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DING-GRADED MODULES AND GORENSTEIN GR-FLAT MODULES

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Abstract. Let R be a graded ring. We introduce the concepts of Ding gr-injective and Ding gr-projective R-modules, which are the graded analogues of Ding injective and Ding projective modules. Several characterizations and properties of Ding gr-injective and Ding gr-projective modules are obtained. In addition, we investigate the relationships among Gorenstein gr-flat, Ding gr-injective and Ding gr-projective modules.

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1. Introduction. The origin of Gorenstein homological algebra may date back to 1960s, when Auslander and Bridger introduced the concept of G-dimension for finitely generated modules over a two-sided Noetherian ring [**5**]. In 1990s, Enochs, Jenda and Torrecillas extended the ideas of Auslander and Bridger and introduced the concepts of Gorenstein injective, Gorenstein projective and Gorenstein flat modules over arbitrary rings [**9**, **11**]. Recently, Ding, Li and Mao [**7**, **22**] considered two special cases of the Gorenstein injective and Gorenstein projective modules, which they called Gorenstein FP-injective and strongly Gorenstein flat modules, respectively. These two classes of modules over Ding-Chen rings possess many nice properties analogous to Gorenstein injective and Gorenstein projective modules over Gorenstein rings (see [**7**, **17**, **22**, **31**]). So, Gillespie later renamed Gorenstein FP-injective as Ding injective, and strongly Gorenstein flat as Ding projective (see [**17**] for details).

The homological theory of graded rings is very important because of its applications in algebraic geometry [**19**]. In particular, the Gorenstein homological theory for graded rings was developed in [**1**, **2**, **4**, **12**], where Gorenstein gr-injective, Gorenstein gr-projective and Gorenstein gr-flat modules were defined and studied. In the present paper, Ding gr-injective and Ding gr-projective modules over a graded ring are introduced and investigated, which are the analogue of the non-graded Ding injective and Ding projective modules. In addition, we explore the connections among Gorenstein gr-flat, Ding gr-injective and Ding gr-projective modules.

Let us describe the contents of the paper in more details.

In Section 2, we recall some basic notions needed in the sequel, especially about graded rings and graded modules.

In Section 3, we first introduce the concept of Ding gr-injective modules, which is a special case of Gorenstein gr-injective modules. Then, we give some characterizations and properties of Ding gr-injective modules. For example, we prove that the following conditions are equivalent for a graded left R-module M over a gr-Ding-Chen ring R: (1) M is Ding gr-injective. (2) $\mathrm{Ext}^1_{R-gr}(N, M) = 0$ for any $N \in \mathcal{W}$, where $\mathcal{W}$ is the class of all graded left R-modules with finite flat dimension. (3) For every exact sequence $0 \to A \to B \to C \to 0$ in R-gr with $A \in \mathcal{W}$ or $C \in \mathcal{W}$, the functor $\mathrm{Hom}_{R-gr}(-, M)$ leaves it exact. As a consequence, we obtain that every graded left R-module M over a gr-Ding-Chen ring R has a Ding gr-injective pre-envelope $f : M \to N$ with $\mathrm{coker}(f) \in \mathcal{W}$. Furthermore, for a Ding-Chen-graded ring R by a finite group G, we prove that a graded left R-module M has a Ding gr-injective precover in R-gr if and only if $U(M)$ has a Ding injective precover in R-Mod, where U is the forgetful functor R-gr $\to R$-Mod.

In Section 4, we introduce the concept of Ding gr-projective modules, which is a dual concept of Ding gr-injective modules. Some characterizations and properties of Ding gr-projective modules are obtained. For example, we prove that the following conditions are equivalent for a graded left R-module M over a gr-Ding-Chen ring R: (1) M is Ding gr-projective. (2) $\mathrm{Ext}^1_{R-gr}(M, N) = 0$ for any $N \in \mathcal{W}$. (3) For every exact sequence $0 \to A \to B \to C \to 0$ in R-gr with $A \in \mathcal{W}$ or $C \in \mathcal{W}$, the functor $\mathrm{Hom}_{R-gr}(M, -)$ leaves it exact. As a consequence, we obtain that every graded left R-module M over a gr-Ding-Chen ring R has a Ding gr-projective precover $g : N \to M$ with $\ker(g) \in \mathcal{W}$. Moreover, for a Ding-Chen-graded ring R by a finite group G, we prove that a graded left R-module M has a Ding gr-projective pre-envelope in R-gr if and only if $U(M)$ has a Ding projective pre-envelope in R-Mod.

In Section 5, we study the relationships among Gorenstein gr-flat modules, Ding gr-injective modules and Ding gr-projective modules. For example, for a left gr-coherent ring R, we prove that: (1) A graded right R-module M is Gorenstein gr-flat if and only if M^+ is Ding gr-injective. (2) Every Ding gr-projective right R-module is Gorenstein gr-flat. Let R be a gr-Ding-Chen ring. We prove that: (1) Every graded right R-module has a Gorenstein gr-flat pre-envelope. (2) Every finitely presented graded right R-module has a Ding gr-projective pre-envelope. (3) Every pure gr-injective left R-module has a Ding gr-injective precover.

2. Preliminaries. All rings considered are associative with identity element and all modules are unitary. By R-Mod (resp. Mod-R) we will denote the category of left (resp. right) R-modules. ${}_RM$ (resp. M_R) denotes a left (resp. right) R-module. $fd(M)$ stands for the flat dimension of M.

Let $\mathcal{A}$ be any category and $\mathcal{B}$ a class of objects in $\mathcal{A}$. Following [**8**, **13**], we say that a morphism $\phi : B \to A$ in $\mathcal{A}$ is a *$\mathcal{B}$-precover* of A if $B \in \mathcal{B}$ and, for any morphism $f : B' \to A$ with $B' \in \mathcal{B}$, there is a morphism $g : B' \to B$ such that $\phi g = f$. A $\mathcal{B}$-precover $\phi : B \to A$ is said to be a *$\mathcal{B}$-cover* if every endomorphism $g : B \to B$ such that $\phi g = \phi$ is an isomorphism. Dually we have the definitions of a $\mathcal{B}$-pre-envelope and a $\mathcal{B}$-envelope.

Recall that a left R-module M is *FP-injective* [**27**] if $\mathrm{Ext}^1_R(N, M) = 0$ for any finitely presented left R-module N.

A left R-module M is called *Ding injective* [**17**] (resp. *Gorenstein injective* [**9**]) if there is an exact sequence $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ of injective left R-modules such that $M = \ker(E^0 \to E^1)$ and $\mathrm{Hom}_R(A, -)$ leaves the sequence exact whenever A is an FP-injective (resp. injective) left R-module.

A left R-module N is called *Ding projective* [**17**] (resp. *Gorenstein projective* [**9**]) if there is an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective left

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R-modules such that $N = \ker(P^0 \to P^1)$ and $\mathrm{Hom}_R(-, B)$ leaves the sequence exact whenever B is a flat (resp. projective) left R-module.

Let G be a multiplicative group with neutral element e. A *graded ring* R is a ring with identity 1, together with a direct decomposition $R = \oplus_{\sigma\in G} R_\sigma$ (as additive subgroups) such that $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ for all $\sigma, \tau \in G$. A *graded left* R*-module* M is a left R-module endowed with an internal direct sum decomposition $M = \oplus_{\sigma\in G} M_\sigma$, where each M_σ is a subgroup of the additive group of M such that $R_\sigma M_\tau \subseteq M_{\sigma\tau}$ for all $\sigma, \tau \in G$. For graded left R-modules M and N, we put $\mathrm{Hom}_{R-gr}(M, N) = \{f : f \in \mathrm{Hom}_R(M, N)$ and $f(M_\sigma) \subseteq N_\sigma, \sigma \in G\}$. $\mathrm{Hom}_{R-gr}(M, N)$ is the group of all morphisms from M to N in the category R-gr of all graded left R-modules (gr-R will denote the category of all graded right R-modules). It is well known that R-gr is a Grothendieck category [**25**]. An R-linear map $f : M \to N$ is said to be a *graded morphism of degree* τ, $\tau \in G$ if $f(M_\sigma) \subseteq N_{\sigma\tau}$ for all $\sigma \in G$. Graded morphisms of degree τ build an additive subgroup $\mathrm{HOM}_R(M, N)_\tau$ of $\mathrm{Hom}_R(M, N)$. Then, $\mathrm{HOM}_R(M, N) = \oplus_{\tau\in G}\mathrm{HOM}_R(M, N)_\tau$ is a graded abelian group of type G. We will denote Ext^i_R, Ext^i_{R-gr} and EXT^i_R as the right derived functors of Hom_R, Hom_{R-gr} and HOM_R.

Let M be a graded right R-module and N a graded left R-module. The abelian group $M \otimes_R N$ may be graded by putting $(M \otimes_R N)_\sigma$, $\sigma \in G$, equal to the additive subgroup generated by elements $x \otimes y$ with $x \in M_\alpha$, $y \in N_\beta$ such that $\alpha\beta = \sigma$. The object of $\mathbb{Z}$-gr thus defined will be called the graded tensor product of M and N.

Let M be a graded right R-module. We can define the *graded character module* of M as $M^+ = \mathrm{HOM}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ [**3**], where $\mathbb{Q}/\mathbb{Z}$ is graded with the trivial grading. We note then that it can be seen as $M^+ = \oplus_{\sigma\in G}\mathrm{Hom}_{\mathbb{Z}}(M_{\sigma^{-1}}, \mathbb{Q}/\mathbb{Z})$.

Let $M = \oplus_{\sigma\in G} M_\sigma$ be a graded left R-module. Then $M(\sigma)$ is the graded left R-module obtained by putting $M(\sigma)_\tau = M_{\tau\sigma}$ for all $\tau \in G$. The graded module $M(\sigma)$ is called the σ*-suspension* of M. We can see the σ-suspension as an isomorphism of categories $T_\sigma : R\text{-gr} \to R\text{-gr}$, given on objects as $T_\sigma(M) = M(\sigma)$ for $M \in R$-gr.

Projective (resp. flat) objects of R-gr will be called *projective* (resp. *flat*) *graded modules* because M is a gr-projective (resp. gr-flat) module if and only if M is a projective (resp. flat) graded module [**25**]. The injective objects of R-gr will be called *gr-injective modules*.

A graded left R-module M is called *FP-gr-injective* [**3**] if $\mathrm{EXT}^1_R(N, M) = 0$ for any finitely presented graded left R-module N. The *FP-gr-injective dimension* of a graded left R-module M, denoted by $FP\text{-gr-}id(M)$, will be the least integer n such that $\mathrm{EXT}^{n+1}_R(N, M) = 0$ for any finitely presented graded left R-module N. If no such n exists, set $FP\text{-gr-}id(M) = \infty$.

A ring R is called *left gr-coherent* [**2**] if every finitely generated graded left ideal is finitely presented, equivalently, the class of gr-flat right R-modules is closed under graded direct products.

A ring R is called *gr-Ding-Chen* or *gr-n-FC* [**2**] if R is a left and right gr-coherent ring and has finite FP-gr-injective dimension as a graded left and right R-module.

For unexplained concepts and notations, we refer the reader to [**10, 18, 25, 26, 29**].

3. Ding gr-injective modules.

Definition 3.1. Let R be a graded ring. A graded left R-module M is called *Ding gr-injective* if there is an exact sequence of gr-injective left R-modules $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ in R-gr such that $M = \ker(E^0 \to E^1)$ and $\mathrm{Hom}_{R-gr}(Q, -)$ leaves the sequence exact whenever Q is an FP-gr-injective left R-module.

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Remark 3.2. (1) Recall that a graded left R-module M is *Gorenstein gr-injective* [**1**] if there is an exact sequence of gr-injective left R-modules $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ in R-gr such that $M = \ker(E^0 \to E^1)$ and $\mathrm{Hom}_{R-gr}(E, -)$ leaves the sequence exact whenever E is a gr-injective left R-module. Obviously, we have the following implications:

gr-injective $\Rightarrow$ Ding gr-injective $\Rightarrow$ Gorenstein gr-injective.

If R is a left gr-noetherian ring, then any FP-gr-injective left module is gr-injective by [**3**, Theorem 3.6]. So the class of Ding gr-injective left R-modules coincides with that of Gorenstein gr-injective left R-modules.

(2) The class of Ding gr-injective modules is closed under graded direct products.

According to [**10**, Definition 8.1.2], a *left* FP*-gr-injective resolution* of a graded left R-module M means that there is a complex $\cdots \to F_1 \to F_0 \to M \to 0$ in R-gr (not necessarily exact) such that each F_i is FP-gr-injective and $\mathrm{Hom}_{R-gr}(Q, -)$ makes it exact whenever Q is an FP-gr-injective left R-module.

If R is a left gr-coherent ring, then every graded left R-module has an FP-gr-injective pre(cover) by [**30**, Theorem 3.3] and so has a left FP-gr-injective resolution by [**10**, Proposition 8.1.3]. Furthermore, the deleted complexes of such left FP-gr-injective resolutions are unique up to homotopy by [**10**, p. 169].

Now, we characterize Ding gr-injective modules over a gr-coherent ring.

THEOREM 3.3. *Let R be a left gr-coherent ring graded by a group G. The following conditions are equivalent for a graded left R-module M:*

(1) *M is Ding gr-injective.*
(2) $\mathrm{Ext}^i_{R-gr}(Q, M) = 0$ *for any $i \geq 1$ and any FP-gr-injective left R-module Q, and every left FP-gr-injective resolution of M is exact.*
(3) $\mathrm{Ext}^i_{R-gr}(N, M) = 0$ *for any $i \geq 1$ and any graded left R-module N with FP-gr-id$(N) < \infty$, and every left FP-gr-injective resolution of M is exact.*

Moreover, if FP-gr-id$({}_RR) < \infty$, then the above conditions are also equivalent to

(4) $\mathrm{Ext}^i_{R-gr}(N, M) = 0$ *for any $i \geq 1$ and any graded left R-module N with FP-gr-id$(N) < \infty$.*

Proof. (1) $\Rightarrow$ (2) By (1), there exists an exact sequence $0 \to M \to E^0 \to E^1 \to \cdots$ in R-gr with each E^i gr-injective such that $\mathrm{Hom}_{R-gr}(Q, -)$ leaves the sequence exact whenever Q is an FP-gr-injective left R-module. Thus $\mathrm{Ext}^i_{R-gr}(Q, M) = 0$ for any $i \geq 1$ by definition.

By (1), there exists also an exact sequence $\mathcal{C} : \cdots \to E_1 \to E_0 \to M \to 0$ in R-gr with each E_i gr-injective such that $\mathrm{Hom}_{R-gr}(Q, -)$ leaves the sequence exact whenever Q is an FP-gr-injective left R-module. It means that M has an exact left FP-gr-injective resolution $\mathcal{C}$. Let $\mathcal{D} : \cdots \to E'_1 \to E'_0 \to M \to 0$ be any left FP-gr-injective resolution of M. Since the two deleted complexes $\cdots \to E'_1 \to E'_0 \to 0$ and $\cdots \to E_1 \to E_0 \to 0$ are homotopic by [**10**, p.169], they have isomorphic homology modules by [**10**, Proposition 1.4.13]. Thus $\mathcal{D} : \cdots \to E'_1 \to E'_0 \to M \to 0$ is exact.

(2) $\Rightarrow$ (1) By [**30**, Theorem 3.3], M has an FP-gr-injective cover $f : E_0 \to M$. There is an exact sequence $0 \to E_0 \xrightarrow{\lambda} E \to L \to 0$ in R-gr with E gr-injective. Note that L is FP-gr-injective by [**30**, Theorem 2.7]. So there exists a graded morphism $g : E \to M$ such that $g\lambda = f$ since $\mathrm{Ext}^1_{R-gr}(L, M) = 0$. Thus there is a graded morphism $h : E \to E_0$ such that $fh = g$ since f is a cover. Therefore, $fh\lambda = g\lambda = f$, and hence $h\lambda$ is an isomorphism. It follows that E_0 is isomorphic to a direct summand of E and so is

gr-injective. By [**13**, Proposition 1.2.2], $\mathrm{Ext}^1_{R-gr}(Q, \ker(f)) = 0$ for any FP-gr-injective left R-module Q. So $\ker(f)$ has an FP-gr-injective cover $E_1 \to \ker(f)$ with E_1 gr-injective by the proof above. Continuing this process, we can get a complex $\cdots \to E_1 \to E_0 \to M \to 0$ in R-gr with each E_i gr-injective such that $\mathrm{Hom}_{R-gr}(Q, -)$ makes it exact for any FP-gr-injective left R-module Q. By (2), the complex $\cdots \to E_1 \to E_0 \to M \to 0$ is exact.

On the other hand, since $\mathrm{Ext}^i_{R-gr}(Q, M) = 0$ for any $i \geq 1$ and any FP-gr-injective left R-module Q, we have an exact sequence $0 \to M \to E^0 \to E^1 \to \cdots$ in R-gr with each E^i gr-injective such that $\mathrm{Hom}_{R-gr}(Q, -)$ leaves it exact. Now, we get an exact sequence $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ of gr-injective left R-modules in R-gr with $M = \ker(E^0 \to E^1)$ such that $\mathrm{Hom}_{R-gr}(Q, -)$ leaves it exact for any FP-gr-injective left R-module Q. Thus M is Ding gr-injective.

(2) $\Rightarrow$ (3) holds by dimension shifting.

(3) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1) By the proof of (2) $\Rightarrow$ (1), we get a complex $\mathcal{E} : \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ of gr-injective left R-modules in R-gr with $M = \ker(E^0 \to E^1)$ such that $\mathrm{Hom}_{R-gr}(Q, -)$ makes it exact for any FP-gr-injective left R-module Q.

Next, we shall show that $\mathrm{HOM}_R(N, \mathcal{E})$ is exact for any graded left R-module N with FP-gr-$id(N) = n < \infty$. We proceed by induction on n.

(i) Let $n = 0$. $\mathrm{HOM}_R(N, \mathcal{E}) = \oplus_{\sigma\in G}\mathrm{HOM}_R(N, \mathcal{E})_\sigma = \oplus_{\sigma\in G}\mathrm{Hom}_{R-gr}(N(\sigma^{-1}), \mathcal{E})$. Since $N(\sigma^{-1})$ is FP-gr-injective by [**30**, Proposition 2.1], $\mathrm{Hom}_{R-gr}(N(\sigma^{-1}), \mathcal{E})$ is exact. So $\mathrm{HOM}_R(N, \mathcal{E})$ is exact.

(ii) Let $n \geq 1$. There is an exact sequence $0 \to N \to E \to L \to 0$ in R-gr with E gr-injective, which induces an exact sequence of complexes

$$0 \to \mathrm{HOM}_R(L, \mathcal{E}) \to \mathrm{HOM}_R(E, \mathcal{E}) \to \mathrm{HOM}_R(N, \mathcal{E}) \to 0.$$

Note that FP-gr-$id(L) = n - 1$ by dimension shifting and so $\mathrm{HOM}_R(L, \mathcal{E})$ is exact by induction. Thus, $\mathrm{HOM}_R(N, \mathcal{E})$ is exact by [**26**, Theorem 6.3]. In particular, $\mathrm{HOM}_R({}_RR, \mathcal{E})$ is exact since FP-gr-$id({}_RR) < \infty$. Note that $\mathrm{Hom}_R({}_RR, \mathcal{E}) = \mathrm{HOM}_R({}_RR, \mathcal{E})$ by [**25**, Corollary I. 2.11]. Therefore, $\mathcal{E}$ is an exact sequence and so M is Ding gr-injective. □

PROPOSITION 3.4. *Let R be a left gr-coherent ring and* $0 \to A \to B \to C \to 0$ *an exact sequence in R-gr.*

(1) *If A and C are Ding gr-injective, then so is B.*
(2) *If A and B are Ding gr-injective, then so is C.*
(3) *If B and C are Ding gr-injective, then A is Ding gr-injective if and only if* $\mathrm{Ext}^1_{R-gr}(Q, A) = 0$ *for any FP-injective left R-module Q.*

Thus the class of Ding gr-injective left R-modules is closed under graded direct summands.

Proof. If A is Ding gr-injective, then $\mathrm{Ext}^1_{R-gr}(Q, A) = 0$ for any FP-gr-injective left R-module Q, which means that the sequence $0 \to A \to B \to C \to 0$ is $\mathrm{Hom}_{R-gr}(Q, -)$ exact. Since every graded left R-module has an FP-gr-injective precover by [**30**, Theorem 3.3], we obtain the long exact sequence of part (1) of [**10**, Theorem 8.2.3] by letting $T = \mathrm{HOM}_R({}_RR, -)$. So (1), (2) and (3) follow from Theorem 3.3.

The last statement follows from the graded version of [**20**, Proposition 1.4]. □

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Given a class $\mathcal{L}$ of graded left R-modules, we write $\mathcal{L}^{\perp} = \{C : \mathrm{Ext}^1_{R-gr}(L, C) = 0 \text{ for all } L \in \mathcal{L}\}$ and ${}^{\perp}\mathcal{L} = \{C : \mathrm{Ext}^1_{R-gr}(C, L) = 0 \text{ for all } L \in \mathcal{L}\}$. Recall that a pair $(\mathcal{F}, \mathcal{C})$ of classes of graded left R-modules is a *cotorsion pair* (also called a *cotorsion theory*) [**14**] if $\mathcal{F}^{\perp} = \mathcal{C}$ and $\mathcal{F} = {}^{\perp}\mathcal{C}$. A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is called *complete* [**13**, **18**] if for every graded left R-module M, there are exact sequences $0 \to M \to A \to B \to 0$ and $0 \to D \to C \to M \to 0$ in R-gr with $B, C \in \mathcal{F}$ and $A, D \in \mathcal{C}$. A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is called *hereditary* [**13**] if whenever $0 \to A \to B \to C \to 0$ is exact in R-gr with $A, B \in \mathcal{C}$, then C is in $\mathcal{C}$, equivalently, whenever $0 \to X \to Y \to Z \to 0$ is exact in R-gr with $Y, Z \in \mathcal{F}$, then X is in $\mathcal{F}$.

LEMMA 3.5. *Let R be a left gr-coherent ring. Then $({}^{\perp}$gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n)$ is a complete and hereditary cotorsion pair, where gr-$\mathcal{FI}_n$ is the class of all graded left R-modules of FP-gr-injective dimension at most n.*

Proof. For any finitely presented graded left R-module A, denote by K_A the n-th graded syzygy module of A. Then

$$\mathrm{Ext}^1_{R-gr}(K_A, N) \cong \mathrm{Ext}^{n+1}_{R-gr}(A, N) = 0,$$

for any $N \in$ gr-$\mathcal{FI}_n$. Let the set $X = \oplus K_A$, where the direct sum is over a representative set of all finitely presented graded left R-modules A. Then gr-$\mathcal{FI}_n = X^{\perp}$. Thus, $({}^{\perp}$gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n)$ is a complete cotorsion pair by [**13**, Corollary 3.1.6] which is obviously hereditary. □

COROLLARY 3.6. *Let R be a left gr-coherent ring. Then a Ding gr-injective left R-module M is either gr-injective or has FP-gr-injective dimension ∞.*

Proof. Assume that FP-gr-$id(M) < \infty$. Then there is an exact sequence $0 \to M \to E \to L \to 0$ in R-gr with E gr-injective. Note that FP-gr-$id(L) < \infty$ by Lemma 3.5 and hence $\mathrm{Ext}^1_{R-gr}(L, M) = 0$ by Theorem 3.3. Thus M is isomorphic to a direct summand of E and so is gr-injective. □

COROLLARY 3.7. *The following conditions are equivalent for a left gr-coherent ring R:*

(1) *R is left gr-noetherian.*
(2) *Every FP-gr-injective left R-module is Ding gr-injective.*

Proof. (1) ⇒ (2) is clear since every FP-gr-injective left R-module is gr-injective by [**3**, Theorem 3.6].

(2) ⇒ (1) Let M be an FP-gr-injective left R-module. Then M is Ding gr-injective by (2). So M is gr-injective by Corollary 3.6. Thus, R is a left gr-noetherian ring by [**3**, Theorem 3.6]. □

Example 3.8. Let K be a field with characteristic $p \neq 0$ and let $G = \cup_{k \geq 1} G_k$, where G_k is the cyclic group with generator a_k, the order of a_k is p^k and $a_k = a^p_{k+1}$. We will denote $R = K[G]$. Consider $R[H]$ with H a group. By [**3**, Example vii], $R[H]$ is a gr-coherent ring but is not gr-noetherian. By [**21**, Proposition 1] or [**3**, Example iii], R is FP-injective, $R[H]$ is FP-gr-injective but is not gr-injective. So $R[H]$ is not Ding gr-injective by Corollary 3.6.

By [**2**, Proposition 2.8], for a gr-Ding-Chen ring R, the class of all graded left R-modules with finite flat dimension and the class of all graded left R-modules with finite gr-FP-injective dimension are the same, and we will use $\mathcal{W}$ to denote these classes in what follows.

The following theorem gives characterizations of Ding gr-injective modules over a gr-Ding-Chen ring.

THEOREM 3.9. *Let R be a gr-Ding-Chen ring. The following conditions are equivalent for a graded left R-module M:*

(1) *M is Ding gr-injective.*
(2) *$M \in \mathcal{W}^{\perp}$.*
(3) *For every exact sequence $0 \to A \to B \to C \to 0$ in R-gr with $C \in \mathcal{W}$, the functor $\mathrm{Hom}_{R-gr}(-, M)$ leaves it exact.*
(4) *For every exact sequence $0 \to A \to B \to C \to 0$ in R-gr with $A \in \mathcal{W}$, the functor $\mathrm{Hom}_{R-gr}(-, M)$ leaves it exact.*

Proof. (1) $\Rightarrow$ (2) follows from Theorem 3.3.

(2) $\Rightarrow$ (1) There is an exact sequence $0 \to M \to E^0 \to E^1 \to \cdots$ of gr-injective left R-modules in R-gr. Let $L^n = \mathrm{coker}(E^{n-2} \to E^{n-1})$. Since $M \in \mathcal{W}^{\perp}$, $L^n \in \mathcal{W}^{\perp}$ for any $n \geq 1$ by [**30**, Theorem 4.3]. So $\mathrm{Ext}^1_{R-gr}(Q, L^n) = 0$, whenever Q is an FP-gr-injective left R-module. Thus, $\mathrm{Hom}_{R-gr}(Q, -)$ leaves the sequence above exact.

On the other hand, there is an exact sequence $0 \to K \to E_0 \to M \to 0$ in R-gr with $E_0 \in \mathcal{W}$ and $K \in \mathcal{W}^{\perp}$ by [**30**, Theorem 4.3]. Since $M \in \mathcal{W}^{\perp}$, we have $E_0 \in \mathcal{W}^{\perp}$. There is an exact sequence $0 \to E_0 \to E \to T \to 0$ in R-gr with E gr-injective. Since E_0 and E have finite gr-FP-injective dimension, T has finite gr-FP-injective dimension. Hence $T \in \mathcal{W}$. Thus the sequence $0 \to E_0 \to E \to T \to 0$ is split and so E_0 is gr-injective. Continuing this process on K, we get an exact sequence $\cdots \to E_1 \to E_0 \to M \to 0$ in R-gr with each E_i gr-injective. Thus, we obtain an exact sequence of gr-injective left R-modules $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ in R-gr such that $M = \ker(E^0 \to E^1)$ and $\mathrm{Hom}_{R-gr}(Q, -)$ leaves the sequence exact whenever Q is an FP-gr-injective left R-module. Hence M is Ding gr-injective.

(2) $\Leftrightarrow$ (3) is easy.

(2) $\Rightarrow$ (4) Let $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ be an exact sequence in R-gr with $A \in \mathcal{W}$. Let ${}_RR \in \text{gr-}\mathcal{FI}_n$. Then, $\mathcal{W} = \text{gr-}\mathcal{FI}_n$ by [**2**, Proposition 2.8]. So there is an exact sequence $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} C \to 0$ in R-gr with $Y \in {}^{\perp}\mathcal{W}$ and $X \in \mathcal{W}$ by Lemma 3.5. Also there is an exact sequence $0 \to X \xrightarrow{\mu} E \xrightarrow{\nu} L \to 0$ in R-gr with E gr-injective. So, there exists $\omega : Y \to E$ such that $\omega\alpha = \mu$. The exact sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ induces the exact sequence

$$\mathrm{Hom}_{R-gr}(Y, B) \to \mathrm{Hom}_{R-gr}(Y, C) \to \mathrm{Ext}^1_{R-gr}(Y, A) = 0.$$

Then there exist $\lambda : Y \to B$ and $\varphi : X \to A$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\lambda} & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{\iota} & B & \xrightarrow{\pi} & C & \longrightarrow & 0. \end{array}$$

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Note that $L \in \mathcal{W}$ by Lemma 3.5. So the exact sequence $0 \to X \stackrel{\mu}{\to} E \stackrel{\nu}{\to} L \to 0$ induces the exact sequence

$$\mathrm{Hom}_{R-gr}(E, M) \to \mathrm{Hom}_{R-gr}(X, M) \to \mathrm{Ext}^1_{R-gr}(L, M) = 0.$$

For any $\xi : A \to M$, there exists $\tau : E \to M$ such that $\tau\mu = \xi\varphi$. Hence $(\tau\omega)\alpha = \tau\mu = \xi\varphi$. By [**16**, Lemma 4.1], the left square above is a pushout. So there exists $\rho : B \to M$ such that $\rho\iota = \xi$.

(4) $\Rightarrow$ (2) Let $0 \to M \stackrel{\kappa}{\to} Z \stackrel{\chi}{\to} N \to 0$ be an exact sequence in R-gr with $N \in \mathcal{W}$. By Lemma 3.5, there is an exact sequence $0 \to D \stackrel{f}{\to} H \stackrel{g}{\to} N \to 0$ in R-gr with $H \in {}^{\perp}\mathcal{W}$ and $D \in \mathcal{W}$. Then $H \in \mathcal{W}$. Also there is an exact sequence $0 \to K \to P \to H \to 0$ in R-gr with P gr-projective. Since $K \in \mathcal{W}$, the exact sequence $0 \to K \to P \to H \to 0$ is split and so H is gr-projective. Therefore, there exist $\phi : H \to Z$ and $\psi : D \to M$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D & \stackrel{f}{\longrightarrow} & H & \stackrel{g}{\longrightarrow} & N & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \phi} & & \Big\| & & \\
0 & \longrightarrow & M & \stackrel{\kappa}{\longrightarrow} & Z & \stackrel{\chi}{\longrightarrow} & N & \longrightarrow & 0
\end{array}$$

By (4), there exists $\gamma : H \to M$ such that $\gamma f = \psi$. Thus there exists $\theta : N \to Z$ such that $\chi\theta = 1$ by [**29**, Lemma 7.16]. Hence the sequence $0 \to M \stackrel{\kappa}{\to} Z \stackrel{\chi}{\to} N \to 0$ is split and so $\mathrm{Ext}^1_{R-gr}(N, M) = 0$. □

COROLLARY 3.10. *Let R be a gr-Ding-Chen ring. Then every graded left R-module M has a Ding gr-injective pre-envelope $f : M \to N$ with* $\mathrm{coker}(f) \in \mathcal{W}$.

Proof. It is an immediate consequence of Theorem 3.9 and [**30**, Theorem 4.3]. □

Next we compare the properties of Ding gr-injective modules with Ding injective modules.

The forgetful functor $U : R$-gr $\to R$-Mod associates to M the ungraded R-module $U(M)$. This functor has a right adjoint F which associated to $M \in R$-Mod the graded R-module $F(M) = \oplus_{\sigma \in G}{}^{\sigma}M$, where each ${}^{\sigma}M$ is a copy of M written $\{{}^{\sigma}x : x \in M\}$ with R-module structure defined by $r *{}^{\tau} x = {}^{\sigma\tau}(rx)$ for each $r \in R_{\sigma}$. If $f : M \to N$ is an R-homomorphism, then $F(f) : F(M) \to F(N)$ is a graded morphism given by $F(f)({}^{\sigma}x) = {}^{\sigma}f(x)$. The natural transformations $\varepsilon : UF \to 1_{R-Mod}$ and $\eta : 1_{R-gr} \to FU$ will mean the co-unit and the unit of the adjunction (U, F). If G is a finite group, (F, U) is also an adjoint pair by [**23**, Theorem 3.1]. The natural transformations $\varepsilon' : FU \to 1_{R-gr}$ and $\eta' : 1_{R-Mod} \to UF$ will mean the co-unit and the unit of the adjunction (F, U).

LEMMA 3.11. *Let R be a graded ring by a finite group G. If M is a Ding injective left R-module, then $F(M)$ is Ding gr-injective.*

Proof. There is an exact sequence of injective left R-modules in R-Mod

$$\mathcal{E} : \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

with $M = \ker(E^0 \to E^1)$ and such that $\mathrm{Hom}_R(H, -)$ leaves the sequence exact whenever H is an FP-injective left R-module. Since the functor F is exact, we have the

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exact sequence

$$F(\mathcal{E}) : \cdots \to F(E_1) \to F(E_0) \to F(E^0) \to F(E^1) \to \cdots$$

such that $F(M) \cong \ker(F(E^0) \to F(E^1))$. Note that $F(E_i)$ and $F(E^i)$ are gr-injective by [**28**, Proposition 9.5 C.IV]. For any FP-gr-injective left R-module Q, $U(Q)$ is an FP-injective left R-module by [**3**, Proposition 3.4] since G is a finite group. Note that $\mathrm{Hom}_R(U(Q), \mathcal{E}) \cong \mathrm{Hom}_{R-gr}(Q, F(\mathcal{E}))$ is exact. So $F(M)$ is Ding gr-injective. □

LEMMA 3.12. *Let R be a left gr-coherent ring graded by a finite group G. If M is a graded left R-module with $U(M)$ Ding injective, then M is Ding gr-injective.*

Proof. By Lemma 3.11, $FU(M)$ is Ding gr-injective. Since $FU(M) \cong \oplus_{\sigma\in G} M(\sigma)$ by [**24**, Lemma 3.1], $M = M(e)$ is isomorphic to a direct summand of a Ding gr-injective left R-module and so is Ding gr-injective by Proposition 3.4. □

Recall that R is a *Ding-Chen ring* [**17**] if R is a left and right coherent ring with finite self-FP-injective dimension on either side. Obviously, any Ding-Chen-graded ring is a gr-Ding-Chen ring.

LEMMA 3.13. *Let R be a Ding-Chen-graded ring by a finite group G. Then a graded left R-module M is Ding gr-injective if and only if $U(M)$ is Ding injective.*

Proof. " $\Rightarrow$ " Let C be a left R-module with $fd(C) < \infty$. There is an exact sequence $0 \to A \to B \to C \to 0$ in R-Mod with B projective. Then, we get the exact sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ in R-gr. Since G is finite, F preserves projective objects and also preserves flat objects by its commutativity with direct limits. So $fd(F(C)) < \infty$. By [**23**, Theorem 3.1], $\mathrm{Hom}_{R-gr}(F(-), M) \cong \mathrm{Hom}_R(-, U(M))$. Thus, we have the following commutative diagram:

$$\begin{array}{ccccccc} \mathrm{Hom}_R(B, U(M)) & \longrightarrow & \mathrm{Hom}_R(A, U(M)) & \longrightarrow & \mathrm{Ext}^1_R(C, U(M)) & \longrightarrow & 0 \\ \downarrow\cong & & \downarrow\cong & & \downarrow & & \\ \mathrm{Hom}_{R-gr}(F(B), M) & \longrightarrow & \mathrm{Hom}_{R-gr}(F(A), M) & \longrightarrow & \mathrm{Ext}^1_{R-gr}(F(C), M) & \longrightarrow & 0. \end{array}$$

Note that $\mathrm{Ext}^1_{R-gr}(F(C), M) = 0$ by Theorem 3.9. So $\mathrm{Ext}^1_R(C, U(M)) = 0$. Thus, $U(M)$ is Ding injective by [**17**, Corollary 4.5].

" $\Leftarrow$ " follows from Lemma 3.12. □

THEOREM 3.14. *Let R be a Ding-Chen-graded ring by a finite group G.*

(1) *If $\alpha : M \to N$ is a Ding injective pre-envelope in R-Mod, then $F(\alpha) : F(M) \to F(N)$ is a Ding gr-injective pre-envelope in R-gr.*
(2) *If $\alpha : M \to N$ is a Ding injective precover in R-Mod, then $F(\alpha) : F(M) \to F(N)$ is a Ding gr-injective precover in R-gr.*

Proof. (1) Note that $F(N)$ is Ding gr-injective by Lemma 3.11. Let $f : F(M) \to A$ with A Ding gr-injective be any morphism in R-gr. Then $U(A)$ is a Ding injective left R-module by Lemma 3.13. Note that (F, U) is an adjoint pair by [**23**, Theorem 3.1]. Thus, there exists $g : N \to U(A)$ such that $g\alpha = U(f)\eta'_M$.

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Consider the following commutative diagram:

$$\begin{array}{ccccc} F(M) & \xrightarrow{F(\eta'_M)} & FUF(M) & \xrightarrow{\varepsilon'_{F(M)}} & F(M) \\ & & \Big\downarrow{\scriptstyle FU(f)} & & \Big\downarrow{\scriptstyle f} \\ & & FU(A) & \xrightarrow{\varepsilon'_A} & A. \end{array}$$

Note that $\varepsilon'_{F(M)}F(\eta'_M) = 1$. Therefore, we have

$$(\varepsilon'_A F(g))F(\alpha) = \varepsilon'_A F(g\alpha) = \varepsilon'_A F(U(f)\eta'_M) = \varepsilon'_A FU(f)F(\eta'_M) = f\varepsilon'_{F(M)}F(\eta'_M) = f.$$

Thus $F(\alpha) : F(M) \to F(N)$ is a Ding gr-injective pre-envelope of $F(M)$ in R-gr.

(2) Note that $F(M)$ is Ding gr-injective by Lemma 3.11. Let $\varphi : A \to F(N)$ with A Ding gr-injective be any morphism in R-gr. Then $U(A)$ is a Ding injective left R-module by Lemma 3.13. Since (U, F) is an adjoint pair, there exists $\psi : U(A) \to M$ such that $\alpha\psi = \varepsilon_N U(\varphi)$.

Consider the following commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{\eta_A} & FU(A) & & \\ \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle FU(\varphi)} & & \\ F(N) & \xrightarrow{\eta_{F(N)}} & FUF(N) & \xrightarrow{F(\varepsilon_N)} & F(N). \end{array}$$

Note that $F(\varepsilon_N)\eta_{F(N)} = 1$. Hence we have

$$F(\alpha)(F(\psi)\eta_A) = F(\alpha\psi)\eta_A = F(\varepsilon_N U(\varphi))\eta_A = F(\varepsilon_N)FU(\varphi)\eta_A = F(\varepsilon_N)\eta_{F(N)}\varphi = \varphi.$$

Thus $F(\alpha) : F(M) \to F(N)$ is a Ding gr-injective precover of $F(N)$. $\square$

THEOREM 3.15. *Let R be a Ding-Chen-graded ring by a finite group G and M a graded left R-module.*

(1) *If $\alpha : M \to N$ is a Ding gr-injective pre-envelope in R-gr, then $U(\alpha) : U(M) \to U(N)$ is a Ding injective pre-envelope of $U(M)$ in R-Mod. Conversely, if $\beta : U(M) \to Y$ is a Ding injective pre-envelope in R-Mod, then $F(\beta)\eta_M : M \to FU(M) \to F(Y)$ is a Ding gr-injective pre-envelope of M in R-gr.*

(2) *If $\mu : N \to M$ is a Ding gr-injective precover of M in R-gr, then $U(\mu) : U(N) \to U(M)$ is a Ding injective precover of $U(M)$ in R-Mod. Conversely, if $\phi : H \to U(M)$ is a Ding injective precover in R-Mod, then $\varepsilon'_M F(\phi) : F(H) \to FU(M) \to M$ is a Ding gr-injective precover in R-gr.*

Proof. (1) Suppose that $\alpha : M \to N$ is a Ding gr-injective pre-envelope in R-gr. Then $U(N)$ is a Ding injective left R-module by Lemma 3.13. We claim that $U(\alpha) : U(M) \to U(N)$ is a Ding injective pre-envelope in R-Mod. In fact, let $f : U(M) \to A$ with A Ding injective be any morphism in R-Mod. Since (U, F) is an adjoint pair, there exists $g : N \to F(A)$ such that $g\alpha = F(f)\eta_M$.

Consider the following commutative diagram:

$$\begin{array}{ccccc} U(M) & \xrightarrow{U(\eta_M)} & UFU(M) & \xrightarrow{\varepsilon_{U(M)}} & U(M) \\ & & \downarrow{\scriptstyle UF(f)} & & \downarrow{\scriptstyle f} \\ & & UF(A) & \xrightarrow{\varepsilon_A} & A. \end{array}$$

Note that $\varepsilon_{U(M)}U(\eta_M) = 1$. So, we have

$$(\varepsilon_A U(g))U(\alpha) = \varepsilon_A U(g\alpha) = \varepsilon_A U(F(f)\eta_M) = \varepsilon_A UF(f)U(\eta_M) = f\varepsilon_{U(M)}U(\eta_M) = f.$$

Thus, $U(\alpha) : U(M) \to U(N)$ is a Ding injective pre-envelope of $U(M)$ in R-Mod.

Conversely, we assume that $U(M)$ has a Ding injective pre-envelope $\beta : U(M) \to Y$ in R-Mod. Then, $F(Y)$ is a Ding gr-injective left R-module by Lemma 3.11. We claim that $F(\beta)\eta_M : M \to FU(M) \to F(Y)$ is a Ding gr-injective pre-envelope in R-gr. In fact, let $\varphi : M \to B$ with B Ding gr-injective be any morphism in R-gr. Then $U(B)$ is Ding injective by Lemma 3.13. So there exists $\gamma : Y \to U(B)$ such that $\gamma\beta = U(\varphi)$. Note that $\eta : 1_{R-gr} \to FU$ is a split monomorphism by [**24**, Proposition 2.4] and [**15**, Proposition 5(2)], i.e., there exists $\overline{\eta} : FU \to 1_{R-gr}$ such that $\overline{\eta}\eta = 1$. Hence we have

$$(\overline{\eta}_B F(\gamma))(F(\beta)\eta_M) = \overline{\eta}_B F(\gamma\beta)\eta_M = \overline{\eta}_B FU(\varphi)\eta_M = (\overline{\eta}\eta)_B\varphi = \varphi.$$

Thus, $F(\beta)\eta_M : M \to FU(M) \to F(Y)$ is a Ding gr-injective pre-envelope of M in R-gr.

(2) Suppose that $\mu : N \to M$ is a Ding gr-injective precover of M in R-gr. Then, $U(N)$ is a Ding injective left R-module by Lemma 3.13. We claim that $U(\mu) : U(N) \to U(M)$ is a Ding injective precover of $U(M)$. In fact, let $h : C \to U(M)$ with C Ding injective be any morphism in R-Mod. Then, $F(C)$ is Ding gr-injective by Lemma 3.11. Since (F, U) is an adjoint pair by [**23**, Theorem 3.1], there exists $j : F(C) \to N$ such that $\mu j = \varepsilon'_M F(h)$.

Consider the following commutative diagram:

$$\begin{array}{ccccc} C & \xrightarrow{\eta'_C} & UF(C) & & \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle UF(h)} & & \\ U(M) & \xrightarrow{\eta'_{U(M)}} & UFU(M) & \xrightarrow{U(\varepsilon'_M)} & U(M). \end{array}$$

Note that $U(\varepsilon'_M)\eta'_{U(M)} = 1$. Thus, we have

$$U(\mu)(U(j)\eta'_C) = U(\mu j)\eta'_C = U(\varepsilon'_M F(h))\eta'_C = U(\varepsilon'_M)UF(h)\eta'_C = U(\varepsilon'_M)\eta'_{U(M)}h = h.$$

Hence $U(\mu) : U(N) \to U(M)$ is a Ding injective precover of $U(M)$ in R-Mod.

Conversely, we assume that $U(M)$ has a Ding injective precover $\phi : H \to U(M)$ in R-Mod. Then $F(H)$ is a Ding gr-injective left R-module by Lemma 3.11. Let $\psi : L \to M$ with L Ding gr-injective be any morphism in R-gr. Then, $U(L)$ is Ding injective by Lemma 3.13. So there exists $\xi : U(L) \to H$ such that $\phi\xi = U(\psi)$. Note that $\varepsilon' : FU \to 1_{R-gr}$ is a split epimorphism by [**24**, Proposition 2.4] and [**15**, Proposition 5(1)].

So there exists $\overline{\varepsilon'}: 1_{R-gr} \to FU$ such that $\varepsilon'\overline{\varepsilon'} = 1$. Hence

$$(\varepsilon'_M F(\phi))(F(\xi)\overline{\varepsilon'_L}) = \varepsilon'_M FU(\psi)\overline{\varepsilon'_L} = \psi\varepsilon'_L\overline{\varepsilon'_L} = \psi.$$

Thus $\varepsilon'_M F(\phi) : F(H) \to FU(M) \to M$ is a Ding gr-injective precover in R-gr. □

4. Ding gr-projective modules. Definition 4.1. Let R be a graded ring. A graded left R-module M is called *Ding gr-projective* if there is an exact sequence of gr-projective left R-modules $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ in R-gr such that $M = \ker(P^0 \to P^1)$ and $\mathrm{Hom}_{R-gr}(-, X)$ leaves the sequence exact whenever X is a gr-flat left R-module.

Remark 4.2. (1) Recall that a graded left R-module M is *Gorenstein gr-projective* [**1**] if there is an exact sequence of gr-projective left R-modules $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ in R-gr such that $M = \ker(P^0 \to P^1)$ and $\mathrm{Hom}_{R-gr}(-, P)$ leaves the sequence exact whenever P is a gr-projective left R-module.

Obviously, we have the following implications:

gr-projective ⇒ Ding gr-projective ⇒ Gorenstein gr-projective.

If R is a left perfect ring, then the class of Ding gr-projective left R-modules coincides with that of Gorenstein gr-projective left R-modules.

In general, Ding gr-projective (resp. Ding gr-injective) R-modules need not be gr-projective (resp. gr-injective). For example, let $R = \mathbb{Z}/4\mathbb{Z}$, then R is a commutative quasi-Frobenius ring. It is clear that $2R$ is a Ding projective and Ding injective R-module, but $2R$ is neither projective nor injective. If we consider R as a trivially graded ring, then $2R$ is a Ding gr-projective and Ding gr-injective R-module, but $2R$ is neither gr-projective nor gr-injective.

(2) The class of Ding gr-projective left R-modules is closed under graded direct sums.

LEMMA 4.3. *Let M be a Ding gr-projective left R-module. The following statements hold:*

(1) $\mathrm{Ext}^i_{R-gr}(M, N) = 0$ *for any $i \geq 1$ and any graded left R-module N with $fd(N) < \infty$.*
(2) *M is either gr-projective or has flat dimension ∞.*

Proof. (1) There is an exact sequence $\cdots \to P_1 \to P_0 \to M \to 0$ in R-gr with each P_i gr-projective such that $\mathrm{Hom}_{R-gr}(-, X)$ leaves the sequence exact whenever X is a gr-flat left R-module. Thus, $\mathrm{Ext}^i_{R-gr}(M, X) = 0$ for any $i \geq 1$. By dimension shifting, $\mathrm{Ext}^i_{R-gr}(M, N) = 0$ for any $i \geq 1$ and any graded left R-module N with $fd(N) < \infty$.

(2) Assume that $fd(M) < \infty$. There is an exact sequence $0 \to K \to P \to M \to 0$ in R-gr with P gr-projective. Note that $fd(K) < \infty$ and hence $\mathrm{Ext}^1_{R-gr}(M, K) = 0$ by (1). So M is gr-projective. □

According to [**10**, Definition 8.1.2], a *right gr-flat resolution* of a graded left R-module M means that there is a complex $0 \to M \to F^0 \to F^1 \to \cdots$ in R-gr (not necessarily exact) such that each F^i is gr-flat and $\mathrm{Hom}_{R-gr}(-, X)$ makes it exact for any gr-flat left R-module X.

If R is a right gr-coherent ring, then every graded left R-module has a gr-flat pre-envelope by [**1**, Proposition 4.2] and so has a right gr-flat resolution by [**10**, Proposition 8.1.3]. Moreover the deleted complexes of such right gr-flat resolutions are unique up to homotopy by [**10**, p.169] and so have isomorphic homology modules by [**10**, Proposition 1.4.13].

We next give some characterizations of Ding gr-projective modules over a gr-coherent ring or gr-Ding-Chen ring.

THEOREM 4.4. *Let R be a right gr-coherent ring graded by a group G. The following conditions are equivalent for a graded left R-module M:*

(1) *M is Ding gr-projective.*
(2) $\mathrm{Ext}^i_{R-gr}(M, X) = 0$ *for any $i \geq 1$ and any gr-flat left R-module X, and every right gr-flat resolution of M is exact.*
(3) $\mathrm{Ext}^i_{R-gr}(M, N) = 0$ *for any $i \geq 1$ and any graded left R-module N with $fd(N) < \infty$, and every right gr-flat resolution of M is exact.*
Moreover, if FP-gr-id$(R_R) < \infty$, then the above conditions are also equivalent to
(4) $\mathrm{Ext}^i_{R-gr}(M, N) = 0$ *for any $i \geq 1$ and any graded left R-module N with $fd(N) < \infty$.*

Proof. (1) $\Rightarrow$ (2) By Lemma 4.3(1), $\mathrm{Ext}^i_{R-gr}(M, X) = 0$ for any $i \geq 1$ and any graded flat left R-module X.

By (1), there is an exact sequence $\mathcal{C} : 0 \to M \to P^0 \to P^1 \to \cdots$ of gr-projective left R-modules in R-gr such that $\mathrm{Hom}_{R-gr}(-, X)$ leaves the sequence exact whenever X is a gr-flat left R-module. Thus M has an exact right gr-flat resolution $\mathcal{C}$. So every right gr-flat resolution of M is exact since its deleted complex is homotopic to the complex $0 \to P^0 \to P^1 \to \cdots$.

(2) $\Rightarrow$ (3) holds by dimension shifting.

(3) $\Rightarrow$ (1) By [**1**, Proposition 4.2], M has a gr-flat pre-envelope $f : M \to F^0$. There is an exact sequence $0 \to K^0 \to P^0 \overset{\pi}{\to} F^0 \to 0$ in R-gr with P^0 gr-projective. Since K^0 is gr-flat, $\mathrm{Ext}^1_{R-gr}(M, K^0) = 0$ by (3). So there exists $g : M \to P^0$ such that $\pi g = f$. It is easy to verify that $g : M \to P^0$ is a gr-flat pre-envelope in R-gr. Thus, for any gr-flat left R-module X, we have the exact sequence

$$\mathrm{Hom}_{R-gr}(P^0, X) \to \mathrm{Hom}_{R-gr}(\mathrm{im}(g), X) \to 0.$$

In addition, the exactness of the sequence $0 \to \mathrm{im}(g) \to P^0 \to \mathrm{coker}(g) \to 0$ yields the exact sequence

$$\mathrm{Hom}_{R-gr}(P^0, X) \to \mathrm{Hom}_{R-gr}(\mathrm{im}(g), X) \to \mathrm{Ext}^1_{R-gr}(\mathrm{coker}(g), X) \to 0.$$

Hence $\mathrm{Ext}^1_{R-gr}(\mathrm{coker}(g), X) = 0$. So $\mathrm{coker}(g)$ has a gr-flat pre-envelope $\mathrm{coker}(g) \to P^1$ with P^1 gr-projective by the proof above. Continuing this process, we can get a complex $0 \to M \to P^0 \to P^1 \to \cdots$ with each P^i gr-projective such that $\mathrm{Hom}_{R-gr}(-, X)$ makes the sequence exact whenever X is a gr-flat left R-module. By (3), the complex is exact.

On the other hand, since $\mathrm{Ext}^i_{R-gr}(M, X) = 0$ for any gr-flat left R-module X and any $i \geq 1$, we have an exact sequence $\cdots \to P_1 \to P_0 \to M \to 0$ in R-gr with each P_i gr-projective such that $\mathrm{Hom}_{R-gr}(-, X)$ leaves the sequence exact whenever X is a gr-flat left R-module. Now we get the exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of gr-projective left R-modules in R-gr such that $M \cong \ker(P^0 \to P^1)$ and $\mathrm{Hom}_{R-gr}(-, X)$ leaves the sequence exact whenever X is a gr-flat left R-module. So M is Ding gr-projective.

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(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1) By the proof of (3) $\Rightarrow$ (1), we get a complex in R-gr

$$\mathcal{P} : \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of gr-projective left R-modules such that $M = \mathrm{coker}(P_1 \to P_0)$ and $\mathrm{Hom}_{R-gr}(-, X)$ makes the sequence exact whenever X is a gr-flat left R-module. Next, we will show that $\mathrm{HOM}_R(\mathcal{P}, N)$ is exact for any graded left R-module N with $fd(N) = n < \infty$. We proceed by induction on n.

(i) Let $n = 0$. $\mathrm{HOM}_R(\mathcal{P}, N) = \oplus_{\sigma\in G}\mathrm{HOM}_R(\mathcal{P}, N)_\sigma = \oplus_{\sigma\in G}\mathrm{Hom}_{R-gr}(\mathcal{P}, N(\sigma))$. Since $N(\sigma)$ is flat, $\mathrm{Hom}_{R-gr}(\mathcal{P}, N(\sigma))$ is exact. So $\mathrm{HOM}_R(\mathcal{P}, N)$ is exact.

(ii) Let $n \geq 1$. There is an exact sequence $0 \to K \to P \to N \to 0$ in R-gr with P gr-projective, which induces an exact sequence

$$0 \to \mathrm{HOM}_R(\mathcal{P}, K) \to \mathrm{HOM}_R(\mathcal{P}, P) \to \mathrm{HOM}_R(\mathcal{P}, N) \to 0$$

of complexes. Note that $fd(K) = n - 1$, so $\mathrm{HOM}_R(\mathcal{P}, K)$ is exact by induction. Thus $\mathrm{HOM}_R(\mathcal{P}, N)$ is exact by [**26**, Theorem 6.3].

It follows that $\mathrm{HOM}_R(\mathcal{P}, (R_R)^+)$ is exact since $fd((R_R)^+) = FP\text{-gr-}id(R_R) < \infty$ by [**3**, Proposition 4.2]. Note that $\mathrm{HOM}_{\mathbb{Z}}(R_R \otimes \mathcal{P}, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{HOM}_R(\mathcal{P}, (R_R)^+)$ by [**25**, I. 2.14]. So $\mathcal{P}^+$ is exact. Therefore, $\mathcal{P}$ is an exact sequence by [**3**, Lemma 2.1]. Thus M is Ding gr-projective. □

PROPOSITION 4.5. *Let R be a right gr-coherent ring and $0 \to A \to B \to C \to 0$ an exact sequence in R-gr.*

(1) *If A and C are Ding gr-projective, then so is B.*
(2) *If B and C are Ding gr-projective, then so is A.*
(3) *If A and B are Ding gr-projective, then C is Ding gr-projective if and only if $\mathrm{Ext}^1_{R-gr}(C, X) = 0$ for any gr-flat left R-module X.*

Thus, the class of Ding gr-projective left R-modules is closed under graded direct summands.

Proof. If C is Ding gr-projective, then $\mathrm{Ext}^1_{R-gr}(C, X) = 0$ for any gr-flat left R-module X, which means that $0 \to A \to B \to C \to 0$ is $\mathrm{Hom}_{R-gr}(-, X)$ exact. Since every graded left R-module has a gr-flat pre-envelope by [**1**, Proposition 4.2], we obtain the long exact sequence of part (1) of [**10**, Theorem 8.2.5 (2)] by letting $T = - \otimes_R R$. So (1), (2) and (3) follow from Theorem 4.4.

The last statement follows from the graded version of [**20**, Proposition 1.4]. □

Let R be a gr-Ding-Chen ring. Recall that $\mathcal{W}$ denotes the class of all graded left R-modules with finite flat dimension, equivalently the class of all graded left R-modules with finite gr-FP-injective dimension.

THEOREM 4.6. *Let R be a gr-Ding-Chen ring. The following conditions are equivalent for a graded left R-module M:*

(1) *M is Ding gr-projective.*
(2) *$M \in {}^\perp\mathcal{W}$.*

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(3) *For every exact sequence* $0 \to A \to B \to C \to 0$ *in* R*-gr with* $A \in \mathcal{W}$*, the functor* $\mathrm{Hom}_{R-gr}(M, -)$ *leaves it exact.*
(4) *For every exact sequence* $0 \to A \to B \to C \to 0$ *in* R*-gr with* $C \in \mathcal{W}$*, the functor* $\mathrm{Hom}_{R-gr}(M, -)$ *leaves it exact.*

Proof. (1) $\Rightarrow$ (2) follows from Lemma 4.3(1).

(2) $\Rightarrow$ (1) There is an exact sequence $\cdots \to P_1 \to P_0 \to M \to 0$ in R-gr with each P_i gr-projective. Let $K_n = \ker(P_{n-1} \to P_{n-2})$. By Lemma 3.5, $K_n \in {}^{\perp}\mathcal{W}$ for any $n \geq 1$ since $M \in {}^{\perp}\mathcal{W}$. So $\mathrm{Ext}^1_{R-gr}(K_n, N) = 0$ for any $N \in \mathcal{W}$. Thus $\mathrm{Hom}_{R-gr}(-, N)$ leaves the sequence above exact.

On the other hand, there is an exact sequence $0 \to M \to P^0 \to L \to 0$ in R-gr with $P^0 \in \mathcal{W}$ and $\mathrm{Ext}^1_{R-gr}(L, N) = 0$ for any $N \in \mathcal{W}$ by Lemma 3.5 and [**2**, Proposition 2.8]. So $\mathrm{Ext}^1_{R-gr}(P^0, N) = 0$. There is the exact sequence $0 \to K \to P \to P^0 \to 0$ in R-gr with P gr-projective. Since $K \in \mathcal{W}$, $0 \to K \to P \to P^0 \to 0$ is split and so P^0 is gr-projective. Continuing this process on L, we get an exact sequence $0 \to M \to P^0 \to P^1 \to \cdots$ in R-gr with P^i gr-projective. Thus, we obtain an exact sequence of gr-projective left R-modules $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ in R-gr such that $M \cong \ker(P^0 \to P^1)$ and $\mathrm{Hom}_{R-gr}(-, X)$ leaves the sequence exact whenever X is a gr-flat left R-module.

(2) $\Leftrightarrow$ (3) is easy.

(2) $\Rightarrow$ (4) Let $0 \to A \overset{\iota}{\to} B \overset{\pi}{\to} C \to 0$ be an exact sequence in R-gr with $C \in \mathcal{W}$. By [**30**, Theorem 4.3], there is an exact sequence $0 \to A \overset{\alpha}{\to} Z \overset{\beta}{\to} H \to 0$ in R-gr with $Z \in \mathcal{W}^{\perp}$ and $H \in \mathcal{W}$. The exact sequence $0 \to A \overset{\iota}{\to} B \overset{\pi}{\to} C \to 0$ induces the exact sequence

$$\mathrm{Hom}_{R-gr}(B, Z) \to \mathrm{Hom}_{R-gr}(A, Z) \to \mathrm{Ext}^1_{R-gr}(C, Z) = 0.$$

Then there exist $\lambda : B \to Z$ and $\varphi : C \to H$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \overset{\iota}{\longrightarrow} & B & \overset{\pi}{\longrightarrow} & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & A & \overset{\alpha}{\longrightarrow} & Z & \overset{\beta}{\longrightarrow} & H & \longrightarrow & 0. \end{array}$$

Also there is an exact sequence $0 \to K \overset{\mu}{\to} P \overset{\nu}{\to} H \to 0$ in R-gr with P gr-projective. So there exists $\omega : P \to Z$ such that $\beta\omega = \nu$. Note that $K \in \mathcal{W}$. Thus by (2), we get the exact sequence

$$\mathrm{Hom}_{R-gr}(M, P) \to \mathrm{Hom}_{R-gr}(M, H) \to \mathrm{Ext}^1_{R-gr}(M, K) = 0.$$

For any $\xi : M \to C$, there exists $\tau : M \to P$ such that $\nu\tau = \varphi\xi$. So $\beta\omega\tau = \nu\tau = \varphi\xi$. By [**16**, Lemma 4.1], the right square above is a pullback. So there exists $\rho : M \to B$ such that $\pi\rho = \xi$.

(4) $\Rightarrow$ (2) Let $0 \to N \overset{\kappa}{\to} X \overset{\phi}{\to} M \to 0$ be an exact sequence in R-gr with $N \in \mathcal{W}$. By [**30**, Theorem 4.3], there is an exact sequence $0 \to N \overset{\zeta}{\to} Y \overset{\sigma}{\to} D \to 0$ with $Y \in \mathcal{W}^{\perp}$ and $D \in \mathcal{W}$. Then $Y \in \mathcal{W}$ and so Y is gr-injective by Theorem 3.9 and Corollary 3.6. Thus there exist $\chi : X \to Y$ and $\psi : M \to D$ such that the following diagram is

commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\kappa} & X & \xrightarrow{\phi} & M & \longrightarrow & 0 \\ & & \| & & \vdots\,\chi & & \vdots\,\psi & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \xrightarrow{\zeta} & Y & \xrightarrow{\sigma} & D & \longrightarrow & 0. \end{array}$$

By (4), there exists $\gamma : M \to Y$ such that $\beta\gamma = \psi$. Thus there exists $\theta : X \to N$ such that $\theta\kappa = 1$ by [**29**, Lemma 7.16]. Hence $\mathrm{Ext}^1_{R-gr}(M, N) = 0$. □

COROLLARY 4.7. *Let R be a gr-Ding-Chen ring. Then every graded left R-module M has a Ding gr-projective precover* $g : N \to M$ *with* $\ker(g) \in \mathcal{W}$.

Proof. It is a consequence of Lemma 3.5 and Theorem 4.6. □

At the end of this section, we compare the properties of Ding gr-projective modules with Ding projective modules.

LEMMA 4.8. *Let R be a graded ring by a finite group G. If M is a Ding projective left R-module, then F(M) is Ding gr-projective.*

Proof. There is an exact sequence of projective left R-modules

$$\mathcal{P} : \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $M = \ker(P^0 \to P^1)$ and such that $\mathrm{Hom}(-, N)$ leaves the sequence exact whenever N is a flat left R-module. Since the functor F is exact, we have the exact sequence

$$F(\mathcal{P}) : \cdots \to F(P_1) \to F(P_0) \to F(P^0) \to F(P^1) \to \cdots$$

such that $F(M) \cong \ker(F(P^0) \to F(P^1))$. Since G is a finite group, $F(P_i)$ and $F(P^i)$ are gr-projective. For any gr-flat left R-module Q, $\mathrm{Hom}_{R-gr}(F(\mathcal{P}), Q) \cong \mathrm{Hom}_R(\mathcal{P}, U(Q))$ is exact. So $F(M)$ is Ding gr-projective. □

LEMMA 4.9. *Let R be a left gr-coherent ring graded by a finite group G. If M is a graded left R-module with U(M) Ding projective, then M is Ding gr-projective.*

Proof. By Lemma 4.8, $F(M)$ is Ding gr-projective. Since $FU(M) \cong \oplus_{\sigma\in G} M(\sigma)$ by [**24**, Lemma 3.1], M is isomorphic to a direct summand of a Ding gr-projective left R-module and so is Ding gr-projective by Proposition 4.5. □

LEMMA 4.10. *Let R be a Ding-Chen graded ring by a finite group G. Then a graded left R-module M is Ding gr-projective if and only if U(M) is Ding projective.*

Proof. “ $\Rightarrow$ " Let C be a left R-module with $fd(C) < \infty$. There is an exact sequence $0 \to C \to E \to L \to 0$ in R-Mod with E injective. Then we get the exact sequence $0 \to F(C) \to F(E) \to F(L) \to 0$ in R-gr with $F(E)$ gr-injective by [**28**, Proposition 9.5 C.IV]. Since G is finite, F preserves projective objects and flat objects. So $F(C) \in \mathcal{W}$. Since $\mathrm{Hom}_R(U(M), -) \cong \mathrm{Hom}_{R-gr}(M, F(-))$, we have the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(U(M), E) & \longrightarrow & \mathrm{Hom}_R(U(M), L) & \longrightarrow & \mathrm{Ext}^1_R(U(M), C) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \\
\mathrm{Hom}_{R-gr}(M, F(E)) & \longrightarrow & \mathrm{Hom}_{R-gr}(M, F(L)) & \longrightarrow & \mathrm{Ext}^1_{R-gr}(M, F(C)) & \longrightarrow & 0.
\end{array}$$

So $\mathrm{Ext}^1_R(U(M), C) \cong \mathrm{Ext}^1_{R-gr}(M, F(C)) = 0$ by Theorem 4.6. Thus, $U(M)$ is Ding projective by [**17**, Corollary 4.5].

" $\Leftarrow$ " follows from Lemma 4.9. □

THEOREM 4.11. *Let R be a Ding-Chen-graded ring by a finite group G.*

(1) *If $\alpha : M \to N$ is a Ding projective precover in R-Mod, then $F(\alpha) : F(M) \to F(N)$ is a Ding gr-projective precover in R-gr.*
(2) *If $\alpha : M \to N$ is a Ding projective pre-envelope in R-Mod, then $F(\alpha) : F(M) \to F(N)$ is a Ding gr-projective pre-envelope in R-gr.*

Proof. The proof is dual to that of Theorem 3.14 using Lemmas 4.8 and 4.10. □

THEOREM 4.12. *Let R be a Ding-Chen graded ring by a finite group G and M a graded left R-module.*

(1) *If $\mu : N \to M$ is a Ding gr-projective precover of M in R-gr, then $U(\mu) : U(N) \to U(M)$ is a Ding projective precover of $U(M)$ in R-Mod. Conversely, if $\phi : H \to U(M)$ is a Ding projective precover in R-Mod, then $\varepsilon'_M F(\phi) : F(H) \to FU(M) \to M$ is a Ding gr-projective precover in R-gr.*
(2) *If $\alpha : M \to N$ is a Ding gr-projective pre-envelope in R-gr, then $U(\alpha) : U(M) \to U(N)$ is a Ding projective pre-envelope of $U(M)$ in R-Mod. Conversely, if $\beta : U(M) \to Y$ is a Ding projective pre-envelope in R-Mod, then $F(\beta)\eta_M : M \to FU(M) \to F(Y)$ is a Ding gr-projective pre-envelope of M in R-gr.*

Proof. The proof is dual to that of Theorem 3.15 using Lemmas 4.8 and 4.10. □

5. Gorenstein gr-flat modules. According to [**2**], a graded right R-module M is called *Gorenstein gr-flat* if there is an exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ of gr-flat right R-modules in gr-R such that $M = \ker(F^0 \to F^1)$ and $- \otimes_R E$ is exact for any gr-injective left R-module E.

Let R be a gr-Ding-Chen ring. Asensio, Lopez Ramos and Torrecillas proved that a graded right R-module M is Gorenstein gr-flat if and only if M^+ is Gorenstein gr-injective [**2**, Theorem 2.10]. Here, we give a new characterization of Gorenstein gr-flat modules over a gr-coherent ring.

Recall that an exact sequence $0 \to A \to B \to C \to 0$ in R-gr is said to be *gr-pure* [**3**] if for any $N \in$ gr-R, the sequence $0 \to N \otimes_R A \to N \otimes_R B \to N \otimes_R C \to 0$ is exact in $\mathbb{Z}$-gr.

THEOREM 5.1. *Let R be a left gr-coherent ring graded by a group G. The following conditions are equivalent for a graded right R-module M:*

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(1) *M is Gorenstein gr-flat.*
(2) *There is an exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ of gr-flat right R-modules such that $M = \ker(F^0 \to F^1)$ and $- \otimes_R Q$ is exact for any FP-gr-injective left R-module Q.*
(3) *M^+ is Ding gr-injective.*

Proof. (1) ⇒ (2) By (1), there is an exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ of gr-flat right R-modules in gr-R such that $M = \ker(F^0 \to F^1)$ and $- \otimes_R E$ is exact for any gr-injective left R-module E. Let Q be an FP-gr-injective left R-module. By [**30**, Proposition 2.1], there exists a gr-pure exact sequence $0 \to Q \to N \to L \to 0$ in R-gr with N gr-injective. Thus, we get the split exact sequence $0 \to L^+ \to N^+ \to Q^+ \to 0$ in gr-R by [**3**, Proposition 2.2], and so Q^+ is isomorphic to a direct summand of N^+. It is easy to see that $\mathrm{Tor}_i^R(M, N) = 0$ for any $i \geq 1$, and so $\mathrm{EXT}^i_R(M, N^+) \cong \mathrm{Tor}_i^R(M, N)^+ = 0$ by [**14**, Lemma 2.1]. Thus, $\mathrm{Tor}_i^R(M, Q)^+ \cong \mathrm{EXT}^i_R(M, Q^+) = 0$, and hence $\mathrm{Tor}_i^R(M, Q) = 0$ by [**3**, Lemma 2.3]. It follows that the sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ is $- \otimes Q$ exact.

(2) ⇒ (1) is trivial.

(2) ⇒ (3) By [**3**, Lemma 2.1 and Theorem 3.5], the exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ of gr-flat right R-modules in gr-R induces the exact sequence

$$\cdots \to (F^1)^+ \to (F^0)^+ \to F_0^+ \to F_1^+ \to \cdots$$

of gr-injective left R-modules in R-gr. It is clear that $M^+ \cong \ker(F_0^+ \to F_1^+)$. For any FP-gr-injective left R-module Q, the exact sequence

$$\cdots \to F_1 \otimes_R Q \to F_0 \otimes_R Q \to F^0 \otimes_R Q \to F^1 \otimes_R Q \to \cdots$$

gives the exactness of the sequence

$$\cdots \to (F^1 \otimes_R Q)^+ \to (F^0 \otimes_R Q)^+ \to (F_0 \otimes_R Q)^+ \to (F_1 \otimes_R Q)^+ \to \cdots .$$

So, we get the exact sequence $\cdots \to \mathrm{HOM}_R(Q, (F^1)^+) \to \mathrm{HOM}_R(Q, (F^0)^+) \to \mathrm{HOM}_R(Q, F_0^+) \to \mathrm{HOM}_R(Q, F_1^+) \to \cdots$. Then, we have the exact sequence $\cdots \to \mathrm{Hom}_{R-gr}(Q, (F^1)^+) \to \mathrm{Hom}_{R-gr}(Q, (F^0)^+) \to \mathrm{Hom}_{R-gr}(Q, F_0^+) \to \mathrm{Hom}_{R-gr}(Q, F_1^+) \to \cdots$. Therefore, M^+ is Ding gr-injective.

(3) ⇒ (2) By (3), there exists an exact sequence $0 \to K \to E \to M^+ \to 0$ in R-gr with E gr-injective, which induces the exact sequence $0 \to M^{++} \to E^+ \to K^+ \to 0$ in gr-R. Note that E^+ is gr-flat by [**3**, Theorem 3.7] and so M embeds in a gr-flat right R-module. By [**1**, Propoition 4.2], M has a gr-flat pre-envelope $f : M \to F^0$ which is a monomorphism. Thus, we get an exact sequence $0 \to M \to F^0 \to L^1 \to 0$ in gr-R, which induces the exact sequence $0 \to (L^1)^+ \to (F^0)^+ \to M^+ \to 0$ in R-gr. We next show that $(L^1)^+$ is Ding gr-injective. For any FP-gr-injective left R-module Q, Q^+ is gr-flat by [**3**, Theorem 3.7] and so the sequence $\mathrm{Hom}_{gr-R}(F^0, Q^+) \to \mathrm{Hom}_{gr-R}(M, Q^+) \to 0$ is exact. Thus, we get the exact sequence $\mathrm{Hom}_{R-gr}(Q, (F^0)^+) \to \mathrm{Hom}_{R-gr}(Q, M^+) \to 0$. So $\mathrm{Ext}^1_{R-gr}(Q, (L^1)^+) = 0$. Since $(F^0)^+$ and M^+ are Ding gr-injective, $(L^1)^+$ is Ding gr-injective by Proposition 3.4. Continuing this process, we get an exact sequence $0 \to M \to F^0 \to F^1 \to F^2 \to \cdots$ in gr-R with each F^i gr-flat such that the sequence $\cdots \to \mathrm{Hom}_{R-gr}(Q, (F^1)^+) \to \mathrm{Hom}_{R-gr}(Q, (F^0)^+) \to \mathrm{Hom}_{R-gr}(Q, M^+) \to 0$ is exact.

On the other hand, there is an exact sequence $\cdots \to F_1 \to F_0 \to M \to 0$ of gr-flat right R-modules in gr-R, which induces the exact sequence $0 \to M^+ \to F_0^+ \to F_1^+ \to \cdots$ in R-gr. By Theorem 3.3, $\mathrm{Ext}^i_{R-gr}(Q, M^+) = 0$ for any

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$i \geq 1$ and any FP-gr-injective left R-module Q. So we get the exact sequence $0 \to \mathrm{Hom}_{R-gr}(Q, M^+) \to \mathrm{Hom}_{R-gr}(Q, F_0^+) \to \mathrm{Hom}_{R-gr}(Q, F_1^+) \to \cdots$. Thus, we have the exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ such that $M \cong \ker(F^0 \to F^1)$ and the sequence $\cdots \to \mathrm{Hom}_{R-gr}(Q, (F^1)^+) \to \mathrm{Hom}_{R-gr}(Q, (F^0)^+) \to \mathrm{Hom}_{R-gr}(Q, F_0^+) \to \mathrm{Hom}_{R-gr}(Q, F_1^+) \to \cdots$ is exact. Note that $\mathrm{HOM}_R(Q, -) = \oplus_{\tau \in G} \mathrm{HOM}_R(Q, -)_\tau = \oplus_{\tau \in G} \mathrm{Hom}_{R-gr}(Q(\tau^{-1}), -)$ and $Q(\tau^{-1})$ is FP-gr-injective by [**30**, Proposition 2.1]. So we have the exact sequence $\cdots \to \mathrm{HOM}_R(Q, (F^1)^+) \to \mathrm{HOM}_R(Q, (F^0)^+) \to \mathrm{HOM}_R(Q, F_0^+) \to \mathrm{HOM}_R(Q, F_1^+) \to \cdots$. Thus, we get the exact sequence $\cdots \to F_1 \otimes_R Q \to F_0 \otimes_R Q \to F^0 \otimes_R Q \to F^1 \otimes_R Q \to \cdots$. □

PROPOSITION 5.2. *Let R be a left gr-coherent ring graded by a group G. Then every Ding gr-projective right R-module is Gorenstein gr-flat.*

Proof. Let M be a Ding gr-projective right R-module. Then there is an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of gr-projective right R-modules in gr-R with $M = \ker(P^0 \to P^1)$ such that $\mathrm{Hom}_{gr-R}(-, X)$ leaves the sequence exact whenever X is a gr-flat right R-module. Let E be a gr-injective left R-module. Then, E^+ is gr-flat by [**3**, Theorem 3.7]. Thus for any $\tau \in G$, $E^+(\tau)$ is gr-flat, and so we get the exact sequence $\cdots \to \mathrm{Hom}_{gr-R}(P^1, E^+(\tau)) \to \mathrm{Hom}_{gr-R}(P^0, E^+(\tau)) \to \mathrm{Hom}_{gr-R}(P_0, E^+(\tau)) \to \mathrm{Hom}_{gr-R}(P_1, E^+(\tau)) \to \cdots$. Note that $\mathrm{HOM}_R(-, E^+) = \oplus_{\tau \in G} \mathrm{HOM}_R(-, E^+)_\tau = \oplus_{\tau \in G} \mathrm{Hom}_{gr-R}(-, E^+(\tau))$. So, we get the exact sequence

$$\cdots \to \mathrm{HOM}_R(P^1, E^+) \to \mathrm{HOM}_R(P^0, E^+) \to \mathrm{HOM}_R(P_0, E^+) \\ \to \mathrm{HOM}_R(P_1, E^+) \to \cdots$$

which gives the exactness of the sequence

$$\cdots \to (P^1 \otimes_R E)^+ \to (P^0 \otimes_R E)^+ \to (P_0 \otimes_R E)^+ \to (P_1 \otimes_R E)^+ \to \cdots.$$

Thus, we have the exact sequence

$$\cdots \to P_1 \otimes_R E \to P_0 \otimes_R E \to P^0 \otimes_R E \to P^1 \otimes_R E \to \cdots.$$

Hence M is Gorenstein gr-flat. □

Recall that a graded left R-module M is *pure gr-injective* [**4**] if the functor $\mathrm{Hom}_{R-gr}(-, M)$ leaves every gr-pure exact sequence in R-gr exact.

PROPOSITION 5.3. *Let R be a gr-Ding-Chen ring. If M is a Ding gr-injective left R-module, then M^+ is Gorenstein gr-flat. The converse holds in case M is a pure gr-injective left R-module.*

Proof. If M is Ding gr-injective, then M is Gorenstein gr-injective. So M^+ is Gorenstein gr-flat by [**2**, Proposition 2.17].

Conversely, if M^+ is Gorenstein gr-flat, then M^{++} is Ding gr-injective by Theorem 5.1. Note that M is gr-pure in M^{++} by [**3**, Lemma 2.3]. So $M \to M^{++}$ is split if M is pure gr-injective. Thus, M is Ding gr-injective by Proposition 3.4. □

Let R be a left gr-coherent ring. It is known that every graded right R-module has a Gorenstein gr-flat cover by [**12**, Theorem 3.2.5]. Next, we discuss the existence of Gorenstein gr-flat pre-envelope.

PROPOSITION 5.4. *Let R be a gr-Ding-Chen ring. Then every graded right R-module has a Gorenstein gr-flat pre-envelope.*

Proof. By [**2**, Corollary 2.11], the class of Gorenstein gr-flat right R-modules is closed under graded direct products.

Now let N be a gr-pure submodule of a Gorenstein gr-flat right R-module M. Then the gr-pure exact sequence $0 \to N \to M \to M/N \to 0$ in gr-R induces the split exact sequence $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$ in R-gr by [**3**, Proposition 2.2]. Since M^+ is Ding gr-injective by Theorem 5.1, N^+ is Ding gr-injective by Proposition 3.4. Hence N is Gorenstein gr-flat by Theorem 5.1 again. Thus the class of Gorenstein gr-flat right R-modules is closed under gr-pure submodules. By [**6**, Theorem 4.1], every graded right R-module has a Gorenstein gr-flat pre-envelope. □

COROLLARY 5.5. *Let R be a gr-Ding-Chen ring. Then every finitely presented graded right R-module has a Ding gr-projective pre-envelope.*

Proof. Let M be a finitely presented graded right R-module. By Proposition 5.4, M has a Gorenstein gr-flat pre-envelope $f : M \to N$. By [**2**, Theorem 2.10], there exist a finitely presented Gorenstein gr-projective right R-module H, $g : M \to H$ and $h : H \to N$ such that $hg = f$. By [**2**, Corollary 2.13], H is Gorenstein gr-flat. For any graded right R-module A with $fd(A) < \infty$, $FP\text{-gr-}id(A^+) < \infty$ by [**3**, Theorem 3.5]. Thus $\mathrm{EXT}^1_R(H, A)^+ \cong \mathrm{Tor}^R_1(H, A^+) = 0$ by [**1**, Lemma 2.3] and [**2**, Theorem 2.10]. Hence $\mathrm{Ext}^1_{gr-R}(H, A) = 0$. So H is Ding gr-projective by Theorem 4.6. It is easy to see that $g : M \to H$ is a Ding gr-projective pre-envelope of M. □

COROLLARY 5.6. *Let R be a gr-Ding-Chen ring. Then every pure gr-injective left R-module has a Ding gr-injective precover.*

Proof. Let M be a pure gr-injective left R-module. Then M^+ has a Gorenstein gr-flat pre-envelope $f : M^+ \to N$ by Proposition 5.4. Note that N^+ is Ding gr-injective by Theorem 5.1. Let $H \to M$ be any morphism in R-gr with H Ding gr-injective. Then H^+ is Gorenstein gr-flat by Proposition 5.2. So $\mathrm{Hom}_{gr-R}(N, H^+) \to \mathrm{Hom}_{gr-R}(M^+, H^+) \to 0$ is exact.

Consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}_{R-gr}(H, N^+) & \longrightarrow & \mathrm{Hom}_{R-gr}(H, M^{++}) & & \\
\cong\downarrow & & \cong\downarrow & & \\
\mathrm{Hom}_{gr-R}(N, H^+) & \longrightarrow & \mathrm{Hom}_{gr-R}(M^+, H^+) & \longrightarrow & 0.
\end{array}$$

Thus, $\mathrm{Hom}_{R-gr}(H, N^+) \to \mathrm{Hom}_{R-gr}(H, M^{++}) \to 0$ is exact. Since M is pure gr-injective, $M \to M^{++}$ is split. Hence $\mathrm{Hom}_{R-gr}(H, M^{++}) \to \mathrm{Hom}_{R-gr}(H, M) \to 0$ is exact. Thus $N^+ \to M^{++} \to M$ is a Ding gr-injective precover of M. □

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