Canad. J. Math. Vol. **44 (6)** 1992, pp. 1272–1288.

COMBINATORIAL LOCAL PLANARITY AND THE WIDTH OF GRAPH EMBEDDINGS

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ABSTRACT. Let G be a graph embedded in a closed surface. The embedding is "locally planar" if for each face, a "large" neighbourhood of this face is simply connected. This notion is formalized, following [RV], by introducing the *width* $\rho(\psi)$ of the embedding ψ. It is shown that embeddings with $\rho(\psi) \geq 3$ behave very much like the embeddings of planar graphs in the 2-sphere. Another notion, "combinatorial local planarity", is introduced. The criterion is independent of embeddings of the graph, but it guarantees that a given cycle in a graph G must be contractible in any minimal genus embedding of G (either orientable, or non-orientable). It generalizes the width introduced before. As application, short proofs of some important recently discovered results about embeddings of graphs are given and generalized or improved. Uniqueness and switching equivalence of graphs embedded in a fixed surface are also considered.

1. **Introduction.** Graphs in this paper are finite, undirected and simple; loops and multiple edges are not allowed. A *surface* is a compact connected 2-manifold without boundary. An *embedding* of a graph G into a surface Σ is a 1-1 continuous mapping $\psi: G \longrightarrow \Sigma$ where G is viewed as endowed with the usual topology as a 1-dimensional simplicial complex. The connected components of $\Sigma \setminus \psi(G)$ are *faces* of ψ, or shortly ψ-*faces*. The Euler-Poincaré formula bounds the number F of ψ-faces:

$$|V(G)| - |E(G)| + F \geq \chi(\Sigma) \tag{1.1}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ. If each ψ-face is homeomorphic to an open disc, then in (1.1) equality holds. Such an embedding is said to be *cellular*, or a *2-cell* embedding. If f is a face of a cellular embedding ψ, its *boundary* is defined as the cyclic sequence of edges of G, as they appear on ∂f, modulo its inverse cyclic sequence. So, the boundary of a face bounded by edges a, b, c, respectively, may be represented by either of (a, b, c), (b, c, a), (a, c, b), *etc.* Two embeddings $\psi: G \longrightarrow \Sigma$ and $\psi': G \longrightarrow \Sigma'$ are (strictly) *equivalent* if there is a homeomorphism $h: \Sigma' \longrightarrow \Sigma$ such that $\psi = h\psi'$. It turns out that two cellular embeddings of a graph G are equivalent if and only if their faces have the same boundaries. See, for example, [HR] where also a combinatorial description of equivalence of non-cellular embeddings is presented. A graph G is *uniquely embeddable* in Σ if there is an embedding of G into Σ and any two such embeddings are equivalent.

It is well-known that 3-connected planar graphs are uniquely embeddable in the 2-sphere. This is a consequence of a more general theorem of Whitney [W] which states

Supported in part by the Ministry of Science and Technology of Slovenia.
Received by the editors January 30, 1991; revised March 30, 1992.
AMS subject classification: Primary: 05C10; secondary: 57M15.

that any two embeddings of a 2-connected graph G into the 2-sphere are 2-switching equivalent. In order to include graphs which are not 2-connected we will define the *switching equivalence* in a slightly weaker form than usual. See Section 3 for details.

The *genus* $g(G)$ of a graph G is the minimal genus of an orientable surface in which G has an embedding. The *non-orientable genus* $\tilde{g}(G)$ of G is the minimal genus (number of cross-caps) of a non-orientable surface in which G can be embedded. An embedding ψ of G into a surface Σ is a *minimal genus embedding* if Σ is either orientable with genus of Σ equal to $g(G)$, or non-orientable with genus $\tilde{g}(G)$. By adding a cross-cap to the orientable surface of genus $g(G)$ one sees immediately that

$$\tilde{g}(G) \leq 2g(G) + 1. \tag{1.2}$$

There are graphs for which the equality holds. Such graphs are called *orientably simple* [WB].

A *closed curve* on Σ is a continuous mapping $\gamma\colon S^1 \longrightarrow \Sigma$. If γ is 1-1, it is *simple*. Any closed walk W in a graph G embedded in Σ determines a closed curve in Σ (up to a change of the "parameter") and this curve is simple if and only if W is a cycle in G. A simple closed curve is also called a *circuit*. It is *essential* if it is non-contractible. A circuit γ is *bounding* if $\Sigma \setminus \gamma(S^1)$ is disconnected, and *non-bounding* otherwise. Contractible circuits are always bounding—one of the components of $\Sigma \setminus \gamma(S^1)$ is a disc. There is another classification of circuits: those having an open neighbourhood homeomorphic to a cylinder are *2-sided*, and others are *1-sided*. The latter have a tubular neighbourhood homeomorphic to the Möbius band. The 1-sided circuits are always non-bounding. The same classification will be used for the cycles of a graph embedded in Σ since they can be viewed as circuits on Σ. It is assumed that the reader is familiar with basic homotopy theory.

A set Γ of circuits on Σ is *homologically independent* if no subset of Γ bounds. If the circuits in Γ are pairwise disjoint this is equivalent to the requirement that $\Sigma \setminus \Gamma = \Sigma \setminus \bigcup\{\gamma(S^1) \mid \gamma \in \Gamma\}$ is connected.

Let H be a subgraph of G. A *(relative) H-component* is a subgraph of G which is either an edge $e \in E(G) \setminus E(H)$ (together with its end-points) which has both end-points in H, or it is a connected component of $G - V(H)$ together with all edges (and their endpoints) between this component and H. Each edge of an H-component R having an end-point in H is a *foot* of R. The vertices of $R \cap V(H)$ are the *vertices of attachment* of R. We will often use relative components of a graph H consisting of a single vertex $x \in V(G)$, or two vertices $x, y \in V(G)$ (not containing a possible edge between them). In such cases we use the name *$\{x\}$-component*, or *$\{x, y\}$-component*, respectively, for their relative components. Note that in a special case when x and y are adjacent, the edge xy gives rise to an $\{x, y\}$-component. If H has only one relative component and $V(H) \neq V(G)$, then H is said to be an *induced non-separating subgraph* of G. In relation to embeddings of 3-connected graphs, induced non-separating cycles play a special role. If such a cycle is bounding (on a surface) then, clearly, it bounds a face. In most important cases (*cf.* Section 3) also the converse is true: the boundary of each face is an induced non-separating cycle.

At the end let us also briefly comment on the main results of this paper. There are very few known methods to prove that some given embedding is indeed a minimal genus embedding. The only general method combines elementary counting techniques with the Euler-Poincaré formula, and it works well on many dense graphs (those with triangular embeddings, for example). But no known methods provide feasible solutions for sparse graphs. In this paper the *local planarity* concept, following [RV], is considered in some detail, and some results concerning genus embeddings of graphs are obtained. The main tool, Theorem 5.1, presents a combinatorial condition which guarantees that a given cycle of G will be contractible in any minimal genus embedding (either orientable, or non-orientable). The condition, which might be viewed as a notion of "combinatorial local planarity", is independent of graph embeddings and it generalizes the local planarity width mentioned above. Our approach gives a new light on the structure of genus embeddings of graphs. It provides simple unified proofs for some recently discovered results about locally planar embeddings of graphs [RV], and some generalizations or improvements are also presented. See the results in Section 6. We also present several results which show that the theory for graphs embedded in higher genus surfaces is similar to the well-known theory for planar graphs. In particular we get results about uniqueness and switching equivalence of graphs embedded in a fixed surface. It should be mentioned that some of the results have similar flavour, although they are essentially different from, as the results of Thomassen [T1] about large edge-width embeddings.

2. **Local planarity.** Let G be a graph and $\psi\colon G \longrightarrow \Sigma$ an embedding in a closed surface Σ. If f is a face of this embedding, the *local planarity width at* f is the minimum

$$\rho(\psi, f) = \rho(f) = \min_{\gamma} \operatorname{cr}\big(\gamma, \psi(G)\big) \tag{2.1}$$

where $\gamma\colon S^1 \longrightarrow \Sigma$ is any essential closed curve in Σ containing a point in $\operatorname{int} f$ and $\operatorname{cr}\big(\gamma, \psi(G)\big)$ is defined as

$$\operatorname{cr}\big(\gamma, \psi(G)\big) = \big|\{z \in S^1 \mid \gamma(z) \in \psi(G)\}\big|.$$

If Σ is the 2-sphere S^2 then there are no essential curves on Σ, and in this case we set $\rho(\psi, f) = \infty$ where ∞ is a "value" greater than any integer. The global version is the *planarity width*, or shorter the *width* of the embedding ψ. This is the number

$$\rho(\psi) = \min_{f} \rho(\psi, f) = \min_{\gamma} \operatorname{cr}\big(\gamma, \psi(G)\big) \tag{2.2}$$

where the first minimum is over all faces of ψ, the second over all essential closed curves γ in Σ. The second minimum in (2.2) can only be taken over all essential simple closed curves γ in Σ which intersect $\psi(G)$ only at vertices of G and use each vertex and each face of ψ at most once. The planarity width was introduced by Robertson and Seymour [RS] in their work on graph minors, where it is called the *representativity* of the embedding.

It was studied later in more detail in [RV]. We point out that for a connected graph G, $\rho(\psi) > 0$ if and only if ψ is a 2-cell embedding.

There is a natural distance function δ between the faces of the embedding ψ. It is defined by $\delta(f,f) = 0$ for each face f, and for $f \neq f'$, $\delta(f,f') = d > 0$ if there is a face f'' with $\delta(f,f'') = d - 1$ such that the closures $\overline{f''}$ and $\overline{f'}$ have a point in common. If k is an integer and f a face, then $B_k(f)$ will denote the union of closed faces which are at δ-distance at most k from f. Note that $B_0(f) = \bar{f}$ and for k large enough, $B_k(f) = \Sigma$.

LEMMA 2.1. *Let f be a ψ-face and k an integer. If $\rho(\psi,f) > 2k+1$ then $B_k(f)$ contains no essential curve of Σ.*

PROOF. It suffices to prove that an arbitrary closed curve γ, which only passes through faces and vertices in $B_k(f)$, is contractible in Σ. Let $f_1, f_2, \ldots, f_n$ be the consecutive faces used by γ, and let z_i $(1 \leq i \leq n)$ be the vertex of G used by γ when passing from f_i to f_{i+1} (index modulo n). Since $f_i \subseteq B_k(f)$, there is a path α_i from the "barycenter" x of f (a chosen point in int$f$) to the "barycenter" of f_i, which uses at most k vertices and no other points of $\psi(G)$. Let γ_i be the closed curve starting at x, following α_i, then using a path in f_i from the end-point of α_i to z_i, continuing in f_{i+1} to the end-point of α_{i+1}, and finally returning to x on α_{i+1}^{-1}. Since $\mathrm{cr}\big(\gamma_i, \psi(G)\big) \leq 2k+1 < \rho(\psi,f)$, the curve γ_i is not essential in Σ for any i. Therefore also the concatenation $\gamma_1\gamma_2\cdots\gamma_n$ is contractible. But this curve is free homotopic to γ, and so γ is contractible. ∎

The bound of Lemma 2.1 is best possible since for any k there exist embeddings ψ with $\rho(\psi,f) = 2k+1$ such that $B_k(f) = \Sigma$ for some face f of ψ. Examples are easy to construct, and details are left to the reader.

The next result justifies the use of the term local planarity width for the number $\rho(\psi,f)$.

COROLLARY 2.2. *Let f be a face of an embedding $\psi\colon G \longrightarrow \Sigma$, $\Sigma \neq S^2$, and let k be an integer such that $\rho(\psi,f) > 2k+1$. Then there is a disc $D_k(f) \subset \Sigma$ which contains $B_k(f)$ and such that $\partial D_k(f) \subseteq \partial B_k(f)$.*

PROOF. Any contractible simple closed curve $\gamma \in \Sigma$ bounds a unique disc since $\Sigma \neq S^2$. Denote this disc by int(γ). Clearly, any disc containing $B_k(f)$ must contain the disc int(γ) for any simple closed curve γ in $B_k(f)$. Let

$$D_k = B_k(f) \bigcup \{\mathrm{int}(\gamma) \mid \gamma \text{ a simple closed curve in } B_k(f)\}.$$

Each closed curve γ in D_k is homotopic to some closed curve contained in $B_k(f)$ since any part of γ in int(γ') can be moved by homotopy to the boundary of int(γ'), which is contained in $B_k(f)$. By Lemma 2.1, D_k is simply connected. It is also connected by construction and $\partial D_k \subseteq \partial B_k(f)$. Since the only simply connected compact surfaces are the 2-sphere and the closed disc, it suffices to show that D_k is a 2-manifold with boundary. By construction it follows that D_k is closed. Moreover, D_k is a union of closed faces. Therefore a singularity can only appear at a vertex of G. But by the following reason a true singularity is excluded. If g, h are faces in $B_k(f)$ meeting at a vertex x, let γ be a closed curve starting at a point in int$f$, leading to g, going through x to h, and returning to f, such that $\mathrm{cr}\big(\gamma, \psi(G)\big) \leq 2k+1$. Since $\rho(f) > 2k+1$, γ bounds a disc in D_k. Consequently, all the faces at x which lie "between" g and h (one or the other side) also lie in D_k. ∎

At the end of this section let us mention another result involving the width whose generalization will be met later in Section 6. This is one of the first known results involving the width of embeddings. It is a bound on the width of non-planar embeddings of planar graphs due to Robertson [RV, T1].

THEOREM 2.3. *Let G be a planar graph and $\psi: G \longrightarrow \Sigma$ an embedding of G into $\Sigma \neq S^2$. Then $\rho(\psi) \leq 2$.*

3. **The core of a graph.** Let x be a vertex of a graph G and B a relative $\{x\}$-component. If B is a planar graph then the deletion of $B - x$ from G, *i.e.* contracting B to the vertex x, is a *1-reduction* of G. Clearly, a 1-reduction is possible only if either G itself is planar, or else x must be a cutvertex of G. Let x, y be distinct vertices of G and B a relative $\{x, y\}$-component which contains both x and y. If $B+xy$ (this is the graph B with the edge xy added) is a planar graph then the replacement of B by a single edge xy (if xy is an already existing edge in $E(G) \setminus E(B)$ then B is replaced just by the vertices x, y) is a *2-reduction* of the graph. A reduction which does not change the graph is said to be *trivial*. Clearly, a 2-reduction is trivial if and only if B is just the edge xy and in G there is no parallel edge to it. G is *2-reduced* if no non-trivial 1- or 2-reduction is possible. In particular, every 3-connected non-planar graph is 2-reduced. Every graph can be made 2-reduced by successive reductions. A graph obtained this way is called the *core* of G. Note that reductions only depend on the graph and no embedding of the graph is needed.

For $x, y \in V(G)$, possibly $x = y$, call a relative $\{x, y\}$-component B *reducible* if B contains x and y and the graph $B+xy$ is planar. In particular, any edge $e = uv$ is a reducible $\{u, v\}$-component. A vertex v of G is said to be *reducible in G* if there exist vertices $x, y \in V(G) \setminus \{v\}$ (possibly $x = y$ to include 1-reductions as well) such that v belongs to a reducible $\{x, y\}$-component.

PROPOSITION 3.1. *The core H of a graph G is uniquely determined. The vertex set of H consists of precisely the non-reducible vertices of G, and two such vertices x, y are adjacent in H if and only if there exists a reducible $\{x, y\}$-component in G. The graph G contains a subgraph H' which is isomorphic to a subdivision of H. The genus of H (and also of H') is equal to the genus of G, more precisely, $g(H) = g(G)$ and $\tilde{g}(H) = \tilde{g}(G)$.*

PROOF. If G is planar then $H = K_1$ and all the claims of the proposition are trivial. Otherwise, let $G = G_1, G_2, \ldots, G_k = H$ be a sequence obtained by making successive 1- and 2-reductions to get H starting with G. To prove the existence of H' it suffices to see that if G_i $(1 < i \leq k)$ contains a subdivision of H then G_{i-1} contains one. The case when G_i is obtained from G_{i-1} by a 1-reduction is easy since then G_i is a subgraph of G_{i-1}. If G_i is obtained by a 2-reduction, let xy be the corresponding edge, and H_1 a subdivision of

H in G_i. If $xy \notin E(H_1)$ then $H_1 \subseteq G_{i-1}$, and otherwise $H_1 - xy \subseteq G_{i-1}$. In the latter case we may add a path between x and y in the deleted $\{x, y\}$-component B to $H_1 - xy$ since B is by definition connected and contains both x and y. This way we get a subdivision of H_1 in G which is also a subdivision of H.

To establish the uniqueness of H we will prove first that $V(H)$ consists precisely of the non-reducible vertices of G. This follows from the fact that each G_i contains all non-reducible vertices of G and that $H = G_k$ contains no vertices reducible in G. Moreover, $x, y \in V(H)$ are adjacent in H if and only if there is a reducible $\{x, y\}$-component in G. For adjacent vertices x, y in H consider the reducible relative $\{x, y\}$-components in $G_k, G_{k-1}, \ldots, G_1 = G$. We want to prove that since in G_k there is such a component (the edge xy), so there must be one in each G_i. But this is obvious since the reductions preserve the reducibility of the components. Conversely, if for $x, y \in V(H)$ there is a reducible $\{x, y\}$-component in G, so $G_1, G_2, \ldots, G_k$ each contains a reducible $\{x, y\}$-component. The only possibility for such a component in H is that it is an edge. This completes the proof of uniqueness.

It remains to prove that $g(G) = g(H)$ and $\tilde{g}(G) = \tilde{g}(H)$. Since G contains a subdivision of H, we clearly have $g(G) \geq g(H)$ and $\tilde{g}(G) \geq \tilde{g}(H)$. But having embedded H one can easily get an embedding of G into the same surface just by embedding each of the 2-reduced $\{x, y\}$-components B "close" to the edge xy by using a plane embedding of $B + xy$ with xy on the "unbounded" face. If necessary, one removes xy afterwards. A similar extension can be performed in case of a 1-reduction. Therefore $g(G) = g(H)$ and $\tilde{g}(G) = \tilde{g}(H)$. ∎

It is worth mentioning that Proposition 3.1 is an immediate consequence of more general theorems—the decomposition theory of Tutte [Tu1] and the genus additivity theorems (*cf.*, *e.g.*, [BHKY, SB, R]).

Later we will need the following lemma.

LEMMA 3.2. *Let $\psi: G \longrightarrow \Sigma$, $\Sigma \neq S^2$, be an embedding with $\rho(\psi) \geq 3$. Then any reducible $\{x\}$-component B, $x \in V(G)$, lies in a disc. More precisely, there is a closed disc $D_B \subset \Sigma$ such that $\psi(B) \subset D_B$ and $\psi(G) \cap \partial D_B = \{\psi(x)\}$. Let $x, y \in V(G)$ be non-reducible vertices of G such that there is a reducible $\{x, y\}$-component. Then there is a closed disc $D_{xy} \subset \Sigma$ such that all reducible $\{x, y\}$-components lie in D_{xy} and $\psi(G) \cap \partial D_{xy} = \{\psi(x), \psi(y)\}$.*

PROOF. Let B be a reducible $\{x\}$-component. Since $\rho(\psi) \geq 3$, G is not a planar graph (by Theorem 2.3), and therefore x must be a cutvertex of G. Since $G - x$ is disconnected, the face F of the embedding $\psi|(G - x)$ containing $\psi(x)$ is not simply connected—it has at least two boundary components. Since G has no loops, $B - x$ is a connected non-empty graph. Let γ_0 be a simple closed curve in F following the boundary component of F corresponding to $\psi(B - x)$. Now modify γ_0 by homotopy in the following way: every time γ_0 crosses $\psi(e)$ for an edge e of G (we may assume every intersection is a crossing and that γ_0 intersects each edge at most once), replace a small part of γ_0 around the intersection by the curve which follows $\psi(e)$ to the vertex x, crosses e at x, and returns on the other side of $\psi(e)$. The changes may be done in such a way that the obtained curve γ is simple up to the multiple crossing of $\psi(x)$ where it may touch itself several times. Also, $\gamma(S^1) \cap \psi(G) = \{\psi(x)\}$. Therefore $\gamma = \alpha_1\alpha_2 \cdots \alpha_k$ where α_i are simple closed curves based at $\psi(x)$. Since $\rho(\psi) \geq 2$, each α_i is contractible and, consequently, also γ and γ_0 are contractible. Let D_i be the closed disc bounded by α_i. If D_i contains an edge of B then $\psi(B) \subset D_i$ and we are done by taking D_i as the required disc D_B. We will prove that this must happen for at lease one index i. Suppose not and consider the local rotation under ψ of edges at x. By construction of γ, coming from α_{i-1} to α_i (indices modulo k) there is only an edge of B between them. Since D_{i-1} and D_i do not contain this edge, all edges incident to x but not in B must lie in $D_1 \cup \cdots \cup D_k$. But this is not possible since G is non-planar.

To prove the second part of the lemma, let x, y be distinct non-reducible vertices of G and $B_0, B_1, \ldots, B_s$ the relative $\{x, y\}$-components each of which contains x and y. Since G is non-planar, at least one B_i is non-reducible. Assume this is B_0. Assume also that $s \geq 1$. We will prove that $B_1, \ldots, B_s$ all lie in a disc D_{xy} (and so each of them is reducible).

Considering the embedding $\psi|G - x - y$ we prove, using the same method as above for the $\{x\}$-component B, that for each B_i, $1 \leq i \leq s$, there is a closed disc D_i containing $\psi(B_i)$ and such that $\partial D_i \cap \psi(G) = \{\psi(x), \psi(y)\}$. In this step we use the non-reducibility of vertices x and y. Also, we may choose the discs in such a way that $D_i \cap D_j = \{\psi(x), \psi(y)\}$ for $1 \leq i < j \leq s$. Let $D = D_1 \cup \cdots \cup D_s$. It is easy to see that if D contains an essential closed curve on Σ then D contains an essential simple curve which intersects $\psi(G)$ at most twice (at $\psi(x)$ and $\psi(y)$). Since $\rho(\psi) \geq 3$, this is not possible. By the same reason any simple closed curve going from $\psi(x)$ to $\psi(y)$ in D_1 and returning back in D_2 is contractible in Σ. Therefore there is a closed disc $D_{12} \subset \Sigma$ containing $D_1 \cup D_2$. For each j $(1 \leq j \leq s)$, D_{12} either contains D_j or else $D_{12} \cap D_j = \{\psi(x), \psi(y)\}$. Therefore we can find a disc D_{123} which contains D_{12} and D_3, using the same method as above. Next we find a disc D_{1234} containing D_{123} and D_4, *etc.*, until finally constructing $D_{12\cdots s} =: D_{xy}$ which satisfies the required properties, and so we are done. ∎

PROPOSITION 3.3. *Let H be the core of G and let H' be a subgraph of G which is isomorphic to a subdivision of H. If $\psi: G \longrightarrow \Sigma$ is an embedding with $\rho(\psi) \geq 3$ then the embedding ψ restricted to H' has the same width, $\rho(\psi|H') = \rho(\psi)$.*

PROOF. Assume that $\rho(\psi) \geq 3$. Clearly $\rho(\psi|H') \leq \rho(\psi)$. To prove the converse inequality, let γ be an essential closed curve in Σ with $\mathrm{cr}\big(\gamma, \psi(H')\big) = \rho(\psi|H')$. It may be assumed that γ only intersects $\psi(H')$ at vertices of degree in H' greater or equal to 3 (*i.e.* the vertices of H). For adjacent vertices $x, y \in V(H) \subseteq V(H')$ denote by D_{xy} the closed disc whose existence is guaranteed by Lemma 3.2. (It follows from Proposition 3.1 that x and y are non-reducible.) D_{xy} will also contain the edge $xy \in E(H)$. It may be assumed that distinct discs D_{xy}, $xy \in E(H)$, pairwise intersect only at a common vertex x or y. One can change γ in such a way that $\gamma(S^1) \cap \operatorname{int} D_{xy} = \emptyset$ for $xy \in E(H)$. This can be done in the following way: Whenever γ enters D_{xy}, find where it comes out of this disc and replace this part of γ in D_{xy} by a path on ∂D_{xy} so that this path uses $\psi(x)$ or $\psi(y)$ only when necessary. The same can be done to guarantee that our curve will not use the interior of any 1-reducible relative component. The obtained curve γ' is homotopic to γ and has no more crossings with $\psi(H')$ than γ. But $\rho(\psi|H') = \mathrm{cr}\big(\gamma, \psi(H')\big) \geq \mathrm{cr}\big(\gamma', \psi(H')\big) = \mathrm{cr}\big(\gamma', \psi(G)\big) \geq \rho(\psi)$. ∎

COROLLARY 3.4. *Let G be a connected 2-reduced graph with an embedding $\psi\colon G \longrightarrow \Sigma$ with $\rho(\psi) \geq 3$ and $\Sigma \neq S^2$. Then G is 3-connected.*

PROOF. We omit the details for 2-connectivity. The proof follows the same lines as the verification of 3-connectedness below.

Assume now that G is 2-connected, but there is a vertex cut-set $\{x, y\}$. None of the non-trivial $\{x, y\}$-components is planar since G is 2-reduced. Consider the induced embedding of $G - x - y$. Since this graph is disconnected, there is a face F which is not homeomorphic to a disc and whose boundary is not connected. Take a simple closed curve γ in F which is essential in this face. Then γ is also essential in Σ, since otherwise it bounds a disc, which would contain a planarly embedded component of $G-x-y$, and so the corresponding relative $\{x, y\}$-component would also be planar.

It may happen that γ intersects some edges of G. Such edges are incident to x or to y. Change γ by a homotopy to intersect all these edges at vertices x and y only. (See the proof of Lemma 3.2 for details.) Call the obtained curve γ'. Since γ' is essential, it also contains an essential simple closed subcurve γ''. But since $\gamma'' \cap \psi(G) \subseteq \{x, y\}$ this contradicts the assumption that $\rho(\psi) \geq 3$. ∎

For 3-connected graphs there is a characterization of planarity due to Tutte [Tu2]: A 3-connected graph G is planar if and only if each edge of G is contained in precisely two induced non-separating cycles. It follows that the faces of a 3-connected graph embedded in the 2-sphere are precisely the induced non-separating cycles of G. The following result from [RV] is an extension of this fact to embeddings in general surfaces.

PROPOSITION 3.5. *Let G be a 3-connected graph and $\psi\colon G \longrightarrow \Sigma$ an embedding into $\Sigma \neq S^2$. Then $\rho(\psi) \geq 3$ if and only if each face of ψ is a disc whose boundary is an induced non-separating cycle of G.*

4. **Switching equivalence of embeddings.** H. Whitney [W] introduced a simple operation on graphs, called a 2-switching. If $x, y \in V(G)$ and R is a relative $\{x, y\}$-component, the *2-switching* of R is the replacement of R in G by an isomorphic copy R' of R in such a way that the vertex x' of R', which is corresponding to x, is identified with y, and $y' \in V(R')$ is identified with x. The importance of this notion lies in the fact that every cycle isomorphism $\phi\colon E(G) \longrightarrow E(H)$ (*i.e.* a map for which $C \subseteq E(G)$ is a cycle in G if and only if $\phi(C)$ is a cycle in H) of a 2-connected graph G is induced by a sequence of 2-switchings (Whitney's 2-isomorphism Theorem [W]).

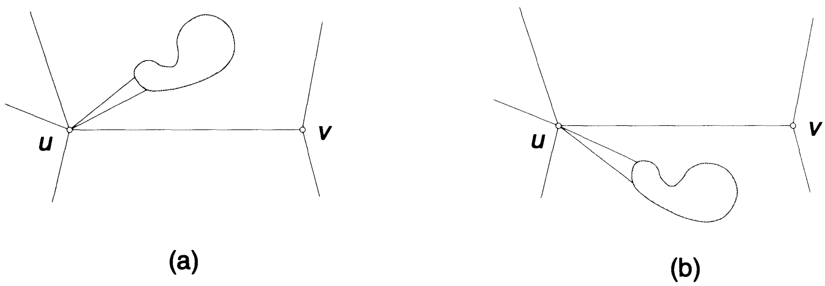

FIGURE 1. Change in the dual after a 2-switching

Let $\psi_1, \psi_2: G \longrightarrow S^2$ be plane embeddings of a 2-connected graph G and let G_1^*, G_2^* be the corresponding geometric duals. Then it can be shown that G_1^* and G_2^* are cycle isomorphic, and so by the Whitney's 2-isomorphism Theorem there is a sequence of 2-switchings transforming G_1^* into G_2^*. But any 2-switching on the dual corresponds to a simple re-embedding of G of the following type: there is a closed disc $D \subset S^2$ with $\psi(G) \cap \partial D = \{\psi(v), \psi(u)\}$ for some $v, u \in V(G)$, and G is re-embedded in such a way that the embedding in $S^2 \setminus D$ remains the same, but in D it changes the "orientation" (see Figure 1). Geometrically, cut out D from S^2, turn it over, leaving $\psi(v)$ and $\psi(u)$ fixed, and paste it back to obtain the 2-sphere. Such a re-embedding is called a *switching*.

The re-embeddings of a graph in S^2 described above are slightly more general than just changing the dual by a 2-switching. For example, if G is not 2-connected, one may use the changes as schematically indicated in Figure 2 to obtain any "placement" of the blocks of G meeting at a vertex. The following is the outcome result (*cf.* [MRV] for details): *Any two embeddings of a graph G in the 2-sphere can be obtained from each other by a sequence of orientation changing re-embeddings on discs in S^2.*

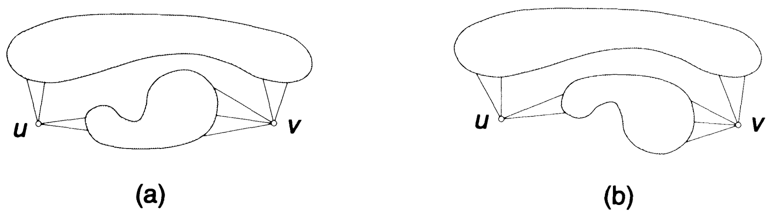

FIGURE 2. Local re-embedding of an $\{x\}$-component

In view of the above result it makes sense to introduce the following equivalence relation between embeddings of a graph G into a surface Σ. Let $\psi: G \longrightarrow \Sigma$ be an embedding and let $D \subset \Sigma$ be a closed disc in Σ such that $\psi(G) \cap \partial D = \{\psi(u), \psi(v)\}$ where $u, v \in V(G)$. If we change the embedding ψ in D but it remains the same out of D, the change is called a *disc switching*. The name "switching" reflects the fact that any re-embedding in D can, in fact, be realized by true "switchings". Two embeddings of a graph into the same surface are *switching equivalent* if one can be obtained from the other by a sequence of disc switchings. Embeddings ψ, ψ' of G into Σ are *weakly equivalent* if there is a homeomorphism $h: \Sigma \longrightarrow \Sigma$ and an embedding ψ'' switching equivalent to ψ such that $h\psi' = \psi''$.

It is easy to see that disc switchings, and hence the switching equivalence and the weak equivalence, preserve the width of embeddings: if ψ and ψ' are weakly equivalent then $\rho(\psi) = \rho(\psi')$.

Let W be a closed walk in G and $\psi, \psi': G \longrightarrow \Sigma$ switching equivalent embeddings. After a disc switching the image of W in Σ changes only in a closed disc. Therefore disc switchings preserve homotopy and thus $\psi(W)$ is homotopic to $\psi'(W)$. It is important that under the assumption of high width, homotopy invariance of contractible cycles already implies the converse – two such embeddings must be weakly equivalent:

THEOREM 4.1. *Let $\psi: G \longrightarrow \Sigma$ and $\psi': G \longrightarrow \Sigma'$ be embeddings of a connected graph G such that $\rho(\psi) \geq 3$ (or $\rho(\psi') \geq 3$). Then the following assertions are equivalent:*

(a) ψ and ψ' are weakly equivalent,

(b) for each cycle C of G, $\psi(C)$ and $\psi'(C)$ are both contractible, or both non-contractible, and

(c) for each induced non-separating cycle C of G, $\psi(C)$ and $\psi'(C)$ are both contractible, or both non-contractible.

PROOF. The implication (a) $\Rightarrow$ (b) is trivial by the homotopy invariance under disc switchings and invariance of contractibility under homeomorphisms. Also (b) $\Rightarrow$ (c) is obvious. To prove that (c) $\Rightarrow$ (a), assume $\rho(\psi) \geq 3$ and that $\psi(C)$ and $\psi'(C)$ are simultaneously contractible, or non-contractible for each induced non-separating cycle C of G. By the extension of the Whitney's theorem about switching equivalence of embeddings into S^2, we may assume henceforth that $\Sigma \neq S^2$. Let H be the core of G and H' a subdivision of H in G. By Proposition 3.4, H is 3-connected and by Proposition 3.5 the face boundaries of $\psi|H'$ are induced non-separating cycles of G. Since each of them is contractible under ψ, it must be contractible under ψ', and the only possibility for this is to bound a face of $\psi'|H'$. This already determines the embedding $\psi'|H'$ up to equivalence. So, up to a homeomorphism of the surface (which we assume to be the identity from now on), the embeddings ψ and ψ' agree on H'. By Proposition 3.3 we also have: $\rho(\psi'|H') = \rho(\psi|H') = \rho(\psi) \geq 3$. By Lemma 3.2 for any two adjacent vertices x, y of H there is a closed disc D_{xy} in Σ containing the ψ-images of all reducible $\{x, y\}$-components of G. This disc is contained in the union of two faces of the two faces of $\psi|H'$ containing the path between x and y corresponding to the edge xy. The same holds for ψ' and we may assume that the disc D_{xy} is the same for both embeddings. A disc switching on D_{xy} therefore makes the reducible $\{x, y\}$-components, embedded under ψ, to be embedded the same as under ψ'. Finally, one may use the disc switchings as shown in Figure 2 to get the ψ-image of any reducible $\{x\}$-component at the same place as under the embedding ψ'. ∎

COROLLARY 4.2. *Let ψ_1, ψ_2 be embeddings of G which coincide on a subdivision of the core $H' \subseteq G$. If $\rho(\psi_1) \geq 3$ then ψ_1 and ψ_2 are switching equivalent.*

5. **The minimal genus embedding lemma.** A sequence $C_1, C_2, \ldots, C_k$ of disjoint cycles in a graph G is *planarly nested* if each cycle C_i $(1 \leq i \leq k)$ has a relative C_i-component H_i such that $H_1 \supset H_2 \supset \cdots \supset H_k$ and the graph obtained from G by contracting to a single vertex all edges in the relative component H_k, except its feet, is planar. It is clear that any subsequence of a planarly nested sequence of cycles is also planarly nested (use the same C_i-components H_i). It is also easy to see that H_i $(1 \leq i \leq k)$ contains the cycles $C_{i+1}, \ldots, C_k$ but does not contain any of the cycles $C_1, \ldots, C_{i-1}$. The proof goes as follows. Since $H_i \supset H_{i+1}$ and the cycles are disjoint, $C_{i+1} \subset H_i$. But then necessarily $C_{i+2} \subset H_i$ since $C_{i+2} \subset H_{i+1} \subset H_i$. Similarly it follows that H_i contains $C_{i+3}, \ldots, C_k$. Suppose now that H_i contains a vertex of C_{i-1}. Then $C_{i-1} \subset H_i \subset H_{i-1}$ which is impossible by the definition of a relative component. Now, since $C_{i-2} \cap H_{i-1} = \emptyset$ and $H_i \subset H_{i-1}$ we have $C_{i-2} \cap H_i = \emptyset$. Similarly we see that $C_{i-3}, \ldots, C_1$ are all disjoint from H_i.

The condition on C_k in the definition of planarly nested sequences, that G with all non-feet edges of H_k contracted is a planar graph, is equivalent to the following one: The overlap graph (for the definition see, *e.g.* [T1]) is bipartite and there is at most one C_k-component H such that $C \cup H$ is non-planar. Our results on the existence of planarly nested sequences, *e.g.* Proposition 5.3, therefore yield a special condition on the structure of relative components for the cycles in the sequences. These results therefore generalize some results by other authors, *e.g.* Theorem 4.5 of [T1].

THEOREM 5.1. *Let $\psi: G \longrightarrow \Sigma$ be a minimal genus embedding of a graph G, and let g denote the genus of Σ. Let $C_1, \ldots, C_k$ be a planarly nested sequence of cycles of G, where $k > g$.*

(a) If Σ is orientable, or Σ is non-orientable and G is not orientably simple, then the cycles $C_1, C_2, \ldots, C_{k-g}$ bound discs.

(b) If the cycles $C_1, C_2, \ldots, C_{k-g}$ do not bound discs, then G is orientably simple and each but at most one of the cycles $C_1, C_2, \ldots, C_r$, $r = k - \frac{g-1}{2}$, bounds either a disc or a Möbius band. There are other embeddings of G in the same surface such that $C_1, \ldots, C_r$ bound discs.

Before giving the proof of this result, let us state a simple lemma.

LEMMA 5.2. *Let Σ be a closed surface of genus g (either orientable, or non-orientable). If $\gamma_1, \gamma_2, \ldots, \gamma_k$ are homologically independent disjoint circuits in Σ, then $k \leq g$.*

PROOF. Since the γ_i are homologically independent it follows that $\Sigma \setminus \big(\gamma_1(S^1) \cup \cdots \cup \gamma_k(S^1)\big)$ is a connected surface with boundary, having at least $2k$ boundary components if Σ is orientable and there are at least k of them in the non-orientable case. Denote their number by b. If we paste a disc on each of the boundary components we get a closed surface Σ' whose Euler characteristic is equal to $\chi(\Sigma') = \chi(\Sigma) + b \leq 2$. If Σ is orientable then $2k \leq b \leq 2 - \chi(\Sigma) = 2 - (2 - 2g) = 2g$. In the non-orientable case we have $k \leq b \leq 2 - \chi(\Sigma) = g$, which proves the claimed inequality in either case. ∎

PROOF (OF THEOREM 5.1). Let $C_1, \ldots, C_k$ be a planarly nested sequence of cycles of G. Assume first that under the embedding ψ none of the cycles C_i bounds a disc. We claim that $C_1, \ldots, C_k$ are homologically independent. Assume the contrary, that $\{C'_1, \ldots, C'_t\} \subseteq \{C_1, \ldots, C_k\}$ bounds but no proper subset does. Then $C'_1, \ldots, C'_t$ divide the surface into exactly two parts, say Σ_1 and Σ_2. Without loss of generality we may assume that $H_k \subseteq \Sigma_1$. If $C'_t = C_s$ has the largest index as a member of the sequence $C_1, \ldots, C_k$ then the graph H_s contains no cycle C'_j ($j \leq t$) except possibly some vertices of C'_t as its vertices of attachment. Notice that $H_s \subseteq \Sigma_1$. Replace Σ_2 in Σ with a union of discs, one for each boundary component of Σ_1. This way we obtain a surface Σ' having larger Euler characteristic than Σ (= smaller genus if the orientability type remains the same). But it is possible to embed G in Σ' as follows. Consider a planar embedding of G with edges in $H_s - V(C_s)$ contracted to a point x. Use this embedding on the disc of Σ' which was pasted in the boundary component of Σ_1 corresponding to C_s. Next embed in Σ_1 the graph H_s as determined by the original embedding ψ. It should be mentioned here that C'_t (and also each other C'_i, $1 \leq i \leq t$) is 2-sided since $\{C'_1, \ldots, C'_{t-1}\}$ does not bound. Notice that local rotation of edges at the vertex x is the same (after a possible exchange of parallel edges) as the sequence of feet of H_s coming to C_s under the embedding ψ. Therefore the two embeddings are easily seen to combine into an embedding of G into Σ'. This is a contradiction to the genus minimality of the embedding ψ, except in the case when G is orientably simple, $t = 1$, and C'_1 bounds a Möbius band. This case will be treated later.

If none of the cycles C_i bounds a disc, and excluding the above exceptional case, $C_1, \ldots, C_k$ are homologically independent. By Lemma 5.2, $k \leq g$. This shows that in general at most g of the cycles C_i do not bound discs (since a subsequence of all non-contractible cycles is also planarly nested). All we have to do to end the proof, is to show that $C_1, \ldots, C_{s-1}$ are all contractible providing C_s bounds a disc ($2 \leq s \leq k$). If not, we may use the same re-embedding procedure as above to get an embedding of G into a surface of lower genus. The details are left to the reader.

Let us return to the orientably simple case. Then we have $t = 1$ and $C'_1 = C_s$ bounds a Möbius band. The surface Σ' is orientable of genus $\frac{g-1}{2}$. By the already proved result for the orientable genus embeddings, $C_1, \ldots, C_{k-\frac{g-1}{2}}$ bound discs in Σ'. Adding a crosscap outside the last one of these discs we get a non-orientable minimal genus embedding as we are to prove. It also follows that among the first $k - \frac{g-1}{2}$ of the cycles C_i at most one is 1-sided, and all other either bound a disc or a Möbius band. ∎

REMARK. From the above proof it also follows this important observation which will be used later: If $k = g$, and either G is not orientably simple or the embedding ψ is orientable, then C_1 must bound a disc unless all of $C_1, C_2, \ldots, C_g$ are non-bounding and homologically independent. In the non-orientable case this means that all of them are 1-sided!

The existence of planarly nested sequences of cycles is a combinatorial condition. It only depends on the combinatorial local structure of the graph and does not involve any embeddings of the graph. Moreover, this condition is local. Based on the local structure of a graph G we may be able to prove results about the structure of minimal genus embeddings of G—or to show that the genus is at least as large as the number of cycles in a given planarly nested sequence. But in several applications we will make use of the following "topological" criterion based on a given embedding which will assure existence of planarly nested sequences $C_1, C_2, \ldots$ with C_1 being a prescribed (usually a facial) cycle in the graph. Let us recall the existence of discs $D_k(f)$ established in Corollary 2.2.

PROPOSITION 5.3. *Let ψ be an embedding of a connected graph G into a closed surface Σ different from the 2-sphere, and let f be a face of ψ. Let $\rho = \rho(\psi, f)$, $r = \lceil \frac{\rho}{2} \rceil - 1$, and $C_i = \partial D_{i-1}(f)$, $1 \le i \le r$. If $\rho \ge 3$ then the sequence of cycles $C_1, C_2, \ldots, C_r$ is planarly nested.*

PROOF. Let $k = \lfloor \frac{\rho}{2} \rfloor - 1$. By Corollary 2.2, C_i, $1 \le i \le k+1$, are disjoint cycles of G. Each of them bounds a disc which contains all the cycles with smaller indices and does not contain any of the others. Denote by H_i the C_i-component containing C_{i+1}. Then H_i are nested: $H_1 \supset H_2 \supset \cdots \supset H_k$. If ρ is even then $k = r$. Since C_{k+1} bounds a disc containing all other cycles C_i, the graph obtained from G by contracting the non-feet edges of H_r to a point, is planar. It follows that our sequence of cycles is planarly nested.

The other case when ρ is odd needs more care. Now $r = k + 1$ and C_r bounds the disc D_k. Let B^0 be the union of D_k together with all *open* faces which are incident to some vertex of C_r and together with the interiors $\operatorname{int}\psi(e)$ of those edges $e \in E(G)$ having an end on C_r. Then B^0 is an open subset of Σ and since $\rho = 2k + 3$, no closed curve in B^0 is essential in Σ. For any simple closed curve γ in B^0, add to B^0 the disc bounded by γ. This way we get an open set $D^0 \subset \Sigma$ which is homeomorphic to an open disc (for details how to verify this, compare with the proof of Corollary 2.2). We claim that exactly one relative C_r-component is embedded out of D^0. There must be at least one such relative component, denote it by H_r, since ψ is a cellular embedding and Σ is not simply connected. Choose a foot e_1 of H_r. Starting at C_r follow $\psi(e_1)$ along the boundary of a face containing this edge. Sooner or later we come back to C_r. Let e_2 be the foot of H_r which was used when returning to C_r back for the first time. Since our walk was in D^0, we have bound a disc (together with C_r) this way, and hence on C_r between e_1 and e_2 there are no feet of any C_r-component whose part is embedded out of D^0. Now we cross to the "other side" of e_2 and traverse the corresponding ψ-face the same way as before until we come back to C_r. We continue the process until we come back to e_1 (on the other side as we started). The same conclusion, as after the first step, can be derived each time. It follows that the only edges that may leave D^0 are the feet of H_r, and this proves our claim. Moreover, if we contract the edges of H_r except its feet, we get a planar graph. Our proof is thus complete. ∎

The result $\lceil \frac{\rho}{2} \rceil - 1$ for the number of the cycles in the obtained planarly nested sequence is best possible. However, we may improve it slightly for non-orientable embeddings in a very special (non-local), but important case.

THEOREM 5.4. *Let G be a graph which is not orientably simple, and let $\psi: G \longrightarrow \Sigma$ be an embedding of width $\rho = \rho(\psi)$. If there is a face f of ψ such that the cycle $C(f) = \partial D_0(f)$ does not bound a disc in a non-orientable minimal genus embedding of G then $\tilde{g}(G) \geq \lfloor \frac{\rho}{2} \rfloor$.*

PROOF. If ρ is odd then Proposition 5.3 guarantees that there is a planarly nested sequence $C_1 = C(f), C_2, \ldots, C_r$ where $r = \lfloor \frac{\rho}{2} \rfloor$ and $C_i = \partial D_{i-1}(f)$. Similarly for ρ even, but in this case we only have $r = \lfloor \frac{\rho}{2} \rfloor - 1$.

Assume that $\tilde{g}(G) < \lfloor \frac{\rho}{2} \rfloor$. By Theorem 5.1, $C_1 = C(f)$ will bound a disc in any non-orientable minimal genus embedding ψ' if $r - \tilde{g}(G) \geq 1$. The only possibility that this will not happen is when ρ is even and $\tilde{g}(G) = r$. It follows from the Remark after Theorem 5.1 that in this case $C_1, \ldots, C_r$ each must be a 1-sided closed curve under the embedding ψ'. By Corollary 2.2, $C_{r+1} = \partial D_r(f)$ also bounds a disc in the embedding ψ. Therefore $\Sigma \setminus D_r(f)$ contains non-contractible curves, and hence for at least one ψ-face f' in $\Sigma \setminus D_r(f)$ its (extended) boundary $C(f')$ does not bound a disc under the embedding ψ'. Similar conclusion as it was made above for $C(f)$ shows that $\psi'\big(C(f')\big)$ is 1-sided. Since this cycle is also disjoint from $C_1, C_2, \ldots, C_r$, this is a contradiction since there are at most r pairwise disjoint 1-sided curves on a surface of genus r. ∎

If Σ is non-orientable then the assumptions of Theorem 5.4 also imply that $g(G) \geq \lfloor \frac{\rho}{2} \rfloor$. But we shall postpone this improvement to the next work [M] where we consider this relation in more details. It should be mentioned that for orientable Σ, the result for $g(G)$ will not hold in general since there are embeddings $\psi: G \longrightarrow \Sigma$ of a toroidal graph G with $\rho(\psi) = 4$ and genus$(\Sigma) > 1$, *cf.* [T1].

6. **Locally flat embeddings.** In this section we present some results about embeddings with large (local) width. Some of the results are recent discoveries of several authors [FHRR, RV, T1]. It should be pointed out that our proofs are direct, in contrast to the known proofs which are done by induction on the genus. Our direct approach is easier since there are no difficulties with the small-genus cases.

Let $\psi: G \longrightarrow \Sigma$ be an embedding of a graph G into $\Sigma \neq S^2$, and let f be a face of ψ. If $\rho(\psi, f) \geq 2$ then $C(f) = \partial D_0(f)$ is a cycle in G since $D_0(f)$ is a disc.

THEOREM 6.1. *Let $\psi: G \longrightarrow \Sigma$ be an embedding of a graph G into $\Sigma \neq S^2$, and let f be a face of ψ. If $\rho(\psi, f) > 2g(G) + 2$ then the cycle $C(f)$ bounds a disc in every orientable minimal genus embedding. If $\rho(\psi, f) > 2\tilde{g}(G) + 2$ and G is not orientably simple then $C(f)$ bounds a disc in every non-orientable minimal genus embedding.*

PROOF. By Proposition 5.3, there is a planarly nested sequence of cycles $C_1, \ldots, C_r$, where $r = \lceil \frac{\rho}{2} \rceil - 1$, $\rho = \rho(\psi, f)$ and $C_1 = C(f)$. By the assumption of our theorem we have $r > g$, or $r > \tilde{g}$, respectively, and we are done by Theorem 5.1. ∎

This was a "local version" of the results which follow. Although some of the following results also have more general local versions we shall not formulate them as separate statements. They are easy to see and we omit details. The following theorem, Case (a), is more or less contained in [RV].

THEOREM 6.2. *Let $\psi\colon G \longrightarrow \Sigma$ be an embedding into $\Sigma \neq S^2$.*

a) *If $\rho(\psi) > 2g(G) + 2$ then ψ is an orientable minimal genus embedding, i.e. Σ is orientable and has genus equal to $g(G)$. Any other embedding of G into Σ has the same width and it is weakly equivalent with ψ.*
b) *If $\rho(\psi) > 2\tilde{g}(G) + 1$ and G is not orientably simple then ψ is a non-orientable minimal genus embedding. Any other embedding of G into Σ has the same width and it is weakly equivalent with ψ.*
c) *If $\rho(\psi) > 2\tilde{g}(G) + 1$ then G is orientably simple if and only if $\rho(\psi) > 2g(G) + 2$, and this is also equivalent to Σ being orientable.*

PROOF. The results of Section 3 enable us to restrict to the case when G is 3-connected. Namely, it is assumed that $\rho(\psi) \geq 3$. By Proposition 3.3 we may consider the restriction $\psi|H'$ to a subdivision $H' \subseteq G$ of the core of G and finally use Theorem 4.1 to prove the general case. So assume from now on that G is 3-connected.

a) Suppose $\rho(\psi) > 2g(G) + 2$. Let f be a face of ψ. Since $C(f) = \partial f$, it follows by Theorem 6.1 that $C(f)$ bounds a disc in every orientable minimal genus embedding. By Proposition 3.5, $C(f)$ is induced and non-separating. Therefore the disc it bounds must be a face. Consequently, every face of ψ is a face in any orientable minimal genus embedding. It follows, therefore, that ψ itself is a minimal genus embedding and that it is unique. This completes the proof.
b) The proof is the same as above. It only needs an improvement of the bound $2g + 2$ of Theorem 6.1. But in the "non-local" and non-orientable case Theorem 5.4 gives precisely such an improvement when used instead of Proposition 5.3 in the proof of Theorem 6.1 above.
c) Assume that $\rho(\psi) > 2\tilde{g}(G) + 1$. If G is orientably simple then $\tilde{g}(G) = 2g(G) + 1$, so $\rho(\psi) > 2\tilde{g}(G) + 1 = 4g(G) + 3 > 2g(G) + 2$. If $\rho(\psi) > 2g(G) + 2$ then Σ is orientable by (a). Finally, if Σ is orientable, then G is orientably simple by (b). We are done. ∎

COROLLARY 6.3 ([RV]). *Let G be a 3-connected graph, and $g' = \min\{g(G), \tilde{g}(G)\}$. If there is an embedding $\psi\colon G \longrightarrow \Sigma$, $\Sigma \neq S^2$, with $\rho(\psi) > 2g' + 2$ then such an embedding is combinatorially unique and necessarily a minimal genus embedding (either orientable, or non-orientable).*

COROLLARY 6.4 ([RV]). *If G has an embedding $\psi\colon G \longrightarrow \Sigma$, $\Sigma \neq S^2$, with $\rho(\psi) > 2$ genus$(\Sigma) + 2$ then this is a minimal genus embedding.*

PROOF. Clear by Theorem 6.2 and the fact that (depending on the orientability of Σ) *genus*$(\Sigma) \geq g(G)$, or $\geq \tilde{g}(G)$, respectively. ∎

It follows from Theorem 6.2 that the embeddings with large width, if there are any, are essentially unique. There is a similar notion to the large width, the so called large edge-width, which also ensures the genus minimality and uniqueness of an embedding. *Large edge-width embeddings* are those for which every cycle in the graph, which is essential on the surface, contains more edges than any face boundary. Thomassen [T1] gives a polynomial-time algorithm to discover such an orientable embedding if there is any, for the case of 3-connected graphs. This is important since this determines the genus of graphs having such embeddings, and since the genus problem for general graphs is NP-hard [T2]. It is also known that to decide, given G and Σ, if G has an embedding ψ into Σ with $\rho(\psi) \geq 3$ is NP-complete [T3]. We believe that it is possible to decide in polynomial time if G has an embedding ψ into some surface Σ such that $\rho(\psi) > 2\,\mathrm{genus}(\Sigma) + 2$.

Another application is an easy recognition of several examples of orientably simple graphs.

COROLLARY 6.5. *If G has an embedding $\psi: G \longrightarrow \Sigma$ with $\rho(\psi) > 4g(G) + 1$, then G is orientably simple.*

PROOF. Every planar graph is orientably simple. Therefore we may assume that $\Sigma \neq S^2$ and $g(G) \geq 1$. Denote by $g = g(G)$. Since $\rho(\psi) > 4g + 1$ it is also $\rho(\psi) > 2g + 2$. If G is not orientably simple then by Theorem 6.2(c), $\rho(\psi) \leq 2\tilde{g}(G) + 1$. It follows that $4g + 1 < 2\tilde{g}(G) + 1$, and $\tilde{g}(G) > 2g$ which implies that G must be orientably simple, contrary to our assumption. ∎

At the end let us mention another special case—embeddings into the projective plane. Uniqueness of such embeddings was largely investigated by several authors [B, N1, N2, L]. The following result in terms of the planarity width seems to be the most natural one.

COROLLARY 6.6. *Let $\psi: G \longrightarrow \tilde{\Sigma}_1$ be an embedding of a graph G into the projective plane. If $\rho(\psi) \geq 4$ then any other embedding of G into the projective plane is weakly equivalent with ψ. In particular, if G is 3-connected then the embedding is combinatorially unique.*

There are 4-connected graphs having more than one embedding into the projective plane with planarity width $\rho = 3$. See Figure 3. The edges are assumed to be on the "boundary" (with the usual identification of opposite points). The embeddings are not equivalent since in (a) the triangle a, b, c is an essential cycle while in (b) it is contractible.

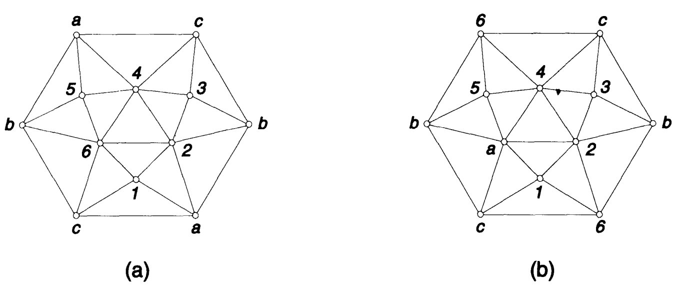

FIGURE 3. Triangulations of the projective plane

https://doi.org/10.4153/CJM-1992-076-8 Published online by Cambridge University Press

It is known [FHRR] that a graph embedded in the projective plane, $\psi: G \longrightarrow \tilde{\Sigma}_1$, has the orientable genus $g(G) = \lfloor \frac{\rho(\psi)}{2} \rfloor$ if $\rho(\psi) \neq 2$. If $\rho(\psi) = 2$, G can either be planar or toroidal. Corollary 6.6 can then be formulated another way:

COROLLARY 6.7. *A 3-connected projective planar graph G with $g(G) \geq 2$ admits a unique embedding into the projective plane.*

Clearly, if $g(G) = 0$ then there are several embeddings of G into the projective plane (*e.g.* non-cellular embeddings and various embeddings with $\rho = 1$). The only open question remains for toroidal graphs.

ACKNOWLEDGEMENT. I am greatly indebted to Steve Fisk for a careful reading of the manuscript.

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