Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 1–29, *2024*
DOI:10.1017/prm.2024.94

Instability of standing waves for fractional NLS with combined nonlinearities

Zaizheng Li
School of Mathematical Sciences, Hebei Center for Applied Mathematics, Hebei Normal University, Shijiazhuang 050024, Hebei, China (zaizhengli@hebtu.edu.cn)

Haijun Luo
School of Mathematics, Hunan Provincial Key Laboratory of Intelligent Information, Processing and Applied Mathematics, Hunan University, Changsha 410082, Hunan, China (luohj@hnu.edu.cn) (corresponding author)

Zhitao Zhang
HLM, Academy of Mathematics and Systems Science, The Chinese Academy of Sciences, Beijing 100190, China and
School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China (zzt@math.ac.cn)

(Received 4 February 2024; revised 29 June 2024; accepted 21 July 2024)

We study the existence and strong instability of standing wave solutions for the fractional nonlinear Schrödinger equation

$$\begin{cases} \mathrm{i}\psi_t = (-\Delta)^s\,\psi - \left(|\psi|^{p-2}\psi + \mu|\psi|^{q-2}\psi\right), & (t,x) \in (0,\infty)\times\mathbb{R}^N, \\ \psi(0,x) = \psi_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 2$, $0 < s < 1$, $2 < q < p < 2_s^* = 2N/(N-2s)$, and $\mu \in \mathbb{R}$. The primary challenge lies in the inhomogeneity of the nonlinearity. We deal with the following three cases: (i) for $2 < q < p < 2 + 4s/N$ and $\mu < 0$, there exists a threshold mass a_0 for the existence of the least energy normalized solution; (ii) for $2 + 4s/N < q < p < 2_s^*$ and $\mu > 0$, we reveal the existence of the ground state solution, explore the strong instability of standing waves, and provide a blow-up criterion; (iii) for $2 < q \leq 2 + 4s/N < p < 2_s^*$ and $\mu < 0$, the strong instability of standing wave solutions is demonstrated. These findings are illuminated through variational characterizations, the profile decomposition, and the virial estimate.

Keywords: fractional NLS; instability; standing wave; variational characterization; ground state solution

2020 *Mathematics Subject Classification:* Primary: 35R11
Secondary: 35B35; 35Q55

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

1. Introduction and main results

We consider the following fractional nonlinear Schrödinger equation (NLS) with combined nonlinearities:

$$\begin{cases} \mathrm{i}\psi_t = (-\Delta)^s \psi - f(|\psi|)\psi, & (t,x) \in (0,\infty) \times \mathbb{R}^N, \\ \psi(0,x) = \psi_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 2$, $0 < s < 1$, $2 < q < p < 2_s^* := 2N/(N-2s)$, $\mu \in \mathbb{R}$, $f(t) = t^{p-2} + \mu t^{q-2}$, and $(-\Delta)^s$ is the fractional Laplacian operator defined by

$$(-\Delta)^s u(x) = C(N,s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(0)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy,$$

where $C(N,s)$ is a dimensional constant (see [9, 11]).

The fractional NLS was initially discovered by Laskin [25]. Its inception traces back to the extension of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. It arises naturally in the continuum limit of discrete models featuring long-range interactions, as explored in [24]. Additionally, its presence is evident in the description of Boson stars and the dynamics of water waves. Beyond the realm of physics, the impact of the fractional NLS extends into interdisciplinary domains. Notably, it finds applications in biology, chemistry, and finance, as documented in [2].

The conservation of mass is a foundational principle for solutions to (1.1), ensuring that $|\psi(t,\cdot)|_2 = |\psi(0,x)|_2$ for any $t > 0$. This motivates the exploration of solutions with a prescribed L^2 norm. We employ the standing wave ansatz $\psi(t,x) = e^{-\mathrm{i}\lambda t} u(x)$. Accordingly, $u(x)$ solves

$$(-\Delta)^s u = \lambda u + \mu |u|^{q-2} u + |u|^{p-2} u \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

Moreover, we impose the mass constraint

$$\int_{\mathbb{R}^N} |u|^2 = a^2, \tag{1.3}$$

where $a > 0$ is a given constant.

For the classical Laplacian case, the existence and stability of normalized solutions to problems (1.2) and (1.3) has attracted considerable attention recently. In the case $f(u) = |u|^{p-2}u$ and $p < p^* := 2 + 4/N$ (L^2-subcritical), the energy functional is bounded below on the constrained manifold. Thus, the global minimizer is a good choice of the normalized solution. Lions [29, 30] developed the concentration compactness principle to obtain the compactness of the minimizing sequences. Recalling the methods developed by Cazenave–Lions [7, 30] and Shibata [38], it is routine to prove the orbital stability. Besides, Hajaiej–Song [20], Hirata–Tanaka [21], and Jeanjean–Lu [23] discussed about multiplicity results. However, the constrained energy functional is unbounded from below in the L^2-supercritical case. Jeanjean [22] exploited the mountain pass lemma and a smart compactness argument to prove the existence of normalized solutions. Berestycki–Cazenave [4] and

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

Le Coz [26] showed that the associated standing wave is strongly unstable. If $f(u)$ contains both L^2-subcritical term and L^2-supercritical term, Soave [39, 40] proved the existence and stability results. Finally, one can find more general nonlinearities in the work of Bartsch–de Valeriola [3], Jeanjean–Lu [23], and Gou–Zhang [18].

For the fractional Laplacian case, it is well known that there exists an L^2-critical exponent

$$\bar{p} := 2 + \frac{4s}{N}.$$

When $2 < q < p < 2_s^*$, the existence of normalized solutions has been widely studied by variational methods in the article [33] of the last two authors. Recently, when $p = 2_s^*, 2 < q < 2_s^*$, Zhen–Zhang [43] proved several existence and nonexistence results for a perturbation term $\mu|u|^{q-2}u$. In addition, Luo–Yang–Yang [35] studied the multiplicity and asymptotics of standing waves for the case $s = 1/2$ and $p = 2_s^*, 2 < q < \bar{p}$. Colorado–Ortega [10] proved the existence of positive radial bound and ground state solutions for fractional systems. One can find more general nonlinearities and more results in [28, 31, 32, 37, 41].

Concerning the stability of standing waves, the existing literature is mainly related to the L^2-subcritical or L^2-critical case (see [19, 36, 44]). For the L^2-supercritical case and $\mu = 0$, Feng–Ren–Wang [15] considered the instability of standing waves to (1.1) when $\bar{p} < p < 2_s^*$, based on the homogeneity of the nonlinearity. A powerful tool for proving the strong instability of standing waves is the virial identity introduced by Bonheure–Casteras–Gou–Jeanjean [5] and Soave [39]. However, virial identity does not hold for non-local operators. Moreover, the instability result of standing waves is unknown for the L^2-supercritical nonlinearity with a perturbation term $\mu|u|^{q-2}u$.

This article deals with the instability of standing waves in this respect. The novelty of our article is as follows: First, for $2 < q < p < 2 + 4s/N$ and $\mu < 0$, we find a threshold value to determine whether the least energy solution exists. If it exists, it is orbitally stable. Second, for $2 + 4s/N < q < p < 2_s^*$ and $\mu > 0$, we give several new kinds of equivalent variational characterizations for ground states. Finally, we obtain the strong instability of the associated standing waves and give the blow-up criterion by constructing the equivalent variational characterization and the viral estimate.

Throughout the article, the $L^p(\mathbb{R}^N)$ $(1 \leq p \leq \infty)$ norm is denoted by $|u|_p$. The Hilbert space $H^s(\mathbb{R}^N, \mathbb{C})$ is defined as

$$H^s(\mathbb{R}^N, \mathbb{C}) = \Big\{ u \in L^2(\mathbb{R}^N, \mathbb{C}) \Big| \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(x)|^2 \mathrm{d}x := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x \mathrm{d}y < +\infty \Big\}.$$

For simplicity, we denote $H^s(\mathbb{R}^N) := H^s(\mathbb{R}^N, \mathbb{C})$. The energy functional of (1.2) and (1.3) is defined by

$$E_\mu : H^s(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R},\ E_\mu(u) := \int_{\mathbb{R}^N} \Big(\frac{1}{2} |(-\Delta)^{\frac{s}{2}} u(x)|^2 - \frac{1}{p} |u(x)|^p - \frac{\mu}{q} |u(x)|^q \Big) \mathrm{d}x.$$

Then, the weak solution of (1.2) corresponds to a critical point of the energy functional E_μ on the manifold

$$S_a = \Big\{ u \in H^s(\mathbb{R}^N) \big| \int_{\mathbb{R}^N} |u(x)|^2 \mathrm{d}x = a^2 \Big\},$$

with $\lambda \in \mathbb{R}$ is determined as the Lagrange multiplier (see, e.g., [42]).

The following is the definition of ground state solutions:

Definition 1.1. *The function $\hat{u}$ is called a ground state solution of (1.2) and (1.3) if*

$$dE_\mu|_{S_a}(\hat{u}) = 0 \quad \textit{and} \quad E_\mu(\hat{u}) = \inf \left\{ E_\mu(u) : dE_\mu|_{S_a}(u) = 0, u \in S_a \right\}.$$

Moreover,

$$\mathcal{G}_{a,\mu} = \{u : u \in S_a \textit{ is a ground state solution of } (1.2) \textit{ and } (1.3) \textit{ with } \mu \textit{ given}\}.$$

Recall the notion of stability and instability as below.

Definition 1.2. *(i) The set $\mathcal{G}_{a,\mu}$ is orbitally stable if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\psi_0 \in H^s$ satisfies $\inf_{v \in \mathcal{G}_{a,\mu}} \|\psi_0 - v\|_{H^s} < \delta$, then*

$$\sup_{t>0} \inf_{v \in \mathcal{G}_{a,\mu}} \|\psi(t,\cdot) - v\|_{H^s} < \varepsilon,$$

where $\psi(t,\cdot)$ is the solution to (1.1) with initial datum ψ_0.

(ii) A standing wave $e^{-\mathrm{i}\lambda t}u$ is strongly unstable if, for every $\varepsilon > 0$, there exists $\psi_0 \in H^s$ such that $\|\psi_0 - u\|_{H^s} < \varepsilon$, but $\psi(t,\cdot)$ blows up in finite time.

Main results. First, we shall study the purely L^2-subcritical and defocusing case, i.e., $2 < q < p < \bar{p} = 2 + 4s/N, \mu < 0$. Due to the Gagliardo–Nirenberg inequality, the energy functional E_μ is bounded from below on S_a, which leads to the following global minimization problem:

$$m_{a,\mu} := \inf_{S_a} E_\mu. \tag{1.4}$$

We call $u \in S_a$ a least energy solution of (1.2) and (1.3) if $E_\mu(u) = m_{a,\mu}$. Define

$$a_0 := \inf\{a > 0 : m_{a,\mu} < 0\}. \tag{1.5}$$

Indeed, the existence of least energy solutions depends on a_0. More precisely, our first result reads as follows.

Theorem 1.3. *Let $2 < q < p < 2 + \frac{4s}{N}$ and $\mu < 0$. Then, for $m_{a,\mu}$, a_0 defined in (1.4) and (1.5), the following statements hold:*

(i) $m_{a,\mu} = 0$ for any $a \in (0, a_0]$, while $m_{a,\mu} < 0$ for any $a > a_0$. Moreover, if $0 < a < a_0$, there exists no global minimizer for $m_{a,\mu}$. In addition, there exists a global minimizer $u \in S_a$ for $a \geq a_0$ and u is a ground state solution of (1.2) and (1.3).

(ii) $a_0 \geq \left(\frac{1}{2C_0 C(s,N)}\right)^{\frac{N}{4s}}$, *where* $C_0 = C_0(p,q,s,|\mu|,N)$ *is given by (2.2) and* $C(s,N)$ *is the best constant in the fractional Gagliardo–Nirenberg inequality for* $\alpha = 2 + \frac{4s}{N}$ *(see lemma A.1).*

(iii) The set $\mathcal{G}_{a,\mu}$ *is orbitally stable for any* $a > a_0$.

REMARK 1.4. Theorem 1.3 fills a gap in the previous work [33, theorem 1.3 (ii)].

We remark that the global well-posedness of (1.1) can be obtained by Guo–Huang [19, theorem 2.6] similarly. In addition, Guo–Huang [19] proved that the set $\mathcal{G}_{a,\mu}$ is orbitally stable for $2 < q < p < 2 + \frac{4s}{N}$ and $\mu \geq 0$. Thus, our result is a complement to [19].

Second, we focus on the case $2 + \frac{4s}{N} < q < p < \frac{2N}{N-2s}$ and $\mu > 0$, i.e., the purely L^2-supercritical and focusing case. The energy functional E_μ is now unbounded from below on S_a. In this situation, we shall introduce the following minimizing problem on the constrained Pohozaev manifold:

$$M_{a,\mu} := \inf_{V_{a,\mu}} E_\mu, \quad V_{a,\mu} := \{u \in S_a : P_\mu(u) = 0\}, \tag{1.6}$$

with

$$P_\mu(u) = \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} u(x)|^2 - \frac{N(p-2)}{2ps} |u(x)|^p - \mu \frac{N(q-2)}{2qs} |u(x)|^q \right] \mathrm{d}x. \tag{1.7}$$

It is well known that any critical point of $E_\mu|_{S_a}$ stays on $V_{a,\mu}$ thanks to the Pohozaev identity (see [8, proposition 4.1]), so $V_{a,\mu}$ is a natural constraint.

In this case, we establish the existence and instability of standing waves as below.

THEOREM 1.5 *Assume* $2 + \frac{4s}{N} < q < p < \frac{2N}{N-2s}$ *and* $\mu > 0$. *Then, the following statements hold for* $M_{a,\mu}$ *defined in (1.6):*

(i) $M_{a,\mu}$ *is achieved for any* $a > 0$. *Moreover, the minimizer* u *is a positive radial function, and* u *is a ground state solution of (1.2) and (1.3) with* $\lambda < 0$.

(ii) $M_{a,\mu}$ *is strictly decreasing with respect to* a *for* $\mu > 0$ *given.*

(iii) Suppose $N/(2N-1) \leq s < 1$ *and* $p < 2 + 4s$ *additionally. Let* u *be the ground state solution obtained in (i), then the standing wave* $e^{-i\lambda t}u$ *of (1.1) is strongly unstable.*

In this case, we stress that we [33] obtained a solution with mountain pass geometry. Here, we give a different proof based on new variational characterizations of ground states. As one will see, these new variational characterizations also play a key role in proving the instability of the standing waves.

REMARK 1.6. The restriction on s ensures the local well-posedness of (1.1) (see lemma A.3). In addition, it always holds that $\frac{2N}{N-2s} < 2 + 4s$ for $N \geq 3$.

Finally, we deal with the combined nonlinearities and defocusing case, i.e., $2 < q \leq \bar{p} < p < 2_s^*, \mu < 0$.

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

Similarly, define

$$\hat{E}_\mu(u):=E_\mu(u)-\frac{2s}{N(p-2)}P_\mu(u)=\Big(\frac{1}{2}-\frac{2s}{N(p-2)}\Big)|(-\Delta)^{\frac{s}{2}}u|_2^2+\frac{\mu}{q}\Big(\frac{q-2}{p-2}-1\Big)|u|_p^p$$

and

$$\hat{M}^r_{a,\mu}:=\inf_{\hat{V}_{a,\mu}}\hat{E}_\mu,\quad \hat{V}^r_{a,\mu}=\{u\in S_a:P_\mu(u)\le 0\}, \tag{1.8}$$

and the existence and instability of the standing waves is established.

THEOREM 1.7. *Let* $2<q\le\bar{p}=2+\frac{4s}{N}<p<2^*_s=\frac{2N}{N-2s}$ *and* $\mu<0$*. We also suppose that*

$$|\mu|a^{\beta(p,q)}<\Big(\frac{2ps}{NC(s,N,p)(p-2)}\Big)^{\frac{\bar{p}-q}{p-\bar{p}}}\Big(\frac{q(2^*_s-p)(N-2s)}{2NC(s,N,q)(p-q)}\Big),$$

where

$$\beta(p,q)=\Big(p-\frac{N(p-2)}{2s}\Big)\frac{\bar{p}-q}{p-\bar{p}}+\Big(q-\frac{N(q-2)}{2s}\Big)>0.$$

Then,

(i) *Problems (1.2) and (1.3) admit a radial solution, denoted by* $\hat{u}$*. Moreover,* $\hat{E}_\mu(\hat{u})>0$ *and the Lagrange multiplier* $\hat{\lambda}<0$*.*

(ii) *Suppose additionally* $N/(2N-1)\le s<1$ *and* $p<2+4s$*. Then, the standing wave* $e^{-i\hat{\lambda}t}\hat{u}$ *is strongly unstable.*

In order to better explain our results, we give the following table 1 roughly.

This article is organized as follows: section 2 is devoted to the purely L^2-subcritical case. In this case, we discuss the existence and orbital stability of standing wave solutions to (1.1). Furthermore, theorem 1.3 will be established based on the concentration compactness principle. In section 3, we show theorem 1.5. In fact, we construct different variational characterizations to search for a solution to (1.2) and prove the strong instability of standing wave solutions. As a by-product, we give two invariant manifolds to determine global existence or blow-up behaviour. Section 4 considers the combined cases and proves theorem 1.7.

2. The purely L^2-subcritical and defocusing case: $2<q<p<\bar{p}=2+4s/N$ and $\mu<0$

By lemma A.6, it holds directly that

$$C_0t^{\bar{p}}-\frac{1}{p}t^p-\frac{\mu}{q}t^q\ge 0,\quad \forall t\ge 0, \tag{2.1}$$

where

$$C_0=\frac{p-q}{p(\bar{p}-q)}\left[\frac{q(\bar{p}-p)}{p|\mu|(\bar{p}-q)}\right]^{\frac{\bar{p}-p}{p-q}}. \tag{2.2}$$

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

Table 1. *Existence and instability*

	$2 < q < p < 2 + \frac{4s}{N}, \mu < 0$	$2 + \frac{4s}{N} < q < p < \frac{2N}{N-2s}, \mu > 0$	$2 < q \leq 2 + \frac{4s}{N} < p < \frac{2N}{N-2s}, \mu < 0$
a_0	$a_0 > 0$	$\forall a \in (0, +\infty)$,	for a small,
$a \in (0, a_0)$	$m_{a,\mu} = 0$; global minimizer ✗	$M_{a,\mu} > 0$;	$\hat{M}^r_{a,\mu} > 0$;
$a = a_0$	$m_{a_0,\mu} = 0$; global minimizer ✓	global minimizer ✓	global minimizer ✓
$a \in (a_0, +\infty)$	$m_{a,\mu} < 0$; global minimizer ✓		
	$\mathcal{G}_{a,\mu}$ is orbitally stable for any $a > a_0$	$e^{-i\lambda t}u$ is strongly unstable	for $s \geq \frac{N}{2N-1}$ and $p < 2 + 4s$,
			$e^{-\mathrm{i}\hat{\lambda} t}\hat{u}$ is strongly unstable

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

Lemma 2.1. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^s(\mathbb{R}^N)$ satisfying $\lim_{n\to\infty}|u_n|_2^2 = a^2 > 0$. Let $\alpha_n = a/|u_n|_2$ and $\tilde{u}_n = \alpha_n u_n$. Then, the following holds:*

$$\tilde{u}_n \in S_a, \quad \lim_{n\to\infty}\alpha_n = 1, \quad \lim_{n\to\infty}|E_\mu(\tilde{u}_n) - E_\mu(u_n)| = 0.$$

Proof. The results could be derived by direct calculations, and we omit the details. □

In what follows, we study the properties of $m_{a,\mu}$.

Lemma 2.2.

(i) $m_{a,\mu}$ is bounded from below. Moreover, $m_{a,\mu} \le 0$ for any $a > 0$.
(ii) $m_{\theta a,\mu} \le \theta^2 m_{a,\mu}$ for any $a > 0, \theta \ge 1$.
(iii) If $a^2 = a_1^2 + a_2^2$ with $a_1, a_2 > 0$, then $m_{a,\mu} \le m_{a_1,\mu} + m_{a_2,\mu}$.
(iv) $a \mapsto m_{a,\mu}$ is non-increasing.
(v) For sufficiently large a, $m_{a,\mu} < 0$ holds.
(vi) $a \mapsto m_{a,\mu}$ is continuous.

Proof. (i) On the one hand, by lemma A.1, we deduce that

$$E_\mu(u) \ge \frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2 - \frac{C(s,N,p)}{p}a^{p-\frac{N(p-2)}{2s}}\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\right)^{\frac{N(p-2)}{4s}}$$

for every $u \in S_a$. Since $2 < p < \bar{p}$, it holds that $0 < \frac{N(p-2)}{4s} < 1$. Hence, E_μ is coercive on S_a and $m_{a,\mu}$ is bounded from below. On the other hand, for $u \in S_a$ and $\tau \in \mathbb{R}$, set $(\tau \star u)(x) = e^{\frac{N}{2}\tau}u(e^\tau x)$, then $\tau \star u \in S_a$ and

$$E_\mu(\tau\star u) = \frac{e^{2s\tau}}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2 - \frac{e^{N\tau(\frac{p}{2}-1)}}{p}\int_{\mathbb{R}^N}|u|^p - \mu\frac{e^{N\tau(\frac{q}{2}-1)}}{q}\int_{\mathbb{R}^N}|u|^q. \quad (2.3)$$

Since $2 < q < p$, we obtain $m_{a,\mu} \le \lim_{\tau\to-\infty}E_\mu(\tau\star u) = 0$.

(ii) Let $\theta \ge 1$ and $u \in S_a$. Set $\tilde{u}(x) = u(\theta^{-2/N}x), x \in \mathbb{R}^N$, then $\tilde{u} \in S_{\theta a}$ and

$$E_\mu(\tilde{u}) \le \theta^2\Big(\frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2 - \frac{1}{p}\int_{\mathbb{R}^N}|u|^p - \mu\frac{1}{q}\int_{\mathbb{R}^N}|u|^q\Big).$$

Since u could be chosen arbitrarily, we obtain $m_{\theta a,\mu} \le \theta^2 m_{a,\mu}$.

(iii) Assume that $a_1 \ge a_2$. As a consequence of (ii),

$$m_{a,\mu} \le \left(\frac{a}{a_1}\right)^2 m_{a_1,\mu} = m_{a_1,\mu} + \frac{a_2^2}{a_1^2}m_{\frac{a_1}{a_2}a_2,\mu} \le m_{a_1,\mu} + m_{a_2,\mu}.$$

(iv) This follows directly from (i) and (iii).

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

(v) For $u \in S_1$ given, we set $u_a(x) = au(x)$ for any $a > 0$, then it holds that $u_a \in S_a$. Furthermore, we obtain

$$E_\mu(u_a) = \frac{a^2}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 - \frac{a^p}{p}\int_{\mathbb{R}^N} |u|^p - \mu\frac{a^q}{q}\int_{\mathbb{R}^N} |u|^q.$$

Since $2 < q < p$, we know $E_\mu(u_a) \to -\infty$ as $a \to \infty$. Thus, we get our conclusion.

(vi) The proof is similar to that of [34, lemma 3.3 (v)] and standard. □

Corollary 2.3.

(i) Assume that there exists a global minimizer $u \in S_a$ with respect to m_a for some $a > 0$. Then, $m_{\theta a,\mu} < \theta^2 m_{a,\mu}$ for any $\theta > 1$.

(ii) If there exists a global minimizer $u \in S_{a_1}$ with respect to m_{a_1} for some $a_1 > 0$, then for $a^2 = a_1^2 + a_2^2$ with $a_2 > 0$, one has $m_{a,\mu} < m_{a_1,\mu} + m_{a_2,\mu}$.

Proof. (i) If $u \in S_a$ satisfies $E_\mu(u) = m_{a,\mu}$, then $u \not\equiv 0$. Recalling the proof of lemma 2.2 (ii), one finds that

$$m_{\theta a,\mu} \le E_\mu(\tilde{u}) < \theta^2 E_\mu(u) = \theta^2 m_{a,\mu}.$$

(ii) If $a_1 \ge a_2 > 0$, by (i) and lemma 2.2 (iii), we have

$$m_{a,\mu} < \left(\frac{a}{a_1}\right)^2 m_{a_1,\mu} = m_{a_1,\mu} + \frac{a_2^2}{a_1^2} m_{a_1,\mu} \le m_{a_1,\mu} + m_{a_2,\mu}.$$

If $a_2 > a_1 > 0$, by (i) and lemma 2.2 (iii) again, we get

$$m_{a,\mu} \le \left(\frac{a}{a_2}\right)^2 m_{a_2,\mu} = m_{a_2,\mu} + \frac{a_1^2}{a_2^2} m_{a_2,\mu} < m_{a_2,\mu} + m_{a_1,\mu}.$$

□

Proposition 2.4. *There exists a constant $a_1 > 0$ such that $m_{a,\mu} = 0$ for any $0 < a \le a_1$. In particular, one has $a_0 \ge a_1 > 0$, where a_0 is given in (1.5).*

Proof. For any $u \in S_a$, by (2.1) and lemma A.1, we obtain

$$E_\mu(u) \ge \left[\frac{1}{2} - C_0 C(s, N, \bar{p}) a^{\frac{4s}{N}}\right] \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2.$$

Set $a_1 := \left(\frac{1}{2C_0 C(s,N,\bar{p})}\right)^{\frac{N}{4s}}$, then $E_\mu(u) \ge 0$ for any $a \in (0, a_1]$ and any $u \in S_a$. It follows by lemma 2.2 (i) that $m_{a,\mu} = 0$ for any $0 < a \le a_1$ and $a_0 \ge a_1 > 0$. □

Proposition 2.5. *Let $a > a_0$. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset S_a$ is a minimizing sequence for $m_{a,\mu}$, i.e., $\lim_{n\to\infty} E_\mu(u_n) = m_{a,\mu}$. Then, up to a subsequence, there exist a family $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and $u \in S_a$ such that $\lim_{n\to\infty} u_n(\cdot - y_n) = u$ in $H^s(\mathbb{R}^N)$. Furthermore, u is a global minimizer for $m_{a,\mu}$.*

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

Proof. First, we claim that

$$\overline{\lim_{n\to\infty}} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |u_n|^2 dx > 0.$$

Otherwise, in virtue of concentration compactness principle [14, lemma 2.2], we know $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for any $p \in (2, 2_s^*)$. Then, it holds that

$$m_{a,\mu} = \lim_{n\to\infty} E_\mu(u_n) \geq \liminf_{n\to\infty} \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \geq 0,$$

which contradicts with $m_{a,\mu} < 0$ for $a > a_0$.

From the claim above, there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that, up to a subsequence,

$$0 < \lim_{n\to\infty} \int_{B(0,1)} |u_n(x - y_n)|^2 dx < \infty. \tag{2.4}$$

By lemma 2.2 (i), the minimizing sequence $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $H^s(\mathbb{R}^N)$. Thus, $\{u_n(\cdot - y_n)\}_{n\in\mathbb{N}}$ is also bounded in $H^s(\mathbb{R}^N)$. As a consequence, there exists a $u \in H^s(\mathbb{R}^N)$ such that, up to a subsequence,

$$u_n(\cdot - y_n) \rightharpoonup u, \quad \text{weakly in } H^s(\mathbb{R}^N). \tag{2.5}$$

Via (2.4) and (2.5), we know $|u|_2 > 0$. Take $v_n = u_n(\cdot - y_n) - u$, it holds that $v_n \rightharpoonup 0$ weakly in $H^s(\mathbb{R}^N)$. Therefore, by Brezis–Lieb lemma, as $n \to \infty$,

$$\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 dx = \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} v_n\right|^2 dx + o(1).$$

$$\int_{\mathbb{R}^N} |u_n|^r dx = \int_{\mathbb{R}^N} |u|^r dx + \int_{\mathbb{R}^N} |v_n|^r dx + o(1), \quad \forall r \in \left[2, \frac{2N}{N-2s}\right].$$

Noting that $E_\mu(u_n) = E_\mu(u_n(\cdot - y_n)) = E_\mu(u + v_n)$, it consequently follows that

$$E_\mu(u_n) = E_\mu(u) + E_\mu(v_n) + o(1), \quad |u_n|_2^2 = |u|_2^2 + |v_n|_2^2 + o(1). \tag{2.6}$$

We claim that $|v_n|_2^2 \to 0$ as $n \to \infty$. Indeed, set $\zeta = |u|_2 > 0$, then $\lim_{n\to\infty} |v_n|_2^2 = a^2 - \zeta^2$. If $\zeta = a$, then the claim holds directly. Suppose that $\zeta < a$ and set $\tilde{v}_n = \frac{\sqrt{a^2-\zeta^2}}{|v_n|_2} v_n$. In virtue of lemma 2.1 and (2.6), we obtain

$$E_\mu(u_n) = E_\mu(u) + E_\mu(v_n) + o(1) = E_\mu(u) + E_\mu(\tilde{v}_n) + o(1) \geq E_\mu(u) + m_{\sqrt{a^2-\zeta^2},\mu} + o(1).$$

Furthermore, letting $n \to \infty$ and by lemma 2.2 (iii), we deduce that

$$m_{a,\mu} \geq E_\mu(u) + m_{\sqrt{a^2-\zeta^2},\mu} \geq m_{\zeta,\mu} + m_{\sqrt{a^2-\zeta^2},\mu} \geq m_{a,\mu}, \tag{2.7}$$

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

which gives $E_\mu(u) = m_{\zeta,\mu}$. According to Corollary 2.3 (ii), it holds that

$$m_{a,\mu} < m_{\zeta,\mu} + m_{\sqrt{a^2-\zeta^2}},$$

which contradicts with (2.7). Thus, the claim holds and $|u|_2^2 = a^2$.

Since $\lim_{n\to\infty} |v_n|_2^2 = 0$, recalling that $\{v_n\}$ is bounded in $H^s(\mathbb{R}^N)$, it follows by Hölder inequality that $\lim_{n\to\infty} |v_n|_p = 0$ and $\lim_{n\to\infty} |v_n|_q = 0$. Moreover,

$$\liminf_{n\to\infty} E_\mu(v_n) = \liminf_{n\to\infty} \frac{1}{2}|(-\Delta)^{\frac{s}{2}} v_n|_2^2 \geq 0. \tag{2.8}$$

In addition, by $|u|_2^2 = a^2$ and (2.6),

$$E_\mu(u_n) = E_\mu(u) + E_\mu(v_n) + o(1) \geq m_{a,\mu} + E_\mu(v_n) + o(1),$$

which implies

$$\limsup_{n\to\infty} E_\mu(v_n) \leq 0. \tag{2.9}$$

It follows from (2.8) and (2.9) that $\lim\limits_{n\to\infty} |(-\Delta)^{\frac{s}{2}} v_n|_2^2 \to 0$. Hence, $u_n(\cdot - y_n) \to u$ strongly in $H^s(\mathbb{R}^N)$. □

REMARK 2.6. For the minimizing sequence with respect to $m_{a_0,\mu}$, either the vanishing case occurs or the compactness case holds.

Proof of theorem 1.3. (i) To begin with, we infer from proposition 2.4 that $a_0 > 0$. By lemma 2.2, we know $m_{a,\mu}$ is non-positive, non-increasing, and continuous in a. Thus, by the definition of a_0, we have $m_{a,\mu} = 0$ for any $0 < a \leq a_0$ and $m_{a,\mu} < 0$ for any $a > a_0$.

Moreover, we claim that $m_{a,\mu}$ cannot be achieved for any $0 < a < a_0$. If not, assume $m_{a,\mu}$ is achieved for some $0 < a < a_0$, then it follows from corollary 2.3 (ii) that $m_{a_0,\mu} < m_{a,\mu} = 0$, which contradicts with the definition of a_0.

For $a > a_0$, the existence of a least energy solution to (1.2) and (1.3) follows directly from proposition 2.5. We know that the least energy solution is also a ground state solution.

Finally, we try to show that $m_{a,\mu}$ is achieved for $a = a_0$. Let u_n be a global minimizer for $m_{a_0+\frac{1}{n},\mu}$ for any $n \in \mathbb{N}$, then using the symmetric arrangement, we can assume that u_n is radially symmetric with respect to the origin and it is non-increasing. Since $E_\mu(u_n)$ and $|u_n|_2$ are uniformly bounded, $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^s(\mathbb{R}^N)$. What is more, $\lim\limits_{n\to\infty} |u_n|_2 = a_0$. Set $v_n = \frac{\sqrt{a_0}}{|u_n|_2} u_n$, then we can deduce from lemma 2.1 that

$$v_n \in S_{a_0}, \quad \lim_{n\to\infty} E_\mu(v_n) = \lim_{n\to\infty} E_\mu(u_n) = \lim_{n\to\infty} m_{a_0+\frac{1}{n},\mu} = 0,$$

where the last equality follows from the continuity of $m_{a,\mu}$ and $m_{a_0,\mu} = 0$. Thus, $\{v_n\}_{n\in\mathbb{N}} \subset S_{a_0}$ is a minimizing sequence for $m_{a_0,\mu}$.

Claim. Up to a subsequence, there exist $v \in S_{a_0}$ and $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that

$$v_n(\cdot - y_n) \to v \text{ in } H^s(\mathbb{R}^N) \text{ as } n \to \infty.$$

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

In particular, v is a global minimizer of $m_{a_0,\mu}$. If the claim fails, by the remark 2.6, since $v_n = \frac{\sqrt{a_0}}{|u_n|_2} u_n$, we obtain

$$\overline{\lim_{n\to\infty}} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |u_n|^2 \, \mathrm{d}x = 0. \tag{2.10}$$

By a similar argument as the proof of Claim 2 in [27, theorem 1.3], we can know $\{u_n\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $C^{\gamma_0}\left(\mathbb{R}^N\right)$ for some small constant $\gamma_0 > 0$. Together with (2.10), it follows that $u_n(0) = \|u_n\|_{L^\infty} \to 0$ as $n \to \infty$.

Define $v_{\eta,n} := \eta^{\frac{N}{2}} u_n(\eta x)$ for $\eta > 1$ large, then $v_{\eta,n} \in S_{a_0+\frac{1}{n}}$. Note that $2 < q < p < 2 + \frac{4s}{N}$, thus $0 < \frac{N(q-2)}{2} < \frac{N(p-2)}{2} < 2s$ and $0 < \eta^{2s} - \eta^{\frac{N(p-2)}{2}} < \eta^{2s} - \eta^{\frac{N(q-2)}{2}}$. Consequently, for n large, $\|u_n\|_{L^\infty}^{p-q} < |\mu|$, and

$$\begin{aligned} E_\mu(v_{\eta,n}) - \eta^{2s} E_\mu(u_n) = \int_{\mathbb{R}^N} |u_n|^q \Bigg[&\left(\eta^{2s} - \eta^{\frac{N(p-2)}{2}}\right) \frac{1}{p} |u_n|^{p-q} \\ &+ \left(\eta^{2s} - \eta^{\frac{N(q-2)}{2}}\right) \frac{\mu}{q} \Bigg] \mathrm{d}x < 0. \end{aligned}$$

We obtain

$$m_{a_0+\frac{1}{n},\mu} \le E_\mu\left(v_\eta, n\right) < \eta^{2s} E_\mu\left(u_n\right) < E_\mu\left(u_n\right) = m_{a_0+\frac{1}{n},\mu},$$

which is a contradiction; thus, there exists a global minimizer for $m_{a_0,\mu}$.

(ii) The lower bound of a_0 is given by proposition 2.4.

(iii) We prove by contradiction. Suppose there exists $\varepsilon_0 > 0$, a sequence of solutions $\{\psi_n\}_{n\in\mathbb{N}}$ of (1.1), and a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that

$$\inf_{v\in\mathcal{G}_{a,\mu}} \|\psi_n(0,\cdot) - v\|_{H^s} < 1/n,$$

but

$$\inf_{v\in\mathcal{G}_{a,\mu}} \|\psi_n(t_n,\cdot) - v\|_{H^s} \ge \varepsilon_0.$$

By the conservation of mass and energy, it holds that

$$|\psi_n(t_n,\cdot)|_2^2 = |\psi_n(0,\cdot)|_2^2 \to a^2, \quad E_\mu(\psi_n(t_n,\cdot)) = E_\mu(\psi_n(0,\cdot)) \to m_{a,\mu}.$$

Let $\alpha_n = a/|\psi_n(t_n,\cdot)|_2$ and $\tilde{\psi}_n(x) = \alpha_n \psi_n(t_n,x)$. Then, by lemma 2.1, the following holds:

$$\tilde{\psi}_n \in S_a, \quad \lim_{n\to\infty} \alpha_n = 1, \quad \lim_{n\to\infty} E_\mu(\tilde{\psi}_n) = m_{a,\mu}.$$

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

By proposition 2.5, there exist a family $\{y_n\} \subset \mathbb{R}^N$ and $u \in \mathcal{G}_{a,\mu}$ such that $\lim_{n\to\infty} \tilde{\psi}_n(\cdot - y_n) = u$ in $H^s(\mathbb{R}^N)$. Thus, $\lim\limits_{n\to\infty} \|\psi_n(t_n, \cdot - y_n) - u\|_{H^s} = 0$. A contradiction follows from the following inequalities:

$$\begin{aligned}\varepsilon_0 &\leq \inf_{v\in\mathcal{G}_{a,\mu}} \|\psi_n(t_n,\cdot) - v\|_{H^s} \leq \|\psi_n(t_n,\cdot) - u(\cdot - y_n)\|_{H^s} \\ &= \|\psi_n(t_n, \cdot - y_n) - u\|_{H^s} = o(1)\end{aligned}$$

as $n \to \infty$. □

3. The purely L^2-supercritical and focusing case: $\bar{p} = 2 + 4s/N < q < p < 2_s^*$ and $\mu > 0$

Recall the minimizing problem:

$$M_{a,\mu} = \inf_{V_{a,\mu}} E_\mu, \quad V_{a,\mu} = \Big\{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 = a^2, P_\mu(u) = 0\Big\}, \tag{3.1}$$

where $P_\mu(u)$ is defined in (1.7). Next, we briefly explain our strategy for proving theorem 1.5(i) and (ii). Actually, to prove $M_{a,\mu}$ is achieved, we consider another minimizing problem:

$$\overline{M}_{a,\mu} := \inf_{\overline{V}_{a,\mu}} E_\mu, \quad \overline{V}_{a,\mu} := \Big\{u \in H^s(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} |u(x)|^2 \mathrm{d}x \leq a^2, P_\mu(u) = 0\Big\}.$$

It is clear that $M_{a,\mu} \geq \overline{M}_{a,\mu}$ since $V_{a,\mu} \subset \overline{V}_{a,\mu}$. For one thing, we will show that $\overline{M}_{a,\mu}$ is achieved based on the profile decomposition of bounded sequences in $H^s(\mathbb{R}^N)$ (see lemma A.2). For another thing, we intend to prove the minimizer u actually stays in $V_{a,\mu}$ by showing

$$\begin{aligned}M_{a,\mu} < E_\mu(u) \quad &\text{for every } u \in \mathring{V}_{a,\mu} \\ &:= \Big\{u \in H^s(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} |u(x)|^2 \mathrm{d}x < a^2, P_\mu(u) = 0\Big\}.\end{aligned}$$

It turns out that $M_{a,\mu}$ is achieved. In addition, we deduce the monotonicity of $M_{a,\mu}$.

First, we analyse the property of $\overline{V}_{a,\mu}$ and $\overline{M}_{a,\mu}$.

Lemma 3.1. *There exists a constant $\delta_0 > 0$ such that*

$$\inf_{u\in\overline{V}_{a,\mu}} |(-\Delta)^{\frac{s}{2}} u|_2^2 \geq \delta_0.$$

Moreover, E_μ is coercive on $\overline{V}_{a,\mu}$, and there exists a constant $\delta_1 > 0$ such that

$$\overline{M}_{a,\mu} \geq \delta_1.$$

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

Proof. First, by $P_\mu(u) = 0$ and lemma A.1, one has, for every $u \in \overline{V}_{a,\mu}$,

$$|(-\Delta)^{\frac{s}{2}} u|_2^2 \leq \frac{N(p-2)}{2ps} C(s,N,p) a^{p-\frac{N(p-2)}{2s}} |(-\Delta)^{\frac{s}{2}} u|_2^{\frac{N(p-2)}{2s}} + \mu \frac{N(q-2)}{2qs} C(s,N,q) a^{p-\frac{N(q-2)}{2s}} |(-\Delta)^{\frac{s}{2}} u|_2^{\frac{N(q-2)}{2s}},$$

which implies

$$1 \leq C_1(s,N,p,a) |(-\Delta)^{\frac{s}{2}} u|_2^{\frac{N(p-2)}{2s}-2} + C_2(s,N,p,a,\mu) |(-\Delta)^{\frac{s}{2}} u|_2^{\frac{N(q-2)}{2s}-2}.$$

Noting that $p > q > 2 + 4s/N$, we have $\frac{N(p-2)}{2s} - 2 > \frac{N(q-2)}{2s} - 2 > 0$. Thus, there exists a constant $\delta_0 > 0$ such that

$$\inf_{u \in \overline{V}_{a,\mu}} |(-\Delta)^{\frac{s}{2}} u|_2^2 \geq \delta_0. \tag{3.2}$$

Moreover, we note that

$$\begin{aligned} \overline{M}_{a,\mu} &= \inf_{u \in \overline{V}_{a,\mu}} \left[\left(\frac{1}{2} - \frac{2s}{N(q-2)} \right) |(-\Delta)^{\frac{s}{2}} u|_2^2 + \frac{1}{p} \left(\frac{p-2}{q-2} - 1 \right) |u|_p^p \right] \\ &\geq \inf_{u \in \overline{V}_{a,\mu}} \left(\frac{1}{2} - \frac{2s}{N(q-2)} \right) |(-\Delta)^{\frac{s}{2}} u|_2^2. \end{aligned}$$

Therefore, $E_\mu|_{\overline{V}_{a,\mu}}$ is coercive, and by (3.2), we obtain $\overline{M}_{a,\mu} \geq \delta_1 := \left(\frac{1}{2} - \frac{2s}{N(q-2)}\right)\delta_0$. □

Proposition 3.2. *$\overline{M}_{a,\mu}$ is achieved at some $u \in \overline{V}_{a,\mu}$. Moreover, the minimizer u is non-negative and radially symmetric.*

Proof. Let $\{v_n\}_{n \in \mathbb{N}}$ be a minimizing sequence of $\overline{M}_{a,\mu}$, then we have

$$E_\mu(v_n) \to \overline{M}_{a,\mu}, \quad P_\mu(v_n) = 0.$$

By lemma 3.1, $\{v_n\}_{n \in \mathbb{N}}$ is bounded in $H^s(\mathbb{R}^N)$ and

$$\liminf_{n \to \infty} |(-\Delta)^{\frac{s}{2}} v_n|_2^2 \geq \delta_0.$$

Applying lemma A.2, we find a profile decomposition of $\{v_n\}_{n \in \mathbb{N}}$ satisfying

$$\limsup_{n \to +\infty} |v_n|_\gamma^\gamma = \sum_{j=1}^{\infty} |V^j|_\gamma^\gamma \quad \text{for every } \gamma \in \left(2, \frac{2N}{(N-2s)^+}\right). \tag{3.3}$$

Let

$$J = \left\{ j \geq 1 : V^j \neq 0 \right\},$$

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

then $J \neq \emptyset$. Otherwise, we can deduce from (3.3) that

$$\limsup_{n\to+\infty} |v_n|_p^p = \limsup_{n\to+\infty} |v_n|_q^q = 0.$$

Noting that $P_\mu(v_n) = 0$, we get

$$\delta_0 \leq \limsup_{n\to+\infty} |\,(-\Delta)^{\frac{s}{2}}\, v_n|_2^2 = 0,$$

which is a contradiction.

We claim that there exists some $j_0 \in J$ such that

$$0 < |\,(-\Delta)^{\frac{s}{2}}\, V^j|_2^2 \leq \frac{N(p-2)}{2ps}|V^j|_p^p + \mu\frac{N(q-2)}{2qs}|V^j|_q^q. \tag{3.4}$$

Otherwise, we suppose that for all $j \in J$,

$$|\,(-\Delta)^{\frac{s}{2}}\, V^j|_2^2 > \frac{N(p-2)}{2ps}|V^j|_p^p + \mu\frac{N(q-2)}{2qs}|V^j|_q^q.$$

Then, by lemma A.2 and $P_\mu(v_n) = 0$, we obtain

$$\begin{aligned}
&\limsup_{n\to+\infty} \left(\frac{N(p-2)}{2ps}|v_n|_p^p + \mu\frac{N(q-2)}{2qs}|v_n|_q^q\right) \\
&\quad \geq \sum_{j\in J} |\,(-\Delta)^{\frac{s}{2}}\, V^j|_2^2 > \sum_{j\in J}\left(\frac{N(p-2)}{2ps}|V^j|_p^p + \mu\frac{N(q-2)}{2qs}|V^j|_q^q\right) \\
&\quad = \limsup_{n\to+\infty} \left(\frac{N(p-2)}{2ps}|v_n|_p^p + \mu\frac{N(q-2)}{2qs}|v_n|_q^q\right),
\end{aligned}$$

which is a contradiction. Thus the claim holds.

Let us define

$$r_u := \left(\frac{\frac{N(p-2)}{2ps}|u|_p^p + \mu\frac{N(q-2)}{2qs}|u|_q^q}{|(-\Delta)^{\frac{s}{2}} u|_2^2}\right)^{\frac{1}{2s}}, \tag{3.5}$$

then we know $P_\mu(u(r_u\cdot)) = 0$. Thus, by (3.4), there exists some $j_0 \in J$ such that $r_{V^{j_0}} \geq 1$ and $P_\mu(V^{j_0}(r_{V^{j_0}}\cdot)) = 0$. Moreover,

$$|V^{j_0}(r_{V^{j_0}}\cdot)|_2^2 = r_{V^{j_0}}^{-N}|V^{j_0}|_2^2 \leq r_{V^{j_0}}^{-N}a^2 \leq a^2,$$

which implies $V^{j_0}(r_{V^{j_0}}\cdot) \in \overline{V}_{a,\mu}$. In addition, we also note that

$$\begin{aligned}
\overline{M}_{a,\mu} &= \inf_{\overline{V}_{a,\mu}} E_\mu = \inf_{u\in\overline{V}_{a,\mu}} \left(E_\mu(u) - \frac{1}{2}P_\mu(u)\right) \\
&= \inf_{u\in\overline{V}_{a,\mu}} \left[\frac{1}{p}\left(\frac{N(p-2)}{4s} - 1\right)|u|_p^p + \frac{\mu}{q}\left(\frac{N(q-2)}{4s} - 1\right)|u|_q^q\right].
\end{aligned}$$

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

Thus, it holds that

$$0<\overline{M}_{a,\mu}\leq E_\mu\left(V^{j_0}(r_{V^{j_0}}\cdot)\right)\leq \frac{1}{p}\Big(\frac{N(p-2)}{4s}-1\Big)|V^{j_0}|_p^p+\frac{\mu}{q}\Big(\frac{N(q-2)}{4s}-1\Big)|V^{j_0}|_q^q$$
$$\leq \limsup_{n\to+\infty}E_\mu(v_n)=\overline{M}_{a,\mu},$$

which implies $r_{V^{j_0}}=1$, $V^{j_0}\in\overline{V}_{a,\mu}$, and $E_\mu\left(V^{j_0}\right)=\overline{M}_{a,\mu}$.

Finally, let $u:=|V^{j_0}|^*$ be the Schwartz symmetrization of $|V^{j_0}|$, then by [1, theorem 9.2], one has

$$|u|_2^2=|V^{j_0}|_2^2,\ |u|_p^p=|V^{j_0}|_p^p,\ |u|_q^q=|V^{j_0}|_q^q,\quad \text{and}\ |\,(-\Delta)^{\frac{s}{2}}\,u|_2^2\leq|\,(-\Delta)^{\frac{s}{2}}\,V^{j_0}|_2^2.$$

By (3.5) and $r_{V^{j_0}}=1$, one has $r_u\geq 1$ and $u(r_u\cdot)\in\overline{V}_{a,\mu}$. Suppose $r_u>1$, then

$$\overline{M}_{a,\mu}\leq E_\mu\left(u(r_u\cdot)\right)<\frac{1}{p}\Big(\frac{N(p-2)}{4s}-1\Big)|u|_p^p+\frac{\mu}{q}\Big(\frac{N(q-2)}{4s}-1\Big)|u|_q^q$$
$$=E_\mu\left(V^{j_0}\right)=\overline{M}_{a,\mu},$$

which is a contradiction. Therefore, we get $r_u=1$ and $E_\mu(u)=E_\mu\left(V^{j_0}\right)=\overline{M}_{a,\mu}$. □

Recall

$$\mathring{V}_{a,\mu}=\overline{V}_{a,\mu}\setminus V_{a,\mu}=\Big\{u\in H^s(\mathbb{R}^N)\setminus\{0\}:\int_{\mathbb{R}^N}|u|^2<a^2,P_\mu(u)=0\Big\}.$$

If $\overline{M}_{a,\mu}$ is achieved at some $u\in V_{a,\mu}$, then $M_{a,\mu}=\overline{M}_{a,\mu}$ and $M_{a,\mu}$ is achieved. To rule out the case that $\overline{M}_{a,\mu}$ is achieved at some $u\in\mathring{V}_{a,\mu}$, we need the following lemma:

Lemma 3.3. *For every $u\in\mathring{V}_{a,\mu}$, it holds that*

$$M_{a,\mu}<E_\mu(u).$$

Proof. Suppose by contradiction that $\overline{M}_{a,\mu}=E_\mu(\tilde{u})\leq M_{a,\mu}$ for some $\tilde{u}\in\mathring{V}_{a,\mu}$. Hence, $\tilde{u}$ is a local minimizer for E_μ on $\mathring{V}_{a,\mu}$, and there exists a Lagrange multiplier $\lambda\in\mathbb{R}$ such that

$$E_\mu'(\tilde{u})-\lambda P_\mu'(\tilde{u})=0,$$

i.e., $\tilde{u}$ is a weak solution to

$$(1-2\lambda)\,(-\Delta)^s\,\tilde{u}=\left[1-\lambda\frac{N(p-2)}{2s}\right]|\tilde{u}|^{p-2}\tilde{u}+\mu\left[1-\lambda\frac{N(q-2)}{2s}\right]|\tilde{u}|^{q-2}\tilde{u}.\quad (3.6)$$

Moreover, $\tilde{u}$ satisfies the Pohozaev identity of equation (3.6), i.e.,

$$\frac{N-2s}{2}\,(1-2\lambda)\,|\,(-\Delta)^{\frac{s}{2}}\,\tilde{u}|_2^2=\frac{N}{p}\left[1-\lambda\frac{N(p-2)}{2s}\right]|\tilde{u}|_p^p+\frac{\mu N}{q}\left[1-\lambda\frac{N(q-2)}{2s}\right]|\tilde{u}|_q^q.\quad (3.7)$$

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

In addition, $\tilde{u}$ satisfies the following Nehari-type identity:

$$(1-2\lambda)\,|\,(-\Delta)^{\frac{s}{2}}\,\tilde{u}|_2^2=\Big[1-\lambda\frac{N(p-2)}{2s}\Big]|\tilde{u}|_p^p+\mu\Big[1-\lambda\frac{N(q-2)}{2s}\Big]|\tilde{u}|_q^q. \tag{3.8}$$

Besides, since $P_\mu(\tilde{u})=0$, we obtain

$$|\,(-\Delta)^{\frac{s}{2}}\,\tilde{u}|_2^2=\frac{N(p-2)}{2ps}|\tilde{u}|_p^p+\mu\frac{N(q-2)}{2qs}|\tilde{u}|_q^q. \tag{3.9}$$

After balancing the coefficients of (3.7), (3.8), and (3.9), we deduce that

$$\lambda\frac{N(p-2)}{p}\Big(1-\frac{N(p-2)}{4s}\Big)|\tilde{u}|_p^p+\lambda\mu\frac{N(q-2)}{q}\Big(1-\frac{N(q-2)}{4s}\Big)|\tilde{u}|_q^q=0.$$

Since $p>q>2+4s/N$, $\tilde{u}\neq 0$, and $\mu>0$, it must hold that $\lambda=0$. Thus, $\tilde{u}$ is a weak solution to

$$(-\Delta)^s\,\tilde{u}=|\tilde{u}|^{p-2}\tilde{u}+\mu|\tilde{u}|^{q-2}\tilde{u}.$$

In particular, $\tilde{u}$ satisfies the following Nehari-type identity:

$$|\,(-\Delta)^{\frac{s}{2}}\,\tilde{u}|_2^2=|\tilde{u}|_p^p+\mu|\tilde{u}|_q^q.$$

We combine the above identity with (3.9) to obtain

$$\frac{2N-p(N-2s)}{2ps}|\tilde{u}|_p^p+\mu\frac{2N-q(N-2s)}{2qs}|\tilde{u}|_q^q=0,$$

which is a contradiction since $q<p<2_s^*$, $\mu>0$, and $\tilde{u}\neq 0$. Thus we conclude the proof. □

With the preparation above at hand, we are now able to prove theorem 1.5(i) and (ii).

Proof of theorem 1.5 (i) and (ii). (i) From proposition 3.2 and lemma 3.3, we immediately have

$$M_{a,\mu}=\overline{M}_{a,\mu}. \tag{3.10}$$

Moreover, $M_{a,\mu}$ is attained by a non-negative and radially symmetric function u in $V_{a,\mu}$. Since it is well known that a critical point for $E_\mu|_{V_{a,\mu}}$ is also a critical point for $E_\mu|_{S_a}$, we apply Lagrange multiplier rules to deduce that there exists $\lambda\in\mathbb{R}$ such that

$$(-\Delta)^s\,u-|u|^{p-2}u-\mu|u|^{p-2}u=\lambda u.$$

Thus,

$$|\,(-\Delta)^{\frac{s}{2}}\,u|_2^2=|u|_p^p+\mu|u|_q^q+\lambda|u|_2^2.$$

Combining with the identity $P_\mu(u)=0$, we get

$$\lambda a^2=\lambda|u|_2^2=-\left[\frac{2N-p(N-2s)}{2sp}|u|_p^p+\mu\frac{2N-q(N-2s)}{2sq}|u|_q^q\right].$$

Since $\mu>0$ and $q<p<2_s^*$, we know $\lambda<0$. Moreover, due to $u\geq 0$ and $u\not\equiv 0$, by the maximum principle, it holds that $u>0$ in $\mathbb{R}^N$.

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

(ii) Let $0 < a_1 < a_2$. There exist two functions u_1 and u_2 such that

$$M_{a_1,\mu} = E_\mu(u_1), \quad |u_1|_2^2 = a_1^2,$$

and

$$M_{a_2,\mu} = E_\mu(u_2), \quad |u_2|_2^2 = a_2^2.$$

Then, we use lemma 3.3 to get

$$M_{a_2,\mu} < E_\mu(u_1) = M_{a_1,\mu},$$

which implies that $M_{a,\mu}$ is strictly decreasing with respect to a. □

In the following, we study the strong instability of standing wave solution $e^{-\mathrm{i}\lambda t}u$ to (1.1), where u is a radial minimizer for $M_{a,\mu}$ obtained in theorem 1.5. Our ideas are as follows. First, we find the third kind of variational characterization for $M_{a,\mu}$. Define

$$\widetilde{E}_\mu(u) := E_\mu(u) - \frac{2s}{N(q-2)} P_\mu(u) = \left(\frac{1}{2} - \frac{2s}{N(q-2)}\right) |(-\Delta)^{\frac{s}{2}} u|_2^2 + \frac{1}{p}\left(\frac{p-2}{q-2} - 1\right)|u|_p^p,$$

and

$$\widetilde{M}_{a,\mu} := \inf_{\widetilde{V}_{a,\mu}} \widetilde{E}_\mu, \quad \widetilde{V}_{a,\mu} := \{u \in S_a : P_\mu(u) \le 0\}. \tag{3.11}$$

We will show $\widetilde{M}_{a,\mu} = M_{a,\mu}$. Second, we give the blow-up criterion (see proposition 3.7) by introducing two invariant manifolds, for which the proof is based on the localized virial action $M_\varphi(\psi(t,\cdot))$ and the virial estimate for $M_\varphi(\psi(t,\cdot))$ (see lemma 3.6). Third, if u is a radial minimizer for $M_{a,\mu}$, letting $\psi_0^\tau(x) = e^{\frac{N}{2}\tau} u(e^\tau x)$ with $\tau > 0$, we derive the strong instability of normalized ground states to (1.1) by the blow-up criterion. Hence, we conclude the proof of theorem 1.5(iii).

First, we prove the third kind of variational characterization of $M_{a,\mu}$. To this aim, we give some notations. For $u \in S_a$ and $\tau \in \mathbb{R}$, we define

$$(\tau \star u)(x) := e^{\frac{N}{2}\tau} u(e^\tau x), \quad \text{for a.e. } x \in \mathbb{R}^N,$$

then $\tau \star u \in S_a$. Moreover, we introduce the fibre map

$$\Psi(\tau) := E_\mu(\tau \star u) = \frac{e^{2s\tau}}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 - \frac{e^{N(\frac{p}{2}-1)\tau}}{p} \int_{\mathbb{R}^N} |u|^p - \mu \frac{e^{N(\frac{q}{2}-1)\tau}}{q} \int_{\mathbb{R}^N} |u|^q. \tag{3.12}$$

Lemma 3.4. *For any $u \in S_a$, there exists a unique constant $\tau_0 \in \mathbb{R}$ such that*

$$E_\mu(\tau_0 \star u) = \max_{\tau \in \mathbb{R}} E_\mu(\tau \star u).$$

Moreover,

(i) It holds that $P_\mu(\tau_0 \star u) = 0$. Furthermore, if $\tau < \tau_0$ (respectively $\tau > \tau_0$), then $P_\mu(\tau \star u) > 0$ (respectively $P_\mu(\tau \star u) < 0$).

(ii) $P_\mu(u) = 0$ *(respectively* $P_\mu(u) < 0$*) if and only if* $\tau_0 = 0$ *(respectively* $\tau_0 < 0$*).*

Proof. By straightforward calculation, one has $\Psi'(\tau) = sP_\mu(\tau \star u)$. In addition, we also see that

$$\Psi'(\tau) = e^{2s\tau}\Big(s|(-\Delta)^{\frac{s}{2}}u|_2^2 - \frac{N(p-2)}{2p}e^{\left(N(\frac{p}{2}-1)-2s\right)\tau}|u|_p^p \\ - \mu\frac{N(q-2)}{2q}e^{\left(N(\frac{q}{2}-1)-2s\right)\tau}|u|_q^q\Big).$$

Since $2 + 4s/N < q < p$ and $\mu > 0$, $\Psi'(\tau)$ is strictly decreasing with respect to τ. Consequently, there exists a unique $\tau_0 \in \mathbb{R}$ such that $\Psi'(\tau_0) = 0$. Other desired results follow directly. □

LEMMA 3.5. *Let* $M_{a,\mu}$ *and* $\widetilde{M}_{a,\mu}$ *be defined by (3.1) and (3.11), respectively, then*

$$\widetilde{M}_{a,\mu} = M_{a,\mu}.$$

Proof. For any $u \in S_a$ with $P_\mu(u) < 0$, by lemma 3.4, there exists a $\tau_0 < 0$ such that $P_\mu(\tau_0 \star u) = 0$, i.e., $\tau_0 \star u \in V_{a,\mu}$ defined in (3.1). Furthermore, it holds that

$$\begin{aligned}\widetilde{E}_\mu(\tau_0 \star u) &= \left(\frac{1}{2} - \frac{2s}{N(q-2)}\right)|(-\Delta)^{\frac{s}{2}}(\tau_0 \star u)|_2^2 + \frac{1}{p}\left(\frac{p-2}{q-2} - 1\right)|\tau_0 \star u|_p^p \\ &= \left(\frac{1}{2} - \frac{2s}{N(q-2)}\right)e^{2s\tau_0}|(-\Delta)^{\frac{s}{2}}u|_2^2 + \frac{1}{p}\left(\frac{p-2}{q-2} - 1\right)e^{N\tau_0(\frac{p}{2}-1)}|u|_p^p \\ &< \left(\frac{1}{2} - \frac{2s}{N(q-2)}\right)|(-\Delta)^{\frac{s}{2}}u|_2^2 + \frac{1}{p}\left(\frac{p-2}{q-2} - 1\right)|u|_p^p \\ &= \widetilde{E}_\mu(u).\end{aligned}$$

Hence, $\widetilde{M}_{a,\mu} = \inf_{\widetilde{V}_{a,\mu}} \widetilde{E}_\mu = \inf_{V_{a,\mu}} \widetilde{E}_\mu = M_{a,\mu}$. □

Let $1/2 \le s < 1$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\nabla\varphi \in W^{3,\infty}(\mathbb{R}^N)$. Assume $\psi \in C([0,T), H^s)$ is a solution to (1.1). The localized virial action of ψ is defined by

$$M_\varphi(\psi(t,\cdot)) := 2\int_{\mathbb{R}^N} \nabla\varphi(x) \cdot \Re\left(\bar{\psi}(t,x)\nabla\psi(t,x)\right) \mathrm{d}x. \tag{3.13}$$

It follows from lemma A.5 that $M_\varphi(\psi(t,\cdot))$ is well-defined. Indeed, by lemma A.5,

$$|M_\varphi(\psi(t,\cdot))| \le C(N, |\nabla\varphi|_{W^{1,\infty}})\,\|\psi(t,\cdot)\|_{H^{1/2}}^2 \le C(N, |\nabla\varphi|_{W^{1,\infty}})\,\|\psi(t,\cdot)\|_{H^s}^2 < \infty.$$

Now let $\varphi\colon \mathbb{R}^N \to \mathbb{R}$ be as above. We assume in addition that φ is radially symmetric and satisfies

$$\varphi(r) := \begin{cases} r^2, & \text{if } r \le 1, \\ const, & \text{if } r \ge 10. \end{cases} \quad \text{and } \varphi''(r) \le 2 \text{ for } r \ge 0.$$

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

Here the precise value of the constant is not important. For $R>0$ given, we define the rescaled function $\varphi_R : \mathbb{R}^N \to \mathbb{R}$ by

$$\varphi_R(x) = \varphi_R(r) := R^2\varphi(r/R), \quad r = |x|.$$

Then, we have the following virial estimate:

LEMMA 3.6 ([13, lemma 4.3], H^s radial virial estimate). *Let $N \geq 2$, $\frac{N}{2N-1} \leq s < 1$ and $2 < q < p < 2_s^*$, φ_R be as above, and $\psi \in C([0,T), H^s)$ be a radial solution to (1.1). Then, for any $t \in [0,T)$,*

$$\begin{aligned}\frac{d}{dt}M_{\varphi_R}(\psi(t,\cdot)) &\leq 8s\,\|\psi(t,\cdot)\|_{\dot H^s}^2 - \frac{4N\mu(q-2)}{q}|\psi(t,\cdot)|_q^q - \frac{4N(p-2)}{p}|\psi(t,\cdot)|_p^p \\ &\quad + O\Big(R^{-2s} + R^{-\frac{(q-2)(N-1)}{2}+\varepsilon_1 s}\,\|\psi(t,\cdot)\|_{\dot H^s}^{\frac{q-2}{2s}+\varepsilon_1} \\ &\quad + R^{-\frac{(p-2)(N-1)}{2}+\varepsilon_2 s}\,\|\psi(t,\cdot)\|_{\dot H^s}^{\frac{p-2}{2s}+\varepsilon_2}\Big) \\ &= 4N(p-2)E_\mu(\psi(t,\cdot)) + (8s - 2N(p-2))\,\|\psi(t,\cdot)\|_{\dot H^s}^2 \\ &\quad + \frac{4N(p-q)\mu}{q}|\psi(t,\cdot)|_q^q \\ &\quad + O\Big(R^{-2s} + R^{-\frac{(q-2)(N-1)}{2}+\varepsilon_1 s}\,\|\psi(t,\cdot)\|_{\dot H^s}^{\frac{q-2}{2s}+\varepsilon_1} \\ &\quad + R^{-\frac{(p-2)(N-1)}{2}+\varepsilon_2 s}\,\|\psi(t,\cdot)\|_{\dot H^s}^{\frac{p-2}{2s}+\varepsilon_2}\Big),\end{aligned}$$

for any $0 < \varepsilon_1 < (2N-1)(q-2)/2s, 0 < \varepsilon_2 < (2N-1)(p-2)/2s$, and where $\|\psi(t,\cdot)\|_{\dot H^s} := |(-\Delta)^{\frac{s}{2}}\psi(t,\cdot)|_2$. Here the implicit constant depends only on $|\psi_0|_2$, N, ε_1, ε_2, s, q, and p.

With the preparation above, we introduce the following two invariant manifolds:

$$\begin{aligned}\mathcal{A}_{a,\mu} &:= \{u \in S_{a,r} : P_\mu(u) > 0, E_\mu(u) < M_{a,\mu}\}, \\ \mathcal{B}_{a,\mu} &:= \{u \in S_{a,r} : P_\mu(u) < 0, E_\mu(u) < M_{a,\mu}\},\end{aligned}$$

where $S_{a,r} = \{u \in S_a : u(x) = u(|x|)\}$.

The following proposition tells us the global existence (respectively blow-up behaviour) of the solution to (1.1) if the initial data belong to $\mathcal{A}_{a,\mu}$ (respectively $\mathcal{B}_{a,\mu}$):

PROPOSITION 3.7. *Under the assumptions of theorem 1.5(iii). Then $\mathcal{A}_{a,\mu}$ and $\mathcal{B}_{a,\mu}$ are two invariant manifolds of (1.1). More precisely,*

(i) if the initial value $\psi_0 \in \mathcal{A}_{a,\mu}$, then the solution $\psi(t,\cdot)$ to (1.1) always stays in $\mathcal{A}_{a,\mu}$ and exists globally over time.

(ii) if the initial value $\psi_0 \in \mathcal{B}_{a,\mu}$, then the solution $\psi(t,\cdot)$ of (1.1) always stays in $\mathcal{B}_{a,\mu}$ but blows up in finite time.

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

Proof. First, we claim that $\mathcal{A}_{a,\mu} \neq \emptyset$ and $\mathcal{B}_{a,\mu} \neq \emptyset$. As a matter of fact, for $u \in S_{a,r}$, recall $(\tau \star u)(x) = e^{\frac{N}{2}\tau} u(e^{\tau}x)$. On the one hand, the following statements hold: (i) $\tau \star u \in S_{a,r}$ for every $\tau \in \mathbb{R}$; (ii) $E_\mu(\tau \star u) \to 0$ as $\tau \to -\infty$ by (2.3); and (iii) $P_\mu(\tau \star u) > 0$ for τ sufficiently negative by lemma 3.4. On the other hand, by lemma 3.1 and (3.10), we know $M_{a,\mu} \geq \delta_1 > 0$. Therefore, $\mathcal{A}_{a,\mu} \neq \emptyset$. Similarly, one has $E_\mu(\tau \star u) \to -\infty$ and $P_\mu(\tau \star u) \to -\infty$ as $\tau \to +\infty$. Thus, $\mathcal{B}_{a,\mu} \neq \emptyset$.

Second, we prove that $\mathcal{A}_{a,\mu}$ and $\mathcal{B}_{a,\mu}$ are two invariant manifolds of (1.1). Let $\psi_0 \in \mathcal{A}_{a,\mu}$, and by lemma A.3, there exists a unique solution $\psi \in C([0,T^*), H^s)$ to (1.1) with initial data ψ_0. Moreover, we have

$$|\psi(t,\cdot)|_2^2 = |\psi_0|_2^2 = a^2, \quad E_\mu(\psi(t,\cdot)) = E_\mu(\psi_0) < M_{a,\mu}$$

for any $t \in (0,T^*)$. If there exists some $t_0 \in [0,T^*)$ such that $P_\mu(\psi(t_0,\cdot)) = 0$, then $E_\mu(\psi(t_0,\cdot)) \geq M_{a,\mu}$, which is a contradiction. Therefore, we deduce from the continuity with respect to t of $\psi(t,\cdot)$ that $P_\mu(\psi(t,\cdot)) > 0$ for any $t \in [0,T^*)$. As a result, $\psi(t,\cdot)$ stays in $A_{a,\mu}$ for any $t \in [0,T^*)$. Similarly, $\mathcal{B}_{a,\mu}$ is invariant under the flow of (1.1).

(i) Due to $\psi(t,\cdot) \in \mathcal{A}_{a,\mu}$ for any $t \in [0,T^*)$, we deduce from the conservation of energy that

$$E_\mu(\psi_0) > \left(\frac{1}{2} - \frac{2s}{N(q-2)}\right) |(-\Delta)^{\frac{s}{2}}\psi(t,\cdot)|_2^2 + \frac{1}{p}\left(\frac{p-2}{q-2} - 1\right) |\psi(t,\cdot)|_p^p,$$

which together with lemma A.3 implies that the solution $\psi(t,\cdot)$ of (1.1) exists globally.

(ii) If $\psi_0 \in \mathcal{B}_{a,\mu}$, then $P_\mu(\psi(t,\cdot)) < 0$ for any $t \in [0,T^*)$. By lemma 3.5, we know

$$\begin{aligned} M_{a,\mu} &= \widetilde{M}_{a,\mu} \leq \widetilde{E}_\mu(\psi(t,\cdot)) \\ &= E_\mu(\psi(t,\cdot)) - \frac{2s}{N(q-2)} P_\mu(\psi(t,\cdot)) = E_\mu(\psi_0) - \frac{2s}{N(q-2)} P_\mu(\psi(t,\cdot)), \end{aligned}$$

which implies

$$P_\mu(\psi(t,\cdot)) \leq \frac{N(q-2)}{2s}\left(E_\mu(\psi_0) - M_{a,\mu}\right) < 0, \quad \forall t \in [0,T^*). \tag{3.14}$$

The proof of blow-up behaviour will be divided into three steps as follows:

Step 1: We prove that there exists $C_1 > 0$ such that

$$|(-\Delta)^{\frac{s}{2}}\psi(t,\cdot)|_2 \geq C_1 \tag{3.15}$$

for every $t \in [0,T^*)$. Indeed, if not, then there exists $\{t_k\} \subseteq [0,T^*)$ such that $|(-\Delta)^{\frac{s}{2}}\psi(t_k,\cdot)|_2 \to 0$. However, we deduce from mass conservation and the sharp Gagliardo-Nirenberg inequality that $|\psi(t_k,\cdot)|_\alpha^\alpha = o(1)$ as $k \to \infty$, where $\alpha = p$ or q.

Therefore, we have

$$P_\mu(\psi(t_k,\cdot)) = |\,(-\Delta)^{\frac{s}{2}}\,\psi(t_k,\cdot)|_2^2 - \frac{N(p-2)}{2ps}|\psi(t_k,\cdot)|_p^p - \mu\frac{N(q-2)}{2qs}|\psi(t_k,\cdot)|_q^q \to 0$$

as $k \to \infty$, which contradicts to (3.14).

Step 2: We claim that there exists $C_2 > 0$ such that

$$\frac{d}{dt}M_{\varphi_R}(\psi(t,\cdot)) \le -C_2|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^2, \tag{3.16}$$

where $M_{\varphi_R}(\psi(t,\cdot))$ is defined by (3.13).

Observe that $\psi(t,\cdot)$ is radial for any $t \in [0,T^*)$, since the initial datum ψ_0 is radial. Therefore, we apply lemma 3.6 to have

$$\begin{aligned}\frac{d}{dt}M_{\varphi_R}(\psi(t,\cdot)) &\le 8s|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^2 - \frac{4N\mu(q-2)}{q}|\psi(t,\cdot)|_q^q - \frac{4N(p-2)}{p}|\psi(t,\cdot)|_p^p \\ &\quad + O\Big(R^{-2s} + R^{-\frac{(q-2)(N-1)}{2}+\varepsilon_1 s}|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^{\frac{q-2}{2s}+\varepsilon_1}\Big) \\ &\quad + O\left(R^{-\frac{(p-2)(N-1)}{2}+\varepsilon_2 s}|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^{\frac{p-2}{2s}+\varepsilon_2}\right)\end{aligned}$$

for all $t \in [0,T^*)$ and $R > 1$. Thanks to the assumption $q < p < 2+4s$, we can apply Young's inequality to obtain for any $\eta > 0$,

$$\begin{aligned}&R^{-\frac{(q-2)(N-1)}{2}+\varepsilon_1 s}|(-\Delta)^{\frac{s}{2}}\psi(t,\cdot)|_2^{\frac{q-2}{2s}+\varepsilon_1} \\ &\quad \le \eta|(-\Delta)^{\frac{s}{2}}\psi(t,\cdot)|_2^2 + \eta^{-\frac{q-2+2\varepsilon_1 s}{2+4s-q-2\varepsilon_1 s}} R^{-\frac{2s[(q-2)(N-1)-2\varepsilon_1 s]}{2+4s-q-2\varepsilon_1 s}}, \\ &R^{-\frac{(p-2)(N-1)}{2}+\varepsilon_2 s}|(-\Delta)^{\frac{s}{2}}\psi(t,\cdot)|_2^{\frac{p-2}{2s}+\varepsilon_2} \\ &\quad \le \eta|(-\Delta)^{\frac{s}{2}}\psi(t,\cdot)|_2^2 + \eta^{-\frac{p-2+2\varepsilon_2 s}{2+4s-p-2\varepsilon_2 s}} R^{-\frac{2s[(p-2)(N-1)-2\varepsilon_2 s]}{2+4s-p-2\varepsilon_2 s}}.\end{aligned}$$

Thus, there exists a constant $C > 0$ such that

$$\begin{aligned}\frac{d}{dt}M_{\varphi_R}(\psi(t,\cdot)) &\le 8s|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^2 - \frac{4N\mu(q-2)}{q}|\psi(t,\cdot)|_q^q - \frac{4N(p-2)}{p}|\psi(t,\cdot)|_p^p \\ &\quad + C\eta|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^2 + I(\eta,R)\end{aligned}$$

for all $t \in [0,T^*)$, any $\eta > 0$, and any $R > 1$, where

$$\begin{aligned}I(\eta,R) := O\Big(&R^{-2s} + \eta^{-\frac{q-2+2\varepsilon_1 s}{2+4s-q-2\varepsilon_1 s}} R^{-\frac{2s[(q-2)(N-1)-2\varepsilon_1 s]}{2+4s-q-2\varepsilon_1 s}} \\ &+ \eta^{-\frac{p-2+2\varepsilon_2 s}{2+4s-p-2\varepsilon_2 s}} R^{-\frac{2s[(p-2)(N-1)-2\varepsilon_2 s]}{2+4s-p-2\varepsilon_2 s}}\Big).\end{aligned}$$

Since $2 + 4s/N < q < p < 2N/(N-2s)$ and $p < 2 + 4s$, we can choose $\varepsilon_1 > 0, \varepsilon_2 > 0$ sufficiently small such that

$$q - 2 + 2\varepsilon_1 s > 0, \quad 2 + 4s - q - 2\varepsilon_1 s > 0, \quad (q-2)(N-1) - 2\varepsilon_1 s > 0,$$
$$p - 2 + 2\varepsilon_2 s > 0, \quad 2 + 4s - p - 2\varepsilon_2 s > 0, \quad (p-2)(N-1) - 2\varepsilon_2 s > 0.$$

In addition, we fix $t \in [0, T^*)$ and denote

$$\kappa := \frac{4N(p-2)|E_\mu(\psi_0)| + 1}{N(p-2) - 4s}.$$

Since $p > 2 + 4s/N$, we know $\kappa > 0$. We consider two cases.

Case 1: $|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^2 \le \kappa$. Noting that

$$8s|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^2 - \frac{4N\mu(q-2)}{q}|\psi(t,\cdot)|_q^q - \frac{4N(p-2)}{p}|\psi(t,\cdot)|_p^p = 8sP_\mu(\psi(t,\cdot)),$$

and (3.14), we have

$$\frac{d}{dt}M_{\varphi_R}(\psi(t,\cdot)) \le 4N(q-2)\left(E_\mu(\psi_0) - M_{a,\mu}\right) + C\eta\kappa + I(\eta, R).$$

By choosing $\eta > 0$ small enough and $R > 1$ large enough depending on η, we can get

$$2N(q-2)\left(E_\mu(\psi_0) - M_{a,\mu}\right) + C\eta\kappa + I(\eta, R) < 0.$$

Thus, we obtain

$$\frac{d}{dt}M_{\varphi_R}(\psi(t,\cdot)) \le \frac{2N(q-2)\left(E_\mu(\psi_0) - M_{a,\mu}\right)}{\kappa}|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^2.$$

Case 2: $|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^2 > \kappa$. By lemma 3.6, we obtain

$$\begin{aligned}\frac{d}{dt}M_{\varphi_R}(\psi(t,\cdot)) &\le 4N(p-2)|E_\mu\left(\psi_0\right)| + 2\left(4s - N(p-2)\right)|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^2 \\ &\quad + C\eta|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^2 + I(\eta, R) \\ &\le -1 + \left(4s - N(p-2)\right)|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^2 \\ &\quad + C\eta|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^2 + I(\eta, R).\end{aligned}$$

Since $p > 2 + 4s/N$, we choose $\eta > 0$ small enough so that

$$N(p-2) - 4s - C\eta \ge \frac{N(p-2) - 4s}{2}.$$

We next choose $R > 1$ large enough depending on η so that

$$-1 + I(\eta, R) < 0.$$

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

We thus obtain

$$\frac{d}{dt}M_{\varphi_R}(\psi(t,\cdot)) \leq -\frac{N(p-2)-4s}{2}|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^2.$$

Combined with the two cases above, we prove our claim (3.16).

Step 3: We are now able to show that the solution $\psi(t,\cdot)$ blows up in a finite time. Assume by contradiction that $T^* = \infty$. It follows from (3.15) and (3.16) that $\frac{d}{dt}M_{\varphi_R}(\psi(t,\cdot)) \leq -C$ with some constant $C > 0$. As a consequence, $M_{\varphi_R}(\psi(t,\cdot)) < 0$ for all $t \geq t_1$ with some sufficiently large t_1. After integrating (3.16) on $[t_1, t]$, we obtain

$$M_{\varphi_R}(\psi(t,\cdot)) \leq -C_2\int_{t_1}^{t}|(-\Delta)^{\frac{s}{2}}\psi(\tau,\cdot)|_2^2 d\tau + M_{\varphi_R}(\psi(t_1)) \leq -C_2\int_{t_1}^{t}|(-\Delta)^{\frac{s}{2}}\psi(\tau,\cdot)|_2^2 d\tau \tag{3.17}$$

for all $t \geq t_1$. On the other hand, we use lemma A.5 and L^2-mass conservation to find that

$$\left|M_{\varphi_R}(\psi(t,\cdot))\right| \leq C(N,s,a,\|\nabla\varphi_R\|_{W^{1,\infty}})\left(|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^{\frac{1}{s}} + |\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^{\frac{1}{2s}}\right), \tag{3.18}$$

where we used the interpolation estimate

$$\left||\nabla|^{1/2}\psi(t,\cdot)\right|_2 \leq |\psi(t,\cdot)|_2^{1-\frac{1}{2s}}|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^{\frac{1}{2s}}$$

for $s > 1/2$. So, we deduce from (3.15) and (3.18) that

$$\left|M_{\varphi_R}(\psi(t,\cdot))\right| \leq C(N,s,a,\|\nabla\varphi_R\|_{W^{1,\infty}})|\,(-\Delta)^{\frac{s}{2}}\,\psi(t,\cdot)|_2^{\frac{1}{s}}.$$

This, together with (3.17), implies that

$$M_{\varphi_R}(\psi(t,\cdot)) \leq -C(N,s,a,\|\nabla\varphi_R\|_{W^{1,\infty}})\int_{t_1}^{t}\left|M_{\varphi_R}(\psi(\tau,\cdot))\right|^{2s}d\tau \quad \text{for} \quad t \geq t_1. \tag{3.19}$$

Let $z(t) := \int_{t_1}^{t}\left|M_{\varphi_R}(\psi(\tau,\cdot))\right|^{2s}d\tau$, noting that $M_{\varphi_R}(\psi(t,\cdot)) < 0$ for $t > t_1$, hence $z(t)$ is strictly increasing for $t > t_1$ and we can find a $t_2 > t_1$ such that $z(t_2) > 0$. Furthermore, by (3.19), we obtain

$$z'(t) \geq [C(N,s,a,\|\nabla\varphi_R\|_{W^{1,\infty}})]^{2s}z^{2s}.$$

Therefore, for $t > t_2$, it holds that $z'(t)z^{-2s} \geq [C(N,s,a,\|\nabla\varphi_R\|_{W^{1,\infty}})]^{2s}$. Since $s > \frac{1}{2}$, we obtain $(1-2s)z'(t)z^{-2s} \leq (1-2s)[C(N,s,a,\|\nabla\varphi_R\|_{W^{1,\infty}})]^{2s}$. After integration on $[t_2, t]$, one has

$$0 < z^{1-2s}(t) \leq z^{1-2s}(t_2) - (2s-1)[C(N,s,a,\|\nabla\varphi_R\|_{W^{1,\infty}})]^{2s}(t-t_2). \tag{3.20}$$

Note that the right-hand side of (3.20) goes to $-\infty$ as $t \to +\infty$, while the left-hand side is positive. Hence, it must hold that $T^* < +\infty$. □

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

Proof of theorem 1.5(iii). Let u be a radial minimizer for $M_{a,\mu}$. Then, we know $P_\mu(u) = 0$. Setting $\psi_0^\tau(x) = e^{\frac{N}{2}\tau} u(e^\tau x)$ with $\tau > 0$, by lemma 3.4, we obtain

$$E_\mu(\psi_0^\tau) = E_\mu(\tau \star u) < E_\mu(u) = M_{a,\mu}, \quad P_\mu(\psi_0^\tau) = P_\mu(\tau \star u) < 0$$

for any $\tau > 0$. Thus, $\psi_0^\tau \in \mathcal{B}_{a,\mu}$ for $\tau > 0$. In addition, let $\tau \to 0^+$, we have $\|\psi_0^\tau - u\|_{H^s} \to 0$. Therefore, we apply proposition 3.7 to get the strong instability of $e^{-i\lambda t}u$. □

Remark 3.8. After completing this article, we learned that Feng and Zhu [16] considered the instability of ground state solutions for the fixed frequency λ. The relationship between the two types of ground state solutions is still a delicate but important open problem.

4. The combined nonlinearities and defocusing case: $2 < q \leq \bar{p} < p < 2_s^*$ and $\mu < 0$

Via lemmas 6.14 and 6.15 and theorem 6.17 in [33], there exists $\hat{u}$ such that the following variational characterization holds:

$$E_\mu(\hat{u}) = M_{a,\mu}^r := \inf_{u \in V_{a,\mu}^r} E_\mu(u), \quad V_{a,\mu}^r := \{u \in S_a : u(x) = u(|x|), P_\mu(u) = 0\}.$$

Once we obtain $\hat{M}_{a,\mu}^r = M_{a,\mu}^r$ (see lemma 4.1), similar to the proof of theorem 1.5(iii), we deduce the strong instability of the standing wave $e^{-i\hat{\lambda}t}\hat{u}$. First, we recall (3.12), and note that lemma 3.4 still holds in this case.

Lemma 4.1. *Let $\hat{M}_{a,\mu}^r$ be defined by (1.8), then*

$$\hat{M}_{a,\mu}^r = \inf_{V_{a,\mu}^r} \hat{E}_\mu = M_{a,\mu}^r.$$

Proof. The proof is similar to the argument of lemma 3.5, so we omit it. □

Similarly, we define the two manifolds by

$$\hat{\mathcal{A}}_{a,\mu} = \left\{u \in S_{a,r} : P_\mu(u) > 0, E_\mu(u) < M_{a,\mu}^r\right\},$$
$$\hat{\mathcal{B}}_{a,\mu} = \left\{u \in S_{a,r} : P_\mu(u) < 0, E_\mu(u) < M_{a,\mu}^r\right\}.$$

Then, we have the following result:

Proposition 4.2. *Under the assumptions of theorem 1.7(ii). Then $\hat{\mathcal{A}}_{a,\mu}$ and $\hat{\mathcal{B}}_{a,\mu}$ are two invariant manifolds of (1.1). More precisely,*

(i) if the initial value $\psi_0 \in \hat{\mathcal{A}}_{a,\mu}$, then the solution $\psi(t,\cdot)$ of (1.1) always stays in $\hat{\mathcal{A}}_{a,\mu}$ and exists globally over time.

(ii) if the initial value $\psi_0 \in \hat{\mathcal{B}}_{a,\mu}$, then the solution $\psi(t,\cdot)$ of (1.1) always stays in $\hat{\mathcal{B}}_{a,\mu}$ but will blow up in finite time.

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

Proof. The proof is similar to that of proposition 3.7. □

Proof of theorem 1.7. The conclusion (i) is established by [33, theorems 1.9, 1.11]. The proof of item (ii) is similar to the argument of theorem 1.5(iii). □

Acknowledgements

We thank the anonymous referee for their helpful comments. No conflict of interest exists in the submission of this manuscript, and all authors approve the manuscript for publication. Z. Li is funded by the Natural Science Foundation of Hebei Province (No. A2022205007), and by the Science and Technology Project of Hebei Education Department (No. QN2022047), and by Hebei Province Yanzhao Golden Stage Talent Gathering Program Key Talent Project (Study Abroad Returning Platform) (No. B2024018). H. Luo is supported by the National Natural Science Foundation of China (Nos. 12471105, 12171143 and 12171144). Z. Zhang is supported by the National Key R&D Program of China (No. 2022YFA1005601) and the National Natural Science Foundation of China (No. 12031015).

References

[1] F. J. Almgren Jr and E. H. Lieb. Symmetric decreasing rearrangement is sometimes continuous. *J. Amer. Math. Soc.* **2** (1989), 683–773.

[2] D. Applebaum. *Lévy Processes and Stochastic Calculus*, 2nd ed. Cambridge Studies in Advanced Mathematics. Vol.116, (Cambridge University Press, Cambridge, 2009).

[3] T. Bartsch and S. de Valeriola. Normalized solutions of nonlinear Schrödinger equations. *Arch. Math. (Basel).* **100** (2013), 75–83.

[4] H. Berestycki and T. Cazenave. Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires. (French) [Instability of stationary states in nonlinear Schrödinger and Klein-Gordon equations]. *C. R. Acad. Sci. Paris Sér. I Math.* **293** (1981), 489–492.

[5] D. Bonheure, J. -B. Casteras, T. Gou, and L. Jeanjean. Strong instability of ground states to a fourth order Schrödinger equation. *Int. Math. Res. Not. IMRN.* **2019** (2019), 5299–5315.

[6] T. Boulenger, D. Himmelsbach and E. Lenzmann. Blowup for fractional NLS. *J. Funct. Anal.* **271** (2016), 2569–2603.

[7] T. Cazenave and P. -L. Lions. Orbital stability of standing waves for some nonlinear Schrödinger equations. *Comm. Math. Phys.* **85** (1982), 549–561.

[8] X. Chang and Z. -Q. Wang. Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity. *Nonlinearity* **26** (2013), 479–494.

[9] W. Chen, C. Li and Y. Li. A direct method of moving planes for the fractional Laplacian. *Adv. Math.* **308** (2017), 404–437.

[10] E. Colorado and A. Ortega. Nonlinear fractional Schrödinger equations coupled by power–type nonlinearities. *Adv. Differ. Equ.* **28** (2023), 113–142.

[11] E. Di Nezza, G. Palatucci and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012), 521–573.

[12] V. Dinh. On instability of standing waves for the mass-supercritical fractional nonlinear Schrödinger equation. *Z. Angew. Math. Phys.* **70** (2019), 17.

[13] V. Dinh and B. Feng. On fractional nonlinear Schrödinger equation with combined power-type nonlinearities. *Discrete Contin. Dyn. Syst.* **39** (2019), 4565–4612.

[14] P. Felmer, A. Quaas and J. Tan. Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian. *Proc. Roy. Soc. Edinb.* A **142** (2012), 1237–1262.

[15] B. Feng, J. Ren and Q. Wang. Existence and instability of normalized standing waves for the fractional Schrödinger equations in the L^2-supercritical case. *J. Math. Phys.* **61** (2020), 19.

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

[16] B. Feng and S. Zhu. Stability and instability of standing waves for the fractional nonlinear Schrödinger equations. *J. Differ. Equ.* **292** (2021), 287–324.

[17] R. L. Frank, E. Lenzmann and L. Silvestre. Uniqueness of radial solutions for the fractional Laplacian. *Comm. Pure Appl. Math.* **69** (2016), 1671–1726.

[18] T. Gou and Z. Zhang. Normalized solutions to the Chern-Simons-Schrödinger system. *J. Funct. Anal.* **280** (2021), 65

[19] B. Guo and D. Huang. Existence and stability of standing waves for nonlinear fractional Schrödinger equations. *J. Math. Phys.* **53** (2012), 15.

[20] H. Hajaiej and L. Song. A general and unified method to prove the uniqueness of ground state solutions and the existence/non-existence, and multiplicity of normalized solutions with applications to various NLS. arXiv:2208.11862v3, (2023).

[21] J. Hirata and K. Tanaka. scalar field equations with L^2 constraint: mountain pass and symmetric mountain pass approaches. *Adv. Nonlinear Stud.* **19** (2019), 263–290.

[22] L. Jeanjean. Existence of solutions with prescribed norm for semilinear elliptic equations. *Nonlinear Anal.* **28** (1997), 1633–1659.

[23] L. Jeanjean and S. Lu. Nonradial normalized solutions for nonlinear scalar field equations. *Nonlinearity* **32** (2019), 4942–4966.

[24] K. Kirkpatrick, E. Lenzmann and G. Staffilani. On the continuum limit for discrete NLS with long-range lattice interactions. *Comm. Math. Phys.* **317** (2013), 563–591.

[25] N. Laskin. Fractional quantum mechanics and Lévy path integrals. *Phys. Lett.* A **268** (2000), 298–305.

[26] S. Le Coz. A note on Berestycki-Cazenave's classical instability result for nonlinear Schrödinger equations. *Adv. Nonlinear Stud.* **8** (2008), 455–463.

[27] Z. Li, Q. Zhang and Z. Zhang. Stable standing waves of nonlinear fractional Schrödinger equations. *Commun. Pure Appl. Anal.* **21** (2022), 4113–4145.

[28] Z. Li, Q. Zhang, and Z. Zhang. Standing waves of fractional Schrödinger equations with potentials and general nonlinearities. *Anal. Theory Appl.* **39** (2023), 357–377.

[29] P. -L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. Ann. Inst. H. Poincaré Anal. Non LinéAire.* **1** (1984), 109–145.

[30] P. -L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. II. *Ann. Inst. H. Poincaré Anal. Non LinéAire.* **1** (1984), 223–283.

[31] Z. Liu, H. Luo and Z. Zhang. Dancer-Fuccčik spectrum for fractional Schrödinger operators with a steep potential well on $\mathbb{R}^N$. *Nonlinear Anal.* **189** (2019), Art. 111565, 26.

[32] H. Luo and D. Wu. Normalized ground states for general pseudo-relativistic Schrödinger equations. *Appl. Anal.* **101** (2022), 3410–3431.

[33] H. Luo and Z. Zhang. Normalized solutions to the fractional Schrödinger equations with combined nonlinearities. *Calc. Var. Partial Differ. Equ.* **59** (2020), 35.

[34] H. Luo and Z. Zhang. Existence and stability of normalized solutions to the mixed dispersion nonlinear Schrödinger equations. *Electron. Res. Arch.* **30** (2022), 2871–2898.

[35] X. Luo, T. Yang and X. Yang. Multiplicity and asymptotics of standing waves for the energy critical half-wave. arXiv:2102.09702v1 (2021).

[36] C. Peng and Q. Shi. Stability of standing wave for the fractional nonlinear Schrödinger equation. *J. Math. Phys.* **59** (2018), 11.

[37] R. Servadei and E. Valdinoci. The Brezis-Nirenberg result for the fractional Laplacian. *Trans. Amer. Math. Soc.* **367** (2015), 67–102.

[38] M. Shibata. Stable standing waves of nonlinear Schrödinger equations with a general nonlinear term. *Manuscripta Math.* **143** (2014), 221–237.

[39] N. Soave. Normalized ground states for the NLS equation with combined nonlinearities. *J. Differ. Equ.* **269** (2020), 6941–6987.

[40] N. Soave. Normalized ground states for the NLS equation with combined nonlinearities: the Sobolev critical case. *J. Funct. Anal.* **279** (2020), 43.

[41] S. Yan, J. Yang and X. Yu. Equations involving fractional Laplacian operator: compactness and application. *J. Funct. Anal.* **269** (2015), 47–79.

[42] Z. Zhang. *Variational, Topological, and Partial Order Methods With Their Applications* (Springer, Heidelberg, 2013).

[43] M. Zhen and B. Zhang. Normalized ground states for the critical fractional NLS equation with a perturbation. *Rev. Mat. Complut.* **35** (2022), 89–132.

[44] S. Zhu. Existence of stable standing waves for the fractional Schrödinger equations with combined nonlinearities. *J. Evol. Equ.* **17** (2017), 1003–1021.

Appendix A.

LEMMA A.1. [17] *Let $2 \leq \alpha < 2_s^*$, then there exists a constant $C(s,N,\alpha)$ such that for any $u \in H^s(\mathbb{R}^N)$, the following inequality holds:*

$$\int_{\mathbb{R}^N} |u(x)|^\alpha \mathrm{d}x \leq C(s,N,\alpha)\Big(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(x)|^2 \mathrm{d}x\Big)^{\frac{N(\alpha-2)}{4s}} \times \Big(\int_{\mathbb{R}^N} |u(x)|^2 \mathrm{d}x\Big)^{\frac{\alpha}{2}-\frac{N(\alpha-2)}{4s}}.$$

LEMMA A.2. [12, lemma 2.2] *Let $N \geq 1$ and $0 < s < 1$. Let $\{v_n\}_{n\geq 1}$ be a bounded sequence in $H^s(\mathbb{R}^N)$. Then, there exist a subsequence of $\{v_n\}_{n\in\mathbb{N}}$, a family $\{x_n^j\}_{n\in\mathbb{N}}$ of sequences in $\mathbb{R}^N$ and a sequence $\{V^j\}_{j\in\mathbb{N}}$ of $H^s(\mathbb{R}^N)$ functions such that for every $k \neq j$, $|x_n^k - x_n^j| \to +\infty$ as $n \to +\infty$. Furthermore, for every $l \geq 1$ and every $x \in \mathbb{R}^N$, $v_n(x)$ can be decomposed into*

$$v_n(x) = \sum_{j=1}^{l} V^j(x - x_n^j) + v_n^l(x),$$

with $\overline{\lim}_{n\to+\infty} |v_n^l|_\gamma \to 0$ as $l \to \infty$ for every $\gamma \in \big(2, \frac{2N}{(N-2s)^+}\big)$. In addition, for every $l \geq 1$, the following expansions hold true as $n \to +\infty$:

$$|v_n|_2^2 = \sum_{j=1}^{l} |V^j|_2^2 + |v_n^l|_2^2 + o_n(1), \qquad |v_n|_\gamma^\gamma = \sum_{j=1}^{l} |V^j|_\gamma^\gamma + |v_n^l|_\gamma^\gamma + o_n(1),$$

$$|(-\Delta)^{\frac{s}{2}} v_n|_2^2 = \sum_{j=1}^{l} |(-\Delta)^{\frac{s}{2}} V^j|_2^2 + |(-\Delta)^{\frac{s}{2}} v_n^l|_2^2 + o_n(1).$$

LEMMA A.3. [13, proposition 3.3] Radial H^s local well-posedness *Assume $N \geq 2$, $\frac{N}{2N-1} \leq s < 1$, and $2 < q < p < \frac{2N}{N-2s}$. Let*

$$q_1 = \frac{4sq}{(q-2)(N-2s)}, \quad q_2 = \frac{Nq}{N+(q-2)s}, \quad p_1 = \frac{4sp}{(p-2)(N-2s)}, \quad p_2 = \frac{Np}{N+(p-2)s}.$$

Then, for any $\psi_0 \in H^s$ radial, there exist $T \in (0,+\infty]$ and a unique solution to (1.1) satisfying

$$\psi \in C([0,T), H^s) \cap L^{q_1}([0,T), W^{s,q_2}) \cap L^{p_1}([0,T), W^{s,p_2}).$$

Moreover, the following properties hold:

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press

(i) $\psi \in L^a_{loc}\left([0,T), W^{s,b}\right)$ *for any fractional admissible pair (a, b).*

(ii) *If* $T < +\infty$*, then* $|(-\Delta)^{\frac{s}{2}}\psi(t,\cdot)|_2^2 \to \infty$ *as* $t \uparrow T$.

(iii) *The laws of conservation of mass and energy hold, i.e.,* $|\psi(t,\cdot)|_2^2 = |\psi_0|_2^2$ *and* $E_\mu(\psi(t,\cdot)) = E_\mu(\psi_0)$ *for all* $t \in [0,T)$.

REMARK A.4. In fact, the $\psi(t,\cdot)$ is also radial for every $t \in [0,T)$.

LEMMA A.5. [6, lemma A.1] *Let* $N \geq 1$ *and* $\varphi\colon \mathbb{R}^N \to \mathbb{R}$ *be such that* $\nabla\varphi \in W^{1,\infty}(\mathbb{R}^N)$. *Then, for all* $u \in H^{1/2}(\mathbb{R}^N)$*, it holds that*

$$\Big|\int_{\mathbb{R}^N} \bar{u}(x)\nabla\varphi(x)\cdot\nabla u(x)\Big| \leq C\Big(\Big||\nabla|^{1/2}u\Big|_2^2 + |u|_2\Big||\nabla|^{1/2}u\Big|_2\Big),$$

for some $C > 0$ *depending only on* $\|\nabla\varphi\|_{W^{1,\infty}}$ *and N.*

LEMMA A.6. *Let* $0 < \gamma < \beta$, $A, B > 0$ *and*

$$g(t) = At^\beta - t^\gamma + B, \quad t \in [0,\infty).$$

Then $g(t) \geq 0$ *for any* $t \in [0,\infty)$ *if and only if* $A \geq \frac{\gamma}{\beta}\left(\frac{\beta-\gamma}{\beta B}\right)^{\frac{\beta-\gamma}{\gamma}}$.

Proof. Since $g'(t) = t^{\gamma-1}\left(\beta A t^{\beta-\gamma} - \gamma\right)$, the minimum of $g(t)$ is attained at $t_0 = \left(\frac{\gamma}{A\beta}\right)^{\frac{1}{\beta-\gamma}}$. Therefore, it is equivalent to $g(t_0) \geq 0$, namely, $A \geq \frac{\gamma}{\beta}\left(\frac{\beta-\gamma}{\beta B}\right)^{\frac{\beta-\gamma}{\gamma}}$. □

https://doi.org/10.1017/prm.2024.94 Published online by Cambridge University Press