ON A GENERALISATION OF MONOTONIC SEQUENCES

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1. Introduction

A bounded monotonic sequence is convergent. Dr J. M. Whittaker recently suggested to me a generalisation of this result, that, if a bounded sequence $\{a_n\}$ of real numbers satisfies the inequality

$$a_{n+2} \leqq \tfrac{1}{2}(a_{n+1}+a_n), \tag{1}$$

then it is convergent. This I was able to prove by considering the corresponding difference equation

$$A_{n+2} = \tfrac{1}{2}(A_{n+1}+A_n).$$

Dr J. B. Tatchell gave me a different proof depending on the fact that (1) is equivalent to saying that the sequence $\{a_{n+1}+\frac{1}{2}a_n\}$ is bounded and decreasing. His argument also applied in the case of the difference inequality

$$a_{n+2} \leqq (1-k)a_{n+1}+ka_n,$$

where k and $1-k$ are strictly positive. This suggested that there should be a more general result in which the mean of a_n and a_{n+1} is replaced by a mean of r consecutive members of the sequence. In this paper I prove the following

Theorem. *If $\{a_n\}$ is a bounded sequence which satisfies the inequality*

$$a_{n+r} \leqq \sum_{s=1}^{r} k_s a_{n+r-s} \tag{2}$$

where the coefficients k_s are strictly positive and $k_1+k_2+\ldots+k_r = 1$, then $\{a_n\}$ is a convergent sequence. But if $\{a_n\}$ is unbounded, it diverges to $-\infty$.

The conclusion does not necessarily follow if some of the coefficients k_s are zero. For example, if $\{a_n\}$ is bounded and

$$a_{n+4} \leqq \tfrac{1}{2}(a_{n+2}+a_n),$$

then the sequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are convergent, but $\{a_n\}$ is not necessarily convergent.

2. A proof of the theorem

My proof depends on the properties of the associated difference equation. But I first give an interesting proof due to Professor R. A. Rankin.

Let us write

$$A_n = \max(a_{n-1}, a_{n-2}, \ldots, a_{n-r}).$$

Then, by (2),

$$a_n \leqq A_n \tag{3}$$

and so $A_{n+1} \leqq A_n$. Therefore, either A_n tends to a finite limit A or A_n diverges to $-\infty$.

If $A_n \to -\infty$, then $a_n \to -\infty$ by (3). We show that, if A is finite, $a_n \to A$. For any positive value of ε, there exists a positive integer N such that

$$A \leqq A_n \leqq A+\varepsilon$$

whenever $n \geqq N$. If $1 \leqq s \leqq r$, we have

$$a_{m+s} \leqq k_s a_m + \sum_{t \neq s} k_t a_{m+s-t} \leqq k_s a_m + \sum_{t \neq s} k_t A_{m+s}$$

$$= (1-k_s)A_{m+s} + k_s a_m \leqq (1-k_s)(A+\varepsilon) + k_s a_m.$$

For each $m \geqq N$, we can find an integer s $(1 \leqq s \leqq r)$ such that

$$a_{m+s} = A_{m+r+1}.$$

Hence

$$A \leqq A_{m+r+1} = a_{m+s} \leqq (1-k_s)(A+\varepsilon)+k_s a_m = a_m+(1-k_s)(A+\varepsilon-a_m).$$

But $a_m \leqq A_m \leqq A+\varepsilon$. Therefore if k is the least of the coefficients k_s,

$$A \leqq a_m+(1-k)(A+\varepsilon-a_m) = ka_m+(1-k)(A+\varepsilon)$$

from which it follows that

$$a_n \geqq A - \frac{1-k}{k}\,\varepsilon,$$

where $0<k<1$. We have thus proved that, for every positive value of ε, there exists an integer N such that, whenever $m \geqq N$,

$$A - \frac{1-k}{k}\,\varepsilon \leqq a_m \leqq A+\varepsilon;$$

hence a_m tends to A as $m \to \infty$.

3. Another proof

Lemma. *Under the conditions of the theorem, every solution A_n of the difference equation*

$$A_{n+r} = \sum_{s=1}^{r} k_s A_{n+r-s}$$

tends to a finite limit as $n \to \infty$.

If the roots $z_1, z_2, \ldots, z_r$ of the equation

$$z^r = \sum_{s=1}^{r} k_s z^{r-s} \tag{4}$$

are distinct, the general solution of the difference equation is

$$A_n = \sum_{s=1}^{r} \alpha_s z_s^n.$$

If the roots are not distinct, the solution has to be modified. For example, if $z_1 = z_2$, the first two terms have to be replaced by $(\alpha+\beta n)z_1^n$; if $z_1 = z_2 = z_3$, the first three terms have to be replaced by $(\alpha+\beta n+\gamma n^2)z_1^n$; and so on. But this does not affect the truth of the lemma.

By a straightforward application of Rouché's Theorem, we can show that all the roots of (4) lie in $|z| \leqq 1$; and, by elementary trigonometry, the only root on $|z| = 1$ is a simple root at $z = 1$. The truth of the lemma is then evident.

The sequence $\{a_n\}$ satisfies

$$a_{n+2} \leqq \sum_{s=1}^{r} k_s a_{n+r-s},$$

where the coefficients k_s are strictly positive and have sum unity. If we replace a_{n+r-1} by

$$\sum_{s=1}^{r} k_s a_{n+r-1-s}$$

in the expression on the right-hand side, we increase the right-hand side, getting

$$a_{n+r} \leqq \sum_{s=1}^{r-1} (k_1 k_s + k_{s+1}) a_{n-r-1-s} + k_1 k_r a_{n-1}.$$

Repeating the process, we obtain

$$a_{n+r} \leqq \sum_{s=1}^{r} A_s(l) a_{n-l+r-s} \tag{5}$$

for every integer $l \leqq n$. Here $A_s(0) = k_s$. The coefficients $A_s(l)$ are given by the recurrence relations

$$A_s(l+1) = k_s A_1(l) + A_{s+1}(l) \tag{6}$$

for $s = 1, 2, \ldots, r-1$, and

$$A_r(l+1) = k_r A_1(l). \tag{7}$$

Evidently

$$\sum_{s=1}^{r} A_s(l+1) = \sum_{s=1}^{r} A_s(l),$$

and so

$$\sum_{s=1}^{r} A_s(l) = \sum_{s=1}^{r} A_s(0) = \sum_{s=1}^{r} k_s = 1. \tag{8}$$

From equations (6) and (7), we find that

$$A_1(l+r) = \sum_{s=1}^{r} k_s A_1(l+r-s),$$

which is the difference equation of the lemma. Hence $A_1(l)$ tends to a finite

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limit α_1 as $l \to \infty$. Making l tend to infinity in (6) and (7), we find that

$$A_2(l) \to \alpha_2 = (1-k_1)\alpha_1,$$

$$A_3(l) \to \alpha_3 = (1-k_1-k_2)\alpha_1$$

and so on;

$$A_s(l) \to \alpha_s = \alpha_1 \sum_{t=s}^{r} k_t.$$

But, by (8),

$$\sum_{s=1}^{r} \alpha_s = 1,$$

from which it follows that

$$\alpha_1 = \frac{1}{k_1+2k_2+3k_3+\ldots+rk_r}.$$

Since the coefficients k_s are strictly positive and have sum unity, we see that $0<\alpha_1<1$.

In the inequality (5), put $l = n+r-m$. Then

$$a_{n+r} \leqq \sum_{s=1}^{r} A_s(n+r-m)a_{m-s}.$$

Now make $n\to\infty$. This gives

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} a_{n+r}$$

$$\leqq \sum_{s=1}^{r} \alpha_s a_{m-s}. \tag{9}$$

Write this as

$$\limsup a_n + \sum_{s=2}^{r} (-\alpha_s)a_{m-s} \leqq \alpha_1 a_{m-1}.$$

Since $\alpha_1>0$,

$$\alpha_1 \liminf_{m\to\infty} a_m = \alpha_1 \liminf_{m\to\infty} a_{m-1}$$

$$\geqq \limsup_{n\to\infty} a_n + \liminf_{m\to\infty} \sum_{s=2}^{r} (-\alpha_s)a_{m-s}.$$

But each α_s is positive. Hence

$$\alpha_1 \liminf_{n\to\infty} a_n \geqq \limsup_{n\to\infty} a_n - \sum_{s=2}^{r} \alpha_s \limsup_{n\to\infty} a_n.$$

But the sum of all the coefficients α_s is unity, and $\alpha_1>0$. Hence

$$\alpha_1 \liminf_{n\to\infty} a_n \geqq \alpha_1 \limsup_{n\to\infty} a_n,$$

or

$$\liminf a_n \geqq \limsup a_n. \tag{10}$$

If $\{a_n\}$ is a bounded sequence, $\limsup a_n$ and $\liminf a_n$ are both finite, and $\liminf a_n \leqq \limsup a_n$. Therefore, by (10), $\limsup a_n$ and $\liminf a_n$ are equal; the sequence converges.

is finite, by (10) so also is lim inf a_n, which is impossible since the sequence is unbounded. Therefore lim sup $a_n = -\infty$; the sequence diverges to $-\infty$.

4. Further remarks on the theorem

The condition of the theorem are sufficient, but not necessary; the coefficients k_s need not be all positive. For example, if $\{a_n\}$ is a bounded sequence satisfying

$$a_{n+3} \leqq -\tfrac{1}{2}a_{n+2}+\tfrac{3}{4}a_{n+1}+\tfrac{3}{4}a_n,$$

then it is a convergent sequence.

The key to the second proof of the theorem is that, if the coefficients k_s are strictly positive and have sum unity, every solution of the difference equation

$$A_{n+r} = \sum_{s=1}^{r} k_s A_{n+r-s}$$

tends to a finite limit as $n \to \infty$, because the equation

$$z^r - \sum_{s=1}^{r} k_s z^{r-s} = 0$$

has one root $z = 1$ on the unit circle and $r-1$ roots in $|z|<1$; or, if we take out the factor $z-1$, all the roots of

$$z^{r-1} + \sum_{s=1}^{r-1} l_s z^{r-s-1} = 0, \tag{11}$$

where

$$l_s = 1-k_1-k_2-\dots-k_s,$$

lie in $|z|<1$.

A polynomial

$$g(z) = \sum_{0}^{m} c_r z^r \quad (c_0 \neq 0,\ c_m \neq 0)$$

whose zeros all lie in $|z|<1$ is called a *Schur polynomial.* Duffin [*SIAM Review*, **11** (1969), 196-213] has shown that $g(z)$ is a Schur polynomial if and only if $|c_0|<|c_m|$ and

$$g_1(z) = \sum_{0}^{m-1} (\bar{c}_m c_{r+1} - c_0 \bar{c}_{m-r-1}) z^r,$$

where bars denote complex conjugates, is also a Schur polynomial. This algorithm enables one to test whether a given polynomial is a Schur polynomial, but it does not provide a simple set of conditions on the coefficients c_r.

If the polynomial on the left-hand side of (11) is a Schur polynomial, the argument of § **3** shows that, as $l \to +\infty$,

$$A_1(l) \to \alpha_1, \quad A_s(l) \to \alpha_s = l_{s-1}\alpha_1,$$

where

$$\alpha_1 = \frac{1}{1+l_1+l_2+\ldots+l_{r-1}}.$$

Since $z = 1$ is not a root of equation (11),

$$1+l_1+l_2+\ldots+l_{r-1} \neq 0.$$

As in § 3, we obtain

$$\limsup_{n\to\infty} a_n \leqq \sum_{s=1}^{r} \alpha_s a_{m-s}.$$

Since the sum of the coefficients α_s is unity, the largest, α_k say, is positive. Write

$$\beta_s = \alpha_s \text{ if } \alpha_s > 0, \quad \gamma_s = 0 \quad \text{ if } a_s > 0,$$
$$= 0 \text{ if } \alpha_s \leqq 0, \quad \gamma_s = -\alpha_s \text{ if } \alpha_s \leqq 0,$$

so that $\alpha_s = \beta_s - \gamma_s$. Then

$$\limsup_{n\to\infty} a_n - \Sigma'\beta_s a_{m-s} + \Sigma\gamma_s a_{m-s} \leqq \alpha_k a_{m-k}, \tag{12}$$

where the prime indicates that the term with $s = k$ is omitted. If only one α_s is positive, the sum Σ' does not occur.

From the inequality (12) it follows that

$$(1-\Sigma'\beta_s) \limsup a_n + \Sigma\gamma_s \liminf a_n \leqq \alpha_k \liminf a_n.$$

But

$$\alpha_k + \Sigma'\beta_s - \Sigma\gamma_s = 1.$$

Hence

$$(1-\Sigma'\beta_s)(\limsup a_n - \liminf a_n) \leqq 0.$$

The conclusion will therefore follow as before if $\Sigma'\beta_s < 1$. This condition is satisfied if there is only one positive α_s or if the sum of all the positive α_s except the greatest is less than unity.

The method of this section will enable one to test whether a bounded sequence $\{a_n\}$ satisfying the equality

$$a_{n+r} \leqq \sum_{s=1}^{r} k_s a_{n+r-s},$$

where the coefficients k_s are not all strictly positive, but have sum unity, is convergent. It does not seem to be possible to give any simple general necessary and sufficient conditions.

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