VIII	$(x, y^2 + xz + \epsilon z^3)$	1
IX	$(x, yz + xy + \epsilon y^3)$	1

If $4b_3b_1 - b_2^2 = 0$, then using the fact that the case above, $4b_3b_1 - b_2^2 \neq 0$, always generates non-simple germs, we can conclude that at any neighbourhood of germs where $4b_3b_1 - b_2^2 = 0$ there are infinite orbits and also in this case the germs are non-simple. For all subcases the codimension is at least 2.

As a consequence of the analysis done before, we have the following results.

THEOREM 3.6. *The map-germs* $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ *with 1-jet equivalent to* $(z, 0)$ *or 2-jet equivalent to* (x, y^2), (x, xy), (x, yz), $(x, xz + z^2)$, (x, xz), (x, z^2), $(x, 0)$ *or 3-jet* $(x, xy + z^2 + \epsilon y^2 z)$, $(x, xy + z^2)$ *or 4-jet* $(x, y^2 + xz)$, $(x, xy + yz)$ *are non-simple germs.*

In Section 4 we study geometrically the germs given by Theorem 3.7. These germs are given in Table 5.

THEOREM 3.7. *The map-germs* $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ *of corank at most 1 and* $\mathcal{B}_e$-*codimension* ≤ 1 *are simple germs.*

4. The geometry of codimension ≤ 1 singularities. In this section we collect together the normal forms of germs $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ of codimension ≤ 1 which arise from parallel projections of hypersurfaces with boundary (so they are of corank at most 1 and also simple germs by Theorem 3.7) and, in order to recognize different cases, we go into some detail on their geometrical properties.

We give, for each non-submersive germ f of codimension 1 in Table 5, a $\mathcal{B}_e$-versal unfolding and a bifurcation diagram to show, for germs close to f in the $\mathcal{B}_e$-versal unfolding, which types of boundary singularities occur. Since the submersions IV and V do not have singular points, the bifurcation diagrams are given for the restriction of f to the boundary. The notation VI^2 indicates the presence of two singularities of Type VI arbitrarily near the origin for germs in an appropriate region of the diagram. Similarly, VI$_-$ and VI$_+$ mean singularities of Type VI with $\epsilon = -1$ and $\epsilon = 1$ respectively.

It is also of interest to find, for each germ in Table 5, a single hypersurface M for which the family of parallel projections realizes a versal unfolding of the germ. More precisely, suppose that

$$i : (x, y, z) \mapsto (X(x, y, z),\ Y(x, y, z), Z(x, y, z),\ W(x, y, z))$$

is a (germ of an) immersion at $(0, 0, 0)$ so that we can regard the image as a small piece of smooth hypersurface M in $\mathbb{R}^4$. We are interested in the restriction to $z \geq 0$.

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If Π is the family of orthogonal projections to 2-spaces given in Section 1 and $u \in G(2, 4)$, then the map Π_u measures the contact between M and the plane orthogonal to u. Let $p \in \mathbb{R}^4$ be the origin and let us suppose that

$$T_pM = \langle(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\rangle.$$

If p is a corank 1 singular point of Π_u, then the orthogonal plane to u is a subset of T_pM. So the generators of the plane u should be taken as a vector a of T_pM and a vector b normal to T_pM at p.

The family of planes in $\mathbb{R}^4$ close to the plane u generated by $a = (1, 0, 0, 0)$ and $b = (0, 0, 0, 1)$ may be given taking $(1, \beta_1, \gamma_1, 0)$ and $(0, \beta_2, \gamma_2, 1)$ as generators of those planes for $\beta_1, \beta_2, \gamma_1, \gamma_2$ close to 0. So, we get the map

$$\Pi_{\beta,\gamma}(X, Y, Z, W) = (X + \beta_1 Y + \gamma_1 Z, W + \beta_2 Y + \gamma_2 Z)\,,$$

where β and γ denote the pairs (β_1, β_2) and (γ_1, γ_2). Note that $\Pi_{0,0} = \Pi_u$.

To realize a versal unfolding with 1-parameter, we take $\beta_1 = \gamma_1 = \beta_2 = 0$ and call $\gamma_2 = \lambda$, where $(x, y, z) \to (X, Y, Z, W)$ is an immersion at $(0, 0, 0)$.

For each normal form we give pictures of the source X together with Σ (i.e. $\Sigma = \Sigma_1$, the critical set of the map f), and the kernel K of $df(0)$. We also draw the critical loci (image of Σ) and the image of the boundary for f and for members of the versal family unfolding f. As aid to the recognition of various cases, we give pictures of the fibre $f^{-1}(0)$ and also information about the singularities of the restriction of f to the boundary (see **[16]**). Parts of Σ or its image that are virtual, in the sense of corresponding to the part $z < 0$ in the source $\mathbb{R}^3$, appear dashed in the figures. The boundary (plane-xy) and its image are drawn with grey colour.

Geometrical information in Cases I–IX.

Cases I–V. These are germs of submersions and so K is a line. Except Case I whose K is transverse to the boundary (that is, on M, the direction of projection is transverse to ∂M), in the other cases K is a subset of the boundary. For Cases I and III to V the image of the boundary is $\mathbb{R}^2$.

- Cases I–III. These cases are stable submersions and so realized simply by the following immersions:

I. $i(x, y, z) = (x, 0, z, y)$. See Figure 1(a).

II. $i(x, y, z) = (x, y, 0, z + \epsilon y^2)$. The image of the boundary is, unlike Cases I–V, a semi-plane. Furthermore, the fibre $f^{-1}(0)$ is a curve tangent to the boundary, for $\epsilon = -1$ (see Figure 1(b)), and $f^{-1}(0) = 0$, for $\epsilon = 1$ (see Figure 1(c)). The singularity of the restriction of f to the boundary is the fold (x, y^2).

III. $i(x, y, z) = (x, y, 0, z + xy + y^3)$. The set $f^{-1}(0)$ is tangent to the boundary (see Figure 1(d)). The singularity of the restriction of f to the boundary $z = 0$ is the cusp $(x, xy + y^3)$.

- Cases IV and V. These cases are codimension 1 submersions. For these cases Σ is empty and we cannot consider the bifurcation diagram for the germ f. Then the bifurcation diagrams are given for the restriction of f to the boundary that has codimension 1 singularity only in these cases.

Case IV. A $\mathcal{B}_e$-versal unfolding is $(x, z + y^3 + \epsilon x^2 y + \lambda y)$. A realization is $i(x, y, z) = (x, y, 0, z + y^3 + \epsilon x^2 y)$. The fibre $f^{-1}(0)$ is also tangent to the boundary, as in Case III

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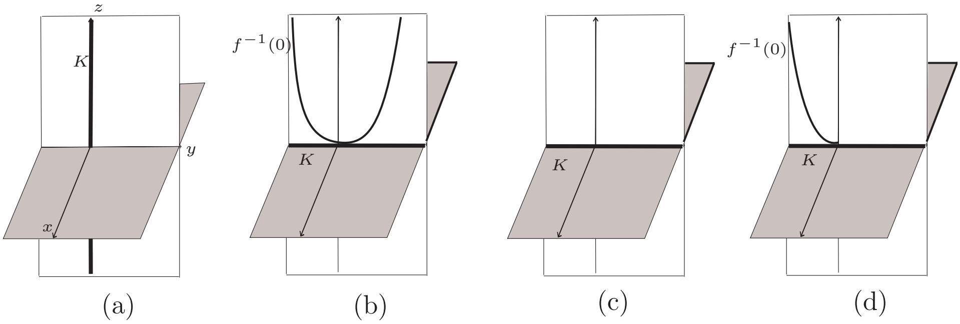

Figure 1. (Colour online) Submersions. (a) Case I. (b) Cases II and V, for $\epsilon = -1$. (c) Cases II and V, for $\epsilon = 1$. (d) Cases III and IV.

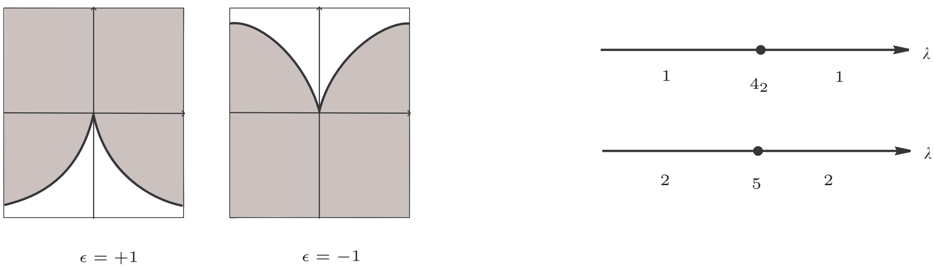

Figure 2. (Colour online) (Left) The image of the boundary of case V. (Right) Bifurcation diagrams of singularities for the restriction of f to the boundary, cases IV and V. (See Table 2 for notation of Rieger.)

(Figure 1(d)) but, unlike this case, the singularity of the restriction of f to the boundary $z = 0$ is the lips/beaks $(x, \epsilon x^2 y + y^3)$, which is a codimension 1 singularity.

Case V. A $\mathcal{B}_e$-versal unfolding is $(x, z + xy + \epsilon y^4 + \lambda y^2)$ and a realization is $i(x, y, z) = (x, y^2, y, z + xy + \epsilon y^4)$. Unlike Cases I–IV, the boundary $z = 0$ is mapped to a curve with an ordinary cusp (see Figure 2). The fibre $f^{-1}(0)$ is as in Case II (Figure 1(b) and (c)) but, unlike this case, the singularity of the restriction of f to the boundary $z = 0$ is the swallowtail $(x, xy + y^4)$, which is a codimension 1 singularity.

Cases VI–IX. These are those germs with $K = \ker df(0)$ being the plane-yz.

- Case VI. This is stable and so realized by an immersion $i(x, y, z) = (x, y, z, y^2 + \epsilon z^2 + xz)$. The sets K and Σ are transverse to each other and to the boundary, whose image is a semi-plane containing $f|_\Sigma$ for $\epsilon = -1$. The fibre $f^{-1}(0)$ also distinguishes cases $\epsilon = 1$ and $\epsilon = -1$: $f^{-1}(0) = 0$ or it is a set transverse to the boundary respectively (see Figure 3). The singularity of the restriction of f to the boundary $z = 0$ is a fold.
- Case VII. A $\mathcal{B}_e$-versal unfolding is $(x, y^2 + \epsilon z^2 + \overline{\epsilon} x^2 z + \lambda z)$ and a realization is $i(x, y, z) = (x, y, z, y^2 + \epsilon z^2 + \overline{\epsilon} x^2 z)$. Sets K and Σ are transverse to each other, and Σ is tangent to the boundary (that is, on M, the critical set of the projection is tangent to ∂M) in the region $z \leq 0$ for $\epsilon\overline{\epsilon} = 1$ and in the region $z \geq 0$ for $\epsilon\overline{\epsilon} = -1$. The boundary image is a semi-plane. See Figures 4 and 5. The fibre $f^{-1}(0)$ is similar to Case VI, according to ϵ, as well as the singularity of the restriction of f to the

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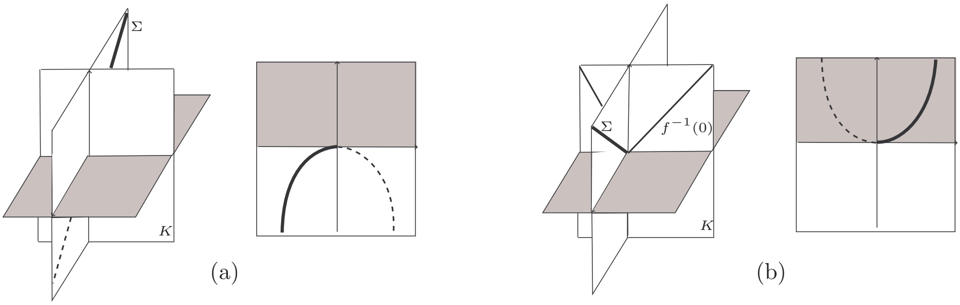

Figure 3. (Colour online) (a) Case VI for $\epsilon = 1$. (b) Case VI for $\epsilon = -1$.

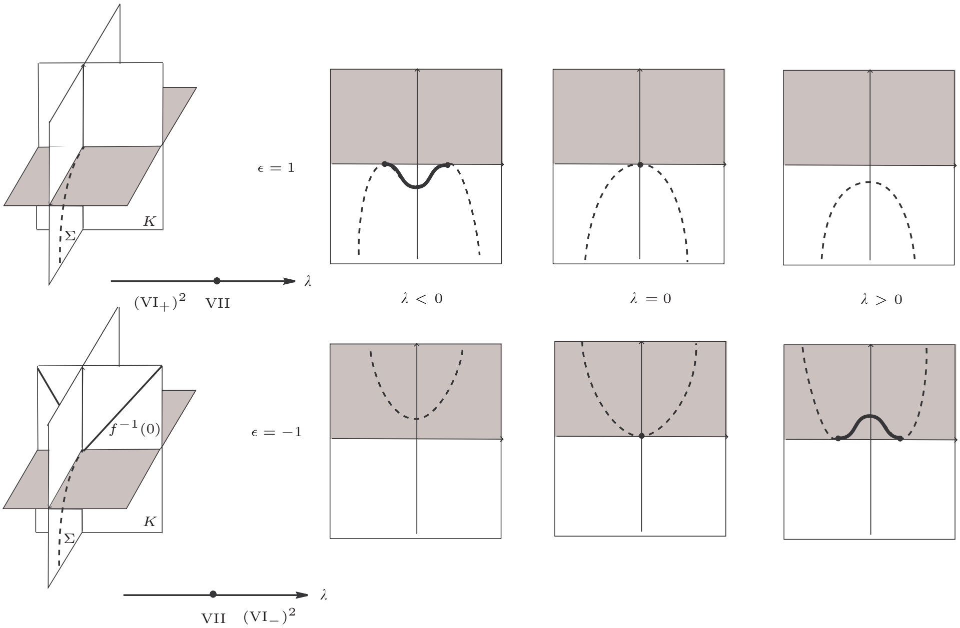

Figure 4. (Colour online) Case VII for $\epsilon\bar{\epsilon} = 1$.

boundary $z = 0$. Note that the singular set Σ distinguishes this case from all other cases.

- Case VIII. A $\mathcal{B}_e$-versal unfolding is $(x, y^2 + xz + \epsilon z^3 + \lambda(z + z^2))$ and a realization is $i(x, y, z) = (x, y, z + z^2, y^2 + xz + \epsilon z^3)$. Sets K and Σ are tangent to each other, and both of these are transverse to the boundary. The fibre $f^{-1}(0)$ also distinguishes cases $\epsilon = -1$ and $\epsilon = 1$: it is a cuspidal curve or $f^{-1}(0) = 0$ respectively (see Figure 6). The singularity of the restriction of f to the boundary $z = 0$ is, as Case VII, the same as Case VI. Note that $f|_\Sigma$, unlike all other cases, has a cusp singularity.
- Case IX. A $\mathcal{B}_e$-versal unfolding is $(x, yz + xy + \epsilon y^3 + \lambda(y^2 + z))$ and a realization is $i(x, y, z) = (x, y, y^2 + z, yz + xy + \epsilon y^3)$. Sets K and Σ are transverse to each other, and also to the boundary. Furthermore, the fibre $f^{-1}(0)$ is, unlike all other cases, the semi-line $(0, 0, z)$ for $\epsilon = 1$ and $(0, 0, z) \cup (0, y, y^2)$, for $\epsilon = -1$, with $z \geq 0$ (see Figure 7). The singularity of the restriction of f to the boundary $z = 0$ is the same as Case III.

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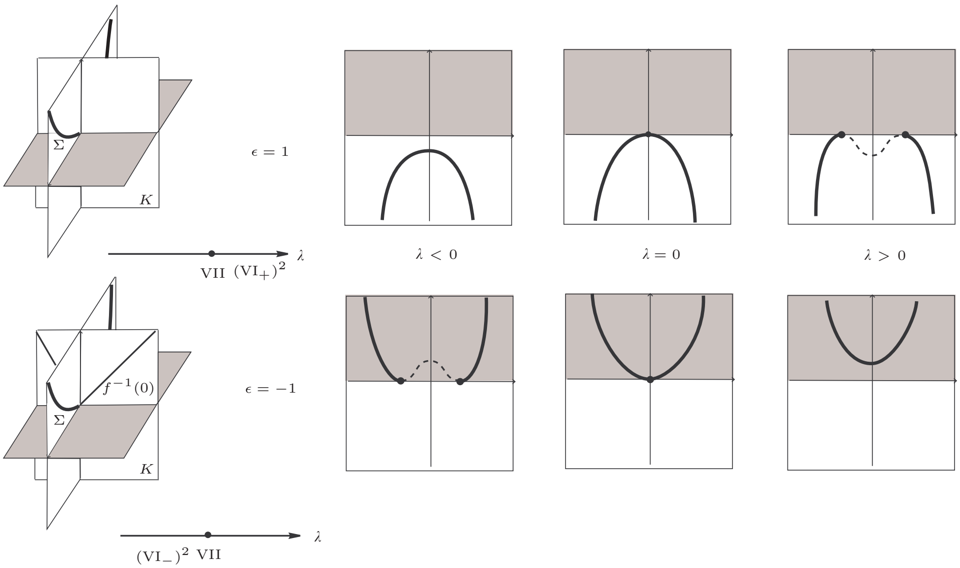

Figure 5. (Colour online) Case VII for $\epsilon\bar{\epsilon} = -1$.

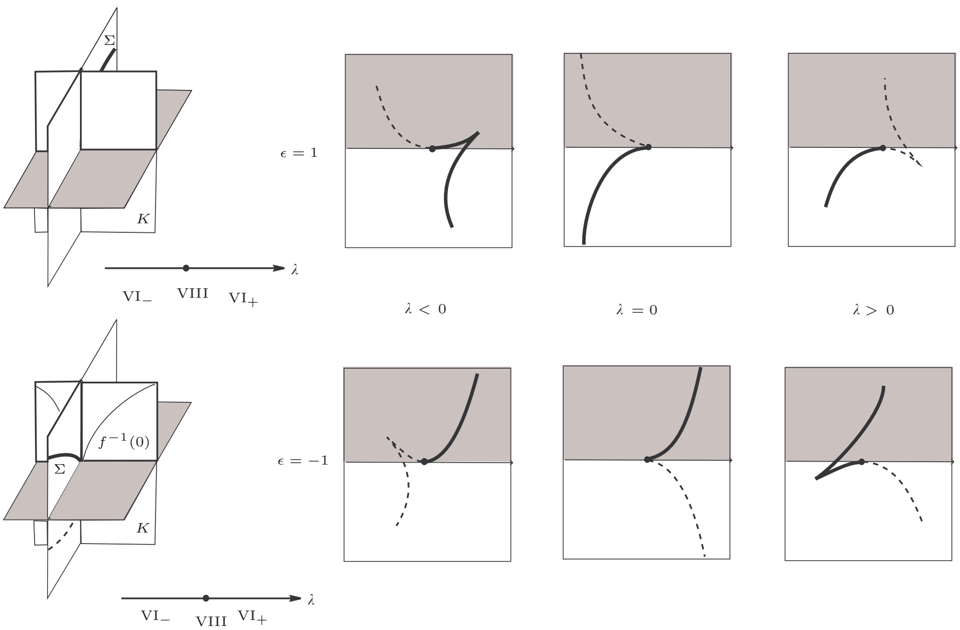

Figure 6. (Colour online) Case VIII ($\epsilon = 1$ and $\epsilon = -1$).

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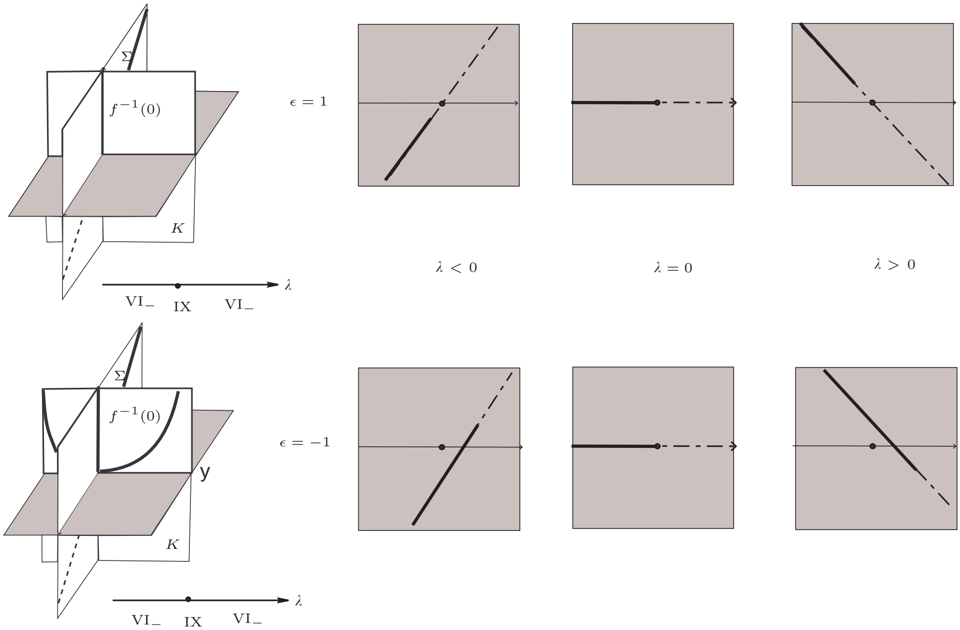

Figure 7. (Colour online) Case IX ($\epsilon = 1$ and $\epsilon = -1$).

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