Can. J. Math., Vol. XXIV, No. 6, 1972, pp. 1145–1153

SERIES EXPANSIONS FOR DUAL LAGUERRE TEMPERATURES

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1. Introduction. In a recent paper [**2**], the author, with F. M. Cholewinski, derived criteria for the series expansions of solutions $u(x, t)$ of the Laguerre differential heat equation $xu_{xx} + (\alpha + 1 - x)u_x = u_t$ in terms of the Laguerre heat polynomials and of their temperature transforms. Our present goal is the characterization of those solutions which are representable in a Maclaurin double series in xe^{-t} and in $1 - e^{-t}$. Some of the results are analogous to those derived by D. V. Widder in [**4**] for the classical heat equation and by the author in [**1**] for the generalized heat equation.

2. Definitions. The Laguerre differential heat equation is given by

$$\nabla_x u(x, t) = (\partial/\partial t)u(x, t) \tag{2.1}$$

where

$$\nabla_x f(x) = xf''(x) + (\alpha + 1 - x)f'(x).$$

We denote by H the class of all C^2 solutions of (2.1) and refer to a member of H as a dual Laguerre temperature.

The fundamental solution of (2.1) is the function

$$g(x; t) = \left[\frac{e^t}{e^t - 1}\right]^{\alpha+1} e^{-x/(e^t-1)}, \quad t > 0, \tag{2.2}$$

whose associate function is

$$g(x, y; t) = \left[\frac{e^t}{e^t - 1}\right]^{\alpha+1} e^{-(x+y)/(e^t-1)} \mathscr{I}\left[\frac{2(xye^t)^{\frac{1}{2}}}{e^t - 1}\right], \quad t > 0, \tag{2.3}$$

where

$$\mathscr{I}(z) = 2^\alpha \Gamma(\alpha + 1) z^{-\alpha} I_\alpha(z),$$

$I_\alpha(z)$ being the ordinary Bessel function of imaginary argument.

The dual Laguerre temperature transform $u^T(x, t)$ of a function $u(x, t) \in H$ is given by

$$u^T(x, t) = g(x; t)u(x/(e^t - 1), \ln(1 - e^{-t})), \quad t > 0. \tag{2.4}$$

Received October 26, 1971 and in revised form, December 21, 1971. This research was supported by the Air Force Office of Scientific Research Grant No. 71-2047.

A subclass H^* of H basic to our needs consists of those dual Laguerre temperatures for which

$$u(x, t) = \int_0^\infty g(x, y; t - t')u(y, t')d \wedge (y), \tag{2.5}$$

$$d \wedge (y) = \frac{1}{\Gamma(\alpha + 1)} e^{-y}y^\alpha dy,$$

for every $t, t', a < t' < t < b$, with the integral converging absolutely. A member of H^* is said to have the Huygens property.

In addition, we need the class (ρ, τ) which includes those entire functions $f(x) = \sum_{n=0}^\infty a_n x^n$ for which

$$\overline{\lim_{n\to\infty}} \frac{n}{e\rho} |a_n|^{\rho/n} \leqq \tau. \tag{2.6}$$

The Laguerre heat polynomials $p_{n,\alpha}(x, t)$ are given by

$$p_{n,\alpha}(x, t) = \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(n + \alpha + 1)}{\Gamma(n - k + \alpha + 1)} (xe^{-t})^{n-k}(1 - e^{-t})^k. \tag{2.7}$$

They are the Cauchy solutions (x, t) of (2.1) satisfying the initial condition $u(x, 0) = x^n$. Their dual Laguerre temperature transforms $w_{n,\alpha}(x, t)$ may be given in the form

$$w_{n,\alpha}(x, t) = (e^t - 1)^{-2n} g(x; t) p_{n,\alpha}(x, -t),\ t > 0. \tag{2.8}$$

From the basic generating relationship

$$g(x, y; t + t') = \sum_{n=0}^\infty \frac{\Gamma(\alpha + 1)}{n!\Gamma(n + \alpha + 1)} p_{n,\alpha}(x, t) w_{n,\alpha}(y, t') \tag{2.9}$$

derived in [**3**], we have, by a direct computation,

$$g(x, y; t + t') = \sum_{m=0}^\infty \frac{(xe^{-t})^m}{m!\Gamma(m + \alpha + 1)} \sum_{k=0}^\infty \frac{(1 - e^{-t})^k}{k!} \Gamma(\alpha + 1) w_{m+k,\alpha}(y, t'). \tag{2.10}$$

3. Region of convergence. We establish the region of convergence of the double Maclaurin series involved in our development.

LEMMA 3.1. *If, for* $\gamma \geqq 0$,

$$\overline{\lim_{n\to\infty}} \frac{e|a_n|^{1/n}}{n} = \gamma, \tag{3.1}$$

then the series

$$\sum_{m=0}^\infty \frac{(xe^{-t})^m}{m!\Gamma(m + \alpha + 1)} \sum_{k=0}^\infty a_{m+k} \frac{(1 - e^{-t})^k}{k!} \tag{3.2}$$

converges for $t \in \mathscr{D}_\gamma$, *where*

(3.3) $$\mathscr{D}_\gamma = \left\{ t \mid \ln \frac{\gamma}{1+\gamma} < t < \infty \text{ if } 0 \leqq \gamma \leqq 1 \text{ and } \ln \frac{\gamma}{1+\gamma} < t < \ln \frac{\gamma}{\gamma - 1} \text{ if } \gamma > 1 \right\}.$$

Proof. For any θ, $0 < \theta < 1$, we have, as a consequence of (3.1),

$$|a_n| < K \left(\frac{\gamma n}{\theta e} \right)^n$$

for some constant K and n sufficiently large. Hence

$$\begin{aligned} I &= \sum_{m=0}^{\infty} \frac{(xe^{-t})^m}{m!\Gamma(m+\alpha+1)} \sum_{k=0}^{\infty} |a_{m+k}| \frac{|1-e^{-t}|^k}{k!} \\ &\leqq K \sum_{m=0}^{\infty} \frac{(xe^{-t})^m}{m!\Gamma(m+\alpha+1)} \sum_{k=0}^{\infty} \left[\frac{\gamma(m+k)}{e\theta} \right]^{m+k} \frac{|1-e^{-t}|^k}{k!} \\ &= K \sum_{k=0}^{\infty} \frac{|1-e^{-t}|^k}{k!} \sum_{m=k}^{\infty} \left[\frac{m\gamma}{e\theta} \right]^m \frac{(xe^{-t})^{m-k}}{(m-k)!\Gamma(m-k+\alpha+1)}. \end{aligned}$$

Now, if $t > 0$,

$$I \leqq K \sum_{m=0}^{\infty} \left(\frac{m\gamma}{e\theta} \right)^m \frac{1}{m!\Gamma(m+\alpha+1)} p_{m,\alpha}(x, t),$$

and an appeal to (4.8) of [**3**] yields the dominating series

$$I \leqq K \sum_{m=0}^{\infty} \left(\frac{m\gamma}{e\theta} \right)^m \frac{1}{m!\Gamma(m+\alpha+1)} m^{\frac{1}{2}\alpha+\frac{1}{4}} \left[\frac{m}{e} (1-e^{-t}) \right]^m e^{2(mx/(e^t-1))^{\frac{1}{2}}}$$

which converges for

$$\gamma(1-e^{-t})/\theta < 1,$$

or, on taking θ arbitrarily close to 1, for

$$\gamma(1-e^{-t}) < 1.$$

Hence, if $0 \leqq \gamma \leqq 1$, the series (3.2) converges for all $t > 0$, whereas if $\gamma \geqq 1$, the series converges for $0 < t < \ln(\gamma/(\gamma - 1))$.

On the other hand, if $t < 0$,

$$I \leqq K \sum_{m=0}^{\infty} \left(\frac{m\gamma}{e\theta} \right)^m \frac{1}{m!\Gamma(m+\alpha+1)} (-1)^m p_{m,\alpha}(-x, t),$$

and an appeal to (4.6) of [**3**] yields the dominating series

$$I \leqq A \sum_{m=0}^{\infty} \left(\frac{m\gamma}{e\theta} \right)^m \frac{1}{m!\Gamma(m+\alpha+1)} m^{\alpha+1} \left[\frac{m}{e} (e^{-t} - 1) \right]^m$$

which converges, since θ may be taken arbitrarily close to 1, for

$$\gamma(e^{-t} - 1) < 1;$$

that is, for $\gamma \geqq 0$, for $\ln(\gamma/(1+\gamma)) < t < 0$. The proof is thus complete.

Note that if the series (3.2) were to converge at some point (x_0, t_0), $t_0 \notin \mathscr{D}_\gamma$, then, in particular, the simple series

$$\sum_{k=0}^{\infty} a_k \frac{(1-e^{-t_0})^k}{k!}$$

must also converge, and it would follow that

$$\overline{\lim_{k\to\infty}} \left|\frac{a_k}{k!}\right|^{1/k} = \overline{\lim_{k\to\infty}} \left|\frac{a_k e}{k}\right| \leqq \frac{1}{|1-e^{-t_0}|}$$

contradicting hypothesis (3.1).

4. Series expansion. We now establish our principal result.

THEOREM 4.1. *A necessary and sufficient condition that a solution* $u(x, t)$ *of the Laguerre differential heat equation* (2.1) *have the double Maclaurin expansion*

$$u(x,t) = \sum_{m=0}^{\infty} \frac{(xe^{-t})^m}{m!\Gamma(m+\alpha+1)} \sum_{k=0}^{\infty} a_{m+k} \frac{(1-e^{-t})^k}{k!} \tag{4.1}$$

with

$$a_k = \Gamma(k+\alpha+1)[(\partial/\partial x)^k u(x,0)]_{x=0} \tag{4.2}$$

for $t \in \mathscr{D}_\gamma$ *is that* $u(x, t) \in H^*$ *for* $t \in \mathscr{D}_\gamma$.

Proof. To prove sufficiency, assume that $u(x, t) \in H^*$ for $t \in \mathscr{D}_\gamma$. Then, for t, t' with $\ln(\gamma/(\gamma+1)) < t' < t < \ln(\gamma/(\gamma-1)) < \infty$, we have

$$u(x,t) = \int_0^\infty g(x, y; t+t')u(y, -t')d\wedge(y) \tag{4.3}$$

with the integral converging absolutely. We choose $t' > 0$. On substituting (2.10) in (4.3) and on interchanging integration with summation, we have

$$u(x,t) = \sum_{m=0}^{\infty} \frac{(xe^{-t})^m}{m!\Gamma(m+\alpha+1)} \sum_{k=0}^{\infty} \frac{(1-e^{-t})^k}{k!} \times \Gamma(\alpha+1)\int_0^\infty u(y, -t')w_{m+k}(y, t')d\wedge(y). \tag{4.4}$$

That termwise integration is justified is a consequence of the fact that an appeal to (4.14) of [3] yields, for $\delta > 0$,

$$\sum_{m=0}^{\infty} \frac{(xe^{-t})^m}{m!\Gamma(m+\alpha+1)} \sum_{k=0}^{\infty} \frac{|1-e^{-t}|^k}{k!} \times \Gamma(\alpha+1)\int_0^\infty |u(y,-t')||w_{m+k}(y,t')|d\wedge(y)$$
$$\leqq A \sum_{m=0}^{\infty} \frac{(xe^{-t})^m}{m!\Gamma(m+\alpha+1)} \sum_{k=0}^{\infty} \frac{(1-e^{-t})^k}{k!} \left(\frac{m+k}{e(e^{t+\delta}-1)}\right)^{m+k} \times \int_0^\infty e^{y(e\delta-2)/(e\delta-1)}|u(y,-t')|d\wedge(y).$$

The rightmost integral converges by Lemma 7.4 of [**3**], and the series clearly converges by an argument similar to that used in the proof of Lemma 3.1.

Now, setting

$$a_k = \Gamma(\alpha + 1) \int_0^\infty u(y, -t')w_k(y, t')d \wedge (y) \tag{4.5}$$

and noting, by Corollary 7.2 of [**3**], that the integral of (4.5) is independent of t, we have, on substituting (4.5) in (4.4), $u(x, t)$ given by the double series as required. Further, since

$$u(x, 0) = \sum_{m=0}^\infty a_m \frac{x^m}{m!\Gamma(m + \alpha + 1)},$$

the determination of the coefficients a_k by (4.2) is immediate.

Conversely, to prove the necessity of the condition, assume that u has the series expansion (4.1) for $t \in \mathscr{D}_\gamma$. Now, for t, t', with

$$\ln (\gamma/(\gamma + 1)) < t' < \ln (\gamma/(\gamma - 1)) \leqq \infty,$$

we have

$$\begin{aligned}
&\int_0^\infty g(x, y; t - t')u(y, t')d \wedge (y) \\
&= \int_0^\infty g(x, y; t - t')d \wedge (y) \sum_{m=0}^\infty \frac{(ye^{-t'})^m}{m!\Gamma(m + \alpha + 1)} \sum_{k=0}^\infty a_{m+k} \frac{(1 - e^{-t'})^k}{k!} \\
&= \int_0^\infty g(x, y; t - t')d \wedge (y) \sum_{k=0}^\infty \frac{(1 - e^{-t'})^k}{k!} \sum_{m=k}^\infty \frac{a_m(ye^{-t'})^{m-k}}{(m - k)!\Gamma(m - k + \alpha + 1)} \\
&= \int_0^\infty g(x, y; t - t')d \wedge (y) \sum_{m=0}^\infty \frac{a_m}{m!\Gamma(m + \alpha + 1)} p_{m,\alpha}(y, t') \\
&= \sum_{m=0}^\infty \frac{a_m}{\mathrm{m}!\Gamma(m + \alpha + 1)} p_{m,\alpha}(x, t),
\end{aligned}$$

where we have used the fact that $p_{n,\alpha}(x, t) \in H^*$ for all t, and where termwise integration can be justified by appeals to (4.4) and (4.8) of [**3**]. Hence, using the definition of $p_{n,\alpha}(x, t)$ we have

$$\begin{aligned}
&\int_0^\infty g(x, y; t - t')u(y, t')d \wedge (y) \\
&\quad = \sum_{m=0}^\infty \frac{a_m}{m!\Gamma(m + \alpha + 1)} \sum_{k=0}^\infty \binom{m}{k} \frac{\Gamma(m + \alpha + 1)}{\Gamma(m - k + \alpha + 1)} (xe^{-t})^{m-k}(1 - e^{-t})^k \\
&\quad = \sum_{k=0}^\infty \frac{(1 - e^{-t})^k}{k!} \sum_{m=0}^\infty \frac{a_{m+k}}{m!} \frac{(xe^{-t})^m}{\Gamma(m + \alpha + 1)} \\
&\quad = u(x, t)
\end{aligned}$$

so that $u(x, t) \in H^*$ as required, and the proof is complete.

https://doi.org/10.4153/CJM-1972-122-8 Published online by Cambridge University Press

Theorem 8.1 of [**3**] provides the following restatement of the theorem.

COROLLARY 4.2. *For* $t \in \mathscr{D}_\gamma$,

$$u(x, t) = \sum_{m=0}^{\infty} \frac{(xe^{-t})^m}{m!\Gamma(m + \alpha + 1)} \sum_{k=0}^{\infty} a_{m+k} \frac{(1 - e^{-t})^k}{k!} \tag{4.6}$$

if and only if

$$u(x, t) = \sum_{n=0}^{\infty} \frac{a_n}{n!\Gamma(n + \alpha + 1)} p_{n,\alpha}(x, t). \tag{4.7}$$

An example illustrating the theorem is given by

$$u(x, t) = e^{a(1-e^{-t})} \mathscr{I} (2(xae^{-t})^{\frac{1}{2}}), \tag{4.8}$$

a function belonging to H^* for all t. We have, in this case,

$$u(x, t) = \Gamma(\alpha + 1) \sum_{k=0}^{\infty} \frac{(1 - e^{-t})^k}{k!} \sum_{m=0}^{\infty} \frac{a^{m+k}(xe^{-t})^m}{m!\Gamma(m + \alpha + 1)}$$

which is (4.1) with

$$a_k = \Gamma(\alpha + 1)a^k$$

as predicted by (4.2).

5. Simple series expansions. We establish the fact that if the double series (4.1) is summed by columns, a dual Laguerre temperature with the Huygens property may be represented by a simple Maclaurin series in x.

THEOREM 5.1. *If* $u(x, t) \in H^*$ *for* $t \in \mathscr{D}_\gamma$, *and if* $g(t) = u(0, t)$, *then, for* $t \in \mathscr{D}_\gamma$,

$$u(x, t) = \sum_{m=0}^{\infty} \frac{\Gamma(\alpha + 1)}{m!\Gamma(m + \alpha + 1)} g^{(m)}(t)x^m. \tag{5.1}$$

Proof. By Theorem 4.1, we have

$$u(x, t) = \sum_{n=0}^{\infty} \frac{(xe^{-t})^n}{n!\Gamma(n + \alpha + 1)} \sum_{m=0}^{\infty} a_{n+m} \frac{(1 - e^{-t})^m}{m!} \tag{5.2}$$

so that

$$\begin{aligned} g(t) &= u(0, t) \\ &= \frac{1}{\Gamma(\alpha + 1)} \sum_{m=0}^{\infty} a_m \frac{(1 - e^{-t})^m}{m!}. \end{aligned} \tag{5.3}$$

Hence, successive differentiation yields

$$g^{(k)}(t) = \frac{1}{\Gamma(\alpha + 1)} \sum_{m=0}^{\infty} \frac{a_{m+k}}{m!} (1 - e^{-t})^m (e^{-t})^k. \tag{5.4}$$

Substituting (5.4) in (5.2), we obtain (5.1) as required.

We note that the example of (4.8) illustrates the theorem since, in this case,

$$g(t) = e^{a(1-e^{-t})}$$

so that

$$g^{(k)}(t) = (ae^{-t})^k e^{a(1-e^{-t})}.$$

We then have

$$e^{a(1-e^{-t})} \mathscr{I} (2(xae^{-t})^{\frac{1}{2}}) = \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+1)}{m!\Gamma(m+\alpha+1)} (ae^{-t})^m e^{a(1-e^{-t})} x^m$$

as expected.

COROLLARY 5.2. *There exists a solution $u(x, t)$ of the Laguerre difference heat equation which is equal to its Maclaurin double series expansion in xe^{-t} and $1 - e^{-t}$ for $t \in \mathscr{D}_\gamma$ and $u(0, t) = g(t)$ if and only if $g(t)$ is equal to its Maclaurin expansion in $(1 - e^{-t})$ for $t \in \mathscr{D}_\gamma$.*

Proof. The necessity of the condition is a consequence of the theorem. To establish sufficiency, set

$$g(t) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(\alpha+1)n!} (1 - e^{-t})^n$$

and form the series

$$\sum_{m=0}^{\infty} \frac{\Gamma(\alpha+1)}{m!\Gamma(m+\alpha+1)} g^{(m)}(t) x^m. \tag{5.5}$$

Since the series defining $g(t)$ is assumed to converge for $t \in \mathscr{D}_\gamma$, it follows that

$$\overline{\lim_{n\to\infty}} \left|\frac{a_n}{n!}\right|^{1/n} \leqq \gamma.$$

But by Lemma 3.1 this inequality is sufficient for the convergence of the series (5.2) for $t \in \mathscr{D}_\gamma$ and for its being equal to the series (5.5) for $t \in \mathscr{D}_\gamma$.

An alternative simple series expansion may be derived if the double series (4.1) is summed by rows as indicated in the following result.

THEOREM 5.3. *Let $u(x, t) \in H^*$ for $t \in \mathscr{D}_\gamma$ and let $f(x) = u(x, 0)$. Then*

$$u(x, t) = \sum_{k=0}^{\infty} \frac{(e^t - 1)^k}{k!} A_x^k f(xe^{-t}) \tag{5.6}$$

where

$$A_x f(x) = xf''(x) + (\alpha + 1)f'(x) \tag{5.7}$$

and f belongs to class $(1, \gamma)$.

Proof. By the principal theorem, we have, for $t \in \mathscr{D}_\gamma$, since $u(x, t) \in H^*$ there, that

$$u(x, t) = \sum_{k=0}^{\infty} \frac{(1 - e^{-t})^k}{k!} \sum_{m=0}^{\infty} \frac{a_{m+k}(xe^{-t})^m}{m!\Gamma(m+\alpha+1)}. \tag{5.8}$$

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Hence

$$f(x) = u(x, 0) = \sum_{m=0}^{\infty} \frac{a_m x^m}{m!\Gamma(m + \alpha + 1)}. \tag{5.9}$$

Now, successive applications of the operator A_x to $f(xe^{-t})$ yield

$$A_x^k f(xe^{-t}) = \sum_{m=0}^{\infty} \frac{a_{m+k}(xe^{-t})^m (e^{-t})^k}{m!\Gamma(m + \alpha + 1)} \tag{5.10}$$

so that on substituting (5.10) in (5.8), we obtain (5.6) as required. Further, since $f(x)$ is given by the series (5.9) which converges for $t \in \mathscr{D}_\gamma$, the conditions that f belong to class $(1, \gamma)$ are satisfied and the proof is complete.

COROLLARY 5.4. *There exists a dual Laguerre temperature $u(x, t)$ which is equal to its Maclaurin double series for $t \in \mathscr{D}_\gamma$ and which reduces to $f(x)$ at $t = 0$ if and only if f belongs to class $(1, \gamma)$.*

Proof. The necessity of the condition follows from the theorem. To establish sufficiency, we assume that f belongs to class $(1, \gamma)$ and is given by the series

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!\Gamma(n + \alpha + 1)}.$$

Then

$$A_x^k f(xe^{-t}) = (e^{-t})^k \sum_{m=0}^{\infty} \frac{a_{m+k}(xe^{-t})^m}{m!\Gamma(m + \alpha + 1)}.$$

Now consider the series

$$\sum_{k=0}^{\infty} \frac{(e^t - 1)^k}{k!} A_x^k f(xe^{-t}). \tag{5.11}$$

Since f belongs to class $(1, \gamma)$, we have that

$$\overline{\lim_{n\to\infty}} \frac{n}{e} \left[\frac{|a_n|}{n!\Gamma(n + \alpha + 1)}\right]^{1/n} = \overline{\lim_{n\to\infty}} \frac{e}{n} |a_n|^{1/n} \leqq \gamma$$

so that the series (5.11) converges for $t \in \mathscr{D}_\gamma$ and represents there the dual Laguerre temperature $u(x, t)$ sought. Clearly $u(x, 0) = f(x)$.

As an example illustrating the corollary, consider, for $t_0 > \ln 2$, the function

$$f(x) = g(x; t_0).$$

It clearly belongs to class $(1, 1/(e^{t_0} - 1))$, and as predicted by the corollary, there is a dual Laguerre temperature

$$u(x, t) = g(x; t + t_0) = \sum_{k=0}^{\infty} \frac{(1 - e^{-t})^k}{k!} \sum_{m=0}^{\infty} \left[\frac{(-1)^{m+k}(e^{t_0})^{\alpha+1}\Gamma(m + k + \alpha + 1)}{(e^{t_0} - 1)^{m+k+\alpha+1}}\right] \times \frac{(e^{-t}x)^m}{m!\Gamma(m + \alpha + 1)}$$

for $t \in \mathscr{D}_{1/(e^{t_0}-1)}$ such that $u(x, 0) = f(x)$.

https://doi.org/10.4153/CJM-1972-122-8 Published online by Cambridge University Press

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