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SINGULAR SOLUTIONS OF A FULLY NONLINEAR 2×2 SYSTEM OF CONSERVATION LAWS

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Abstract Existence and admissibility of δ-shock solutions is discussed for the non-convex strictly hyperbolic system of equations

$$\partial_t u + \partial_x(\tfrac{1}{2}(u^2 + v^2)) = 0,$$
$$\partial_t v + \partial_x(v(u - 1)) = 0.$$

The system is fully nonlinear, i.e. it is nonlinear with respect to both unknowns, and it does not admit the classical Lax-admissible solution for certain Riemann problems. By introducing complex-valued corrections in the framework of the weak asymptotic method, we show that a compressive δ-shock solution resolves such Riemann problems. By letting the approximation parameter tend to zero, the corrections become real valued, and the solutions can be seen to fit into the framework of weak singular solutions defined by Danilov and Shelkovich. Indeed, in this context, we can show that every 2×2 system of conservation laws admits δ-shock solutions.

Keywords: conservation laws; Riemann problem; singular solutions; weak asymptotics; magnetohydrodynamics

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1. Introduction

The main subject of this paper is a system of conservation laws appearing in the study of plasmas. The system is known as the *Brio* system and has the form

$$\left.\begin{aligned} \partial_t u + \partial_x(\tfrac{1}{2}(u^2 + v^2)) &= 0, \\ \partial_t v + \partial_x(v(u - 1)) &= 0. \end{aligned}\right\} \tag{1.1}$$

The system is strictly hyperbolic; it is genuinely nonlinear at $\{(u, v) \colon u \in \mathbb{R},\ v > 0\}$ and $\{(u, v) \colon u \in \mathbb{R},\ v < 0\}$, but not on the whole of $\mathbb{R}^2$. The system was introduced in [**2**] and thoroughly considered in [**14**], where it was found that for certain initial data no solution consisting of the Lax-admissible elementary waves (shock and rarefaction waves) exists. In [**14**], Riemann problems for (1.1) were compared with Riemann problems for the system

$$\left.\begin{aligned}\partial_t u + \partial_x(\tfrac{1}{2}u^2) = 0,\\ \partial_t v + \partial_x(v(u-1)) = 0.\end{aligned}\right\} \qquad (1.2)$$

Numerical computations of appropriate viscous profiles for (1.1) and (1.2) demonstrated surprising similarities. In [**14**], it was shown that certain Riemann problems for (1.2) admit δ-shock-wave solutions. However, the same fact could not be established for any Riemann problem corresponding to (1.1). Here, we aim to resolve the question of existence of δ-shock-wave solutions of (1.1), and the question of physical justifiability of such solutions to the Riemann problem associated to (1.1). We remark that for (1.2), if the δ distribution is a part if the solution then it is adjoined to the function v (with respect to which the system is linear). However, in the case of system (1.1), our investigation shows that it is more natural for the δ distribution to be a part of the function u.

The study of singular solutions of systems of conservation laws was initiated by Korchinski [**21**] and Keyfitz and Kranzer [**19**, **20**]. In the last few years, interest in the topic has grown, and a sample of results may be found in [**3**, **8**, **12**, **14**, **15**, **17**, **18**, **23**–**25**, **28**, **32**, **34**, **36**]. One convenient tool for constructing singular solutions is the method of weak asymptotics [**7**, **11**, **31**]. This method has been used recently to understand the evolution of nonlinear waves in scalar conservation laws as well as the interaction and formation of δ-shock waves in the case of a triangular system of conservation laws [**8**–**10**]. We refer the reader to [**30**] and the references therein for further applications of the weak asymptotic method.

In the present paper, we introduce an extension of the weak asymptotic method to the case where complex-valued corrections are considered for the approximate solutions. Even though the imaginary parts of the solutions so constructed vanish in an appropriate limit, it appears that the use of complex-valued weak asymptotic solutions significantly extends the range of possible singular solutions.

It appears that the weak asymptotic method has so far only been used to construct singular solutions of systems for which the flux functions were *linear* with respect to the unknown function which contains the δ-distribution. In contrast, note that the flux $(f(u,v), g(u,v)) = (\frac{1}{2}(u^2+v^2), v(u-1))$ associated with (1.1) is nonlinear in both u and v, and none of the existing methods yield singular solutions of this system. Thus, it appears that the use of complex-valued corrections is essential in the construction of singular solutions for (1.1).

Let us next define what we mean by a complex-valued weak asymptotic solution, and highlight some methods to restrict the notion of solution with the goal of obtaining uniqueness. First, we define a vanishing family of distributions.

Definition 1.1. Let $f_\varepsilon(x) \in \mathcal{D}'(\mathbb{R})$ be a family of distributions depending on $\varepsilon \in (0,1)$, We say that $f_\varepsilon = o_{\mathcal{D}'}(1)$ if, for any test function $\phi(x) \in \mathcal{D}(\mathbb{R})$, the estimate

$$\langle f_\varepsilon, \phi\rangle = o(1) \quad \text{as } \varepsilon \to 0$$

holds.

The estimate on the right-hand side is understood in the usual Landau sense. Thus, we may say that a family of distributions approach zero in the sense defined above if, for a given test function ϕ, the pairing $\langle f_\varepsilon, \phi\rangle$ converges to zero as ε approaches zero. For families of distributions $f_\varepsilon(x,t)$, we write $f_\varepsilon = o_{\mathcal{D}'}(1) \subset \mathcal{D}'(\mathbb{R})$ if the estimate above holds uniformly in t. More succinctly, we require that

$$\langle f_\varepsilon(\cdot,t), \varphi\rangle \leqslant C_T g(\varepsilon) \quad \text{for } t \in [0,T],$$

where the function g depends on the test function $\varphi(x,t)$ and tends to zero as $\varepsilon \to 0$, and where C_T is a constant depending only on T. We define weak asymptotic solutions to a general system of two conservation laws

$$\left.\begin{aligned} \partial_t u + \partial_x f(u,v) &= 0,\\ \partial_t v + \partial_x g(u,v) &= 0, \end{aligned}\right\} \tag{1.3}$$

as follows.

Definition 1.2. We say that the families of smooth complex-valued distributions (u_ε) and (v_ε) represent a weak asymptotic solution to (1.3) if there exist real-valued distributions $u, v \in C(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}))$, such that, for every fixed $t \in \mathbb{R}_+$,

$$u_\varepsilon \rightharpoonup u, \quad v_\varepsilon \rightharpoonup v \quad \text{as } \varepsilon \to 0,$$

in the sense of distributions in $\mathcal{D}'(\mathbb{R})$, and

$$\left.\begin{aligned} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon, v_\varepsilon) &= o_{\mathcal{D}'}(1),\\ \partial_t v_\varepsilon + \partial_x g(u_\varepsilon, v_\varepsilon) &= o_{\mathcal{D}'}(1). \end{aligned}\right\} \tag{1.4}$$

It is evident that this definition requires some additional assumptions of the fluxes f and g. In particular, f and g must have an extension into the complex plane. One may, for instance, restrict to fluxes that are real analytic, though in principle a wider class of fluxes is possible. The main issue in the requirement on the fluxes, and indeed with this method of constructing solutions, is the question of uniqueness. For example, by adding a constant term of order $\mathcal{O}(\varepsilon)$ to any weak asymptotic solution, one immediately obtains two different weak asymptotic solutions which correspond to the same solution if a more restrictive concept is used.

One way to narrow the class of solution candidates is to require distributional solutions to satisfy the equations in a stronger sense than the one defined in Definition 1.2. This approach entails substituting them into (1.1), and checking directly whether the equations are satisfied. This strategy involves the problem of multiplication of singular distributions. The problem of taking products of singular distributions was overcome by Danilov and Shelkovich in [**10**] in a rather elegant way. In their work, the weak asymptotic solution is constructed such that the terms that do not have a distributional limit cancel in the limit as ε approaches zero. As a result, it is not necessary to include singular terms in the definition of the weak solution. Thus, the problem of multiplication of distributions is automatically eliminated, and the class of possible solutions is significantly reduced.

There are also several other reasonable ways to multiply Heaviside and Dirac distributions. In [**4**, **6**, **16**, **35**], a number of definitions of weak solutions of (1.3) are introduced. Among the latter approaches, we emphasize the measure-type solution concept introduced in [**6**, **16**]. Moreover, the framework from [**16**] yields uniqueness of solutions if an additional condition of Oleinik type is required, and that is probably the only work so far which obtains a uniqueness result for arbitrary initial data in a class of distributional solutions weak enough to allow delta distributions. However, uniqueness has also been obtained for special classes of initial data by LeFloch [**22**] and Nedeljkov [**28**].

We remark that in [**4**, **13**, **29**] the multiplication of distributions problem was systematically investigated in the Colombeau algebra framework. In these works, problems of the type considered here were also investigated. Actually, Definition 1.2 can be understood as a variant of appropriate definitions in [**5**, **26**, **27**]. The main difference is that in the present case a solution is found pointwise with respect to $t \in \mathbb{R}_+$, and it is required that the distributional limit of the weak asymptotic solution be a distribution. The latter is not necessary in the framework of the Colombeau algebra, though it may be tacitly assumed.

The plan of our paper is as follows. We shall provide a review of the definition of weak singular solutions from [**10**] in § 2. It turns out that a somewhat more general statement is appropriate here. Moreover, it will be proved that any 2×2 system of hyperbolic conservation admits singular solutions of this type. In § 3, weak asymptotic solutions of the Brio system are found. The results of that section are very important since they represent a justification of the concept introduced in § 2 which will be applied in § 4. In the latter, it is shown that the limit of the weak asymptotic solutions satisfies the equation in the sense of Definitions 2.1 and 2.2. Also, an adaptation of the Lax admissibility concept is proposed which provides physically sustainable solutions to corresponding Riemann problems. We consider other possibilities for existence of δ-shock solutions in Appendix A.

2. Generalized weak solutions

In this section, the definition of weak singular solutions of a 2×2 system of conservation laws provided in [**10**] is reviewed. Indeed, we shall show that any 2×2 system of the form

$$\begin{aligned} \partial_t u + \partial_x f(u,v) &= 0, \\ \partial_t v + \partial_x g(u,v) &= 0 \end{aligned}$$

admits a δ-type solution in the framework introduced in [**10**]. While the definition in [**10**] is given only for solutions singular in the second variable, while assuming that the flux functions f and g are linear in the second variable, it appears that the definition can actually be made more general. Suppose $\Gamma = \{\gamma_i \mid i \in I\}$ is a graph in the closed upper half-plane, containing Lipschitz continuous arcs γ_i, $i \in I$, where I is a finite index set. Let I_0 be the subset of I containing all indices of arcs that connect to the x-axis, and let $\Gamma_0 = \{x_k^0 \mid k \in I_0\}$ be the set of initial points of the arcs γ_k with $k \in I_0$. Define the singular part by

$$\alpha(x,t)\delta(\Gamma) = \sum_{i \in I} \alpha_i(x,t)\delta(\gamma_i).$$

Let (u,v) be a pair of distributions, where v is represented in the form

$$v(x,t) = V(x,t) + \alpha(x,t)\delta(\Gamma),$$

and where $u, V \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$. Finally, the expression $\partial\varphi(x,t)/\partial \boldsymbol{l}$ denotes the tangential derivative of a function φ on the graph γ_i, and $\int_{\gamma_i}$ connotes the line integral over the arc γ_i.

Definition 2.1. The pair of distributions u and $v = V + \alpha(x,t)\delta(\Gamma)$ are called a generalized δ-shock-wave solution of system (1.3) with the initial data $U_0(x)$ and $V_0(x) + \sum_{I_0} \alpha_k(x_0^k, 0)\delta(x - x_k^0)$ if the integral identities

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} (u\partial_t\varphi + f(u,V)\partial_x\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} U_0(x)\varphi(x,0)\,\mathrm{d}x = 0 \tag{2.1}$$

and

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} (V\partial_t\varphi + g(u,V)\partial_x\varphi)\,\mathrm{d}x\,\mathrm{d}t + \sum_{i \in I} \int_{\gamma_i} \alpha_i(x,t)\frac{\partial\varphi(x,t)}{\partial \boldsymbol{l}} + \int_{\mathbb{R}} V_0(x)\varphi(x,0)\,\mathrm{d}x + \sum_{k \in I_0} \alpha_k(x_k^0, 0)\varphi(x_k^0, 0) = 0 \tag{2.2}$$

hold for all test functions $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$.

The next definition concerns the similar situation where the singular solution is contained in u, and v is a regular distribution. Thus, we assume the representation

$$u(x,t) = U(x,t) + \alpha(x,t)\delta(\Gamma),$$

where now $U, v \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$, and $\alpha(x,t)\delta(\Gamma)$ is defined as before.

Definition 2.2. The pair of distributions $u = U + \alpha(x,t)\delta(\Gamma)$ and v is a generalized δ-shock-wave solution of (1.3) with the initial data $U_0(x) + \sum_{I_0} \alpha_k(x_0^k, 0)\delta(x - x_k^0)$ and $V_0(x)$ if the integral identities

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} (U\partial_t\varphi + f(U,v)\partial_x\varphi)\,\mathrm{d}x\,\mathrm{d}t + \sum_{i \in I} \int_{\gamma_i} \alpha_i(x,t)\frac{\partial\varphi(x,t)}{\partial \boldsymbol{l}} + \int_{\mathbb{R}} U_0(x)\varphi(x,0)\,\mathrm{d}x + \sum_{k \in I_0} \alpha_k(x_k^0, 0)\varphi(x_k^0, 0) = 0 \tag{2.3}$$

and

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} (v\partial_t\varphi + g(U,v)\partial_x\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} V_0(x)\varphi(x,0)\,\mathrm{d}x = 0 \tag{2.4}$$

hold for all test functions $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$.

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This definition may be interpreted as an extension of the classical weak solution concept. Moreover, as noted, for example, in [**1**], the definition is consistent with the concept of measure solutions [**6**, **16**].

Definitions 2.1 and 2.2 are quite general, allowing a combination of initial steps and delta distributions, but their effectiveness can be demonstrated by considering the Riemann problem with a single jump. Indeed, for this configuration it can be shown that a δ-shock-wave solution exists for any 2×2 system of conservation laws. Consider the Riemann problem for (1.3) with initial data $u(x,0) = U_0(x)$ and $v(x,0) = V_0(x)$, where

$$U_0(x) = \begin{cases} u_1, & x < 0, \\ u_2, & x > 0, \end{cases} \qquad V_0(x) = \begin{cases} v_1, & x < 0, \\ v_2, & x > 0. \end{cases} \tag{2.5}$$

Then, the following theorem holds.

Theorem 2.3.

(a) *If $u_1 \neq u_2$, then the pair of distributions*

$$u(x,t) = U_0(x - ct), \tag{2.6}$$

$$v(x,t) = V_0(x - ct) + \alpha(t)\delta(x - ct), \tag{2.7}$$

where

$$c = \frac{[f(u,V)]}{[u]} = \frac{f(u_2,v_2) - f(u_1,v_1)}{u_2 - u_1} \quad \text{and} \quad \alpha(t) = (c[V] - [g(u,V)])t,$$

represents the δ-shock-wave solution of (1.3) with initial data $U_0(x)$ and $V_0(x)$ in the sense of Definition 2.1.

(b) *If $v_1 \neq v_2$, then the pair of distributions*

$$u(x,t) = U_0(x - ct) + \alpha(t)\delta(x - ct), \tag{2.8}$$

$$v(x,t) = V_0(x - ct), \tag{2.9}$$

where

$$c = \frac{[g(U,v)]}{[v]} = \frac{g(u_2,v_2) - g(u_1,v_1)}{v_2 - v_1} \quad \text{and} \quad \alpha(t) = (c[U] - [f(U,v)])t,$$

represents the δ-shock solution of (1.3) with initial data $U_0(x)$ and $V_0(x)$ in the sense of Definition 2.2.

Proof. We shall prove only part (a), as part (b) can be proved analogously. We immediately see that u and v given by (2.6) and (2.7) satisfy (2.1) since c is given exactly by the Rankine–Hugoniot condition derived from that system. By substituting u and v into (2.2), after standard transformations we get

$$\int_{\mathbb{R}_+} (-c[V] + [g(u,V)])\varphi(ct,t)\,\mathrm{d}t - \int_{\mathbb{R}_+} \alpha'(t)\varphi(ct,t)\,\mathrm{d}t = 0.$$

From here and since $\alpha(0) = 0$, the conclusion follows immediately. $\square$

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As the solution framework of Definitions 2.1 and 2.2 is very weak, one might expect non-uniqueness issues to arise. This is indeed the case, and the proof of the following proposition is an easy exercise.

Proposition 2.4. *System (1.3) with the zero initial data: $u|_{t=0} = v|_{t=0} = 0$ admits δ-shock solutions of the form*

$$u(x,t) = 0, \qquad v(x,t) = \beta\delta(x - c_1 t) - \beta\delta(x - c_2 t),$$

for arbitrary constants β, c_1 and c_2.

At the moment, we do not have a general concept for resolving such and similar non-uniqueness issues. In the case of the Brio system which we shall consider in the following, we are also not able to obtain uniqueness, but we can prove that there always exists a physically reasonable solution to the corresponding Riemann problem.

Finally, let us remark that it is of course possible and reasonable to give a definition along the lines of Definitions 2.1 and 2.2 which allows for simultaneous concentration effects in both unknowns u and v. In this case, a generalized δ-shock-wave solution of (1.3) with the initial data

$$U_0(x) + \sum_{I_0} \alpha_k(x_0^k, 0)\delta(x - x_k^0) \quad \text{and} \quad V_0(x) + \sum_{I_0} \beta_k(x_0^k, 0)\delta(x - x_k^0)$$

would have the form $u = U + \alpha(x,t)\delta(\Gamma)$ and $v = V + \beta(x,t)\delta(\Gamma)$, and satisfy

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} (U\partial_t\varphi + f(U,V)\partial_x\varphi)\,\mathrm{d}x\,\mathrm{d}t + \sum_{i\in I}\int_{\gamma_i} \alpha_i(x,t)\frac{\partial\varphi(x,t)}{\partial \boldsymbol{l}} + \int_{\mathbb{R}} U_0(x)\varphi(x,0)\,\mathrm{d}x + \sum_{k\in I_0}\alpha_k(x_k^0,0)\varphi(x_k^0,0) = 0, \tag{2.10}$$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} (V\partial_t\varphi + g(U,V)\partial_x\varphi)\,\mathrm{d}x\,\mathrm{d}t + \sum_{i\in I}\int_{\gamma_i} \beta_i(x,t)\frac{\partial\varphi(x,t)}{\partial \boldsymbol{l}} + \int_{\mathbb{R}} V_0(x)\varphi(x,0)\,\mathrm{d}x + \sum_{k\in I_0}\beta_k(x_k^0,0)\varphi(x_k^0,0) = 0, \tag{2.11}$$

for all test functions $\varphi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}_+)$. An example of such a situation can be found in [**33**].

3. Weak asymptotics for the Brio system

In this section, we shall construct weak asymptotic solutions for the Riemann problem associated to the Brio system (1.1) and then show that the weak asymptotic solution converges to the generalized weak solution to the system in the sense of Definitions 2.1 and 2.2. This construction is very important, since the fact that it is possible to find a sequence of smooth approximating solutions to (1.1), (2.5) converging to the δ-shock solution represents a justification of the concept laid down in § 2. In particular, observe

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that the vanishing viscosity approximation is a special case of the weak asymptotic approximation since the term εu_{xx} is clearly of order $\mathcal{O}_{\mathcal{D}'}(\varepsilon)$.

To find the weak asymptotic solutions we need to find families of smooth functions (u_ε), (v_ε) such that

$$\left.\begin{aligned}\partial_t u_\varepsilon + \partial_x(\tfrac{1}{2}(u_\varepsilon^2 + v_\varepsilon^2)) = o_{\mathcal{D}'}(1),\\ \partial_t v_\varepsilon + \partial_x(v_\varepsilon(u_\varepsilon - 1)) = o_{\mathcal{D}'}(1),\end{aligned}\right\} \tag{3.1}$$

$$u_\varepsilon \rightharpoonup u, \quad v_\varepsilon \rightharpoonup v \quad \text{as } \varepsilon \to 0, \tag{3.2}$$

and such that $u(x,0) = U_0(x)$ and $v(x,0) = V_0(x)$ are given by (2.5). We shall prove the following theorem.

Theorem 3.1.

(a) *If $u_1 \neq u_2$, then there exist weak asymptotic solutions (u_ε), (v_ε) of the Brio system (1.1) such that the families (u_ε) and (v_ε) have distributional limits*

$$u(x,t) = U_0(x - ct), \tag{3.3}$$

$$v(x,t) = V_0(x - ct) + \alpha(t)\delta(x - ct), \tag{3.4}$$

where

$$c = \frac{u_1^2 + v_1^2 - u_2^2 - v_2^2}{2(u_1 - u_2)} \quad \text{and} \quad \alpha(t) = \tfrac{1}{2}(c(v_2 - v_1) + (v_1(u_1 - 1) - v_2(u_2 - 1)))t. \tag{3.5}$$

(b) *If $v_1 \neq v_2$, then there exist weak asymptotic solutions (u_ε), (v_ε) of the Brio system (1.1), such that the families (u_ε) and (v_ε) have distributional limits*

$$u(x,t) = U_0(x - ct) + \alpha(t)\delta(x - ct), \tag{3.6}$$

$$v(x,t) = V_0(x - ct), \tag{3.7}$$

where

$$c = \frac{v_1(u_1 - 1) - v_2(u_2 - 1)}{v_1 - v_2} \quad \text{and} \quad \alpha(t) = (c(u_2 - u_1) + \tfrac{1}{2}(u_1^2 + v_1^2 - u_2^2 - v_2^2))t. \tag{3.8}$$

Proof. (a) Let $\rho \in C_c^\infty(\mathbb{R})$ be an even, non-negative, smooth, compactly supported function such that

$$\operatorname{supp} \rho \subset (-1, 1), \quad \int_{\mathbb{R}} \rho(z)\,\mathrm{d}z = 1, \quad \rho \geqslant 0.$$

We take

$$\left.\begin{aligned}R_\varepsilon(x,t) &= \frac{\mathrm{i}}{\varepsilon}\rho\left(\frac{x - ct - 2\varepsilon}{\varepsilon}\right) - \frac{\mathrm{i}}{\varepsilon}\rho\left(\frac{x - ct + 2\varepsilon}{\varepsilon}\right),\\ \delta_\varepsilon(x,t) &= \frac{1}{\varepsilon}\rho\left(\frac{x - ct - 4\varepsilon}{\varepsilon}\right) + \frac{1}{\varepsilon}\rho\left(\frac{x - ct + 4\varepsilon}{\varepsilon}\right).\end{aligned}\right\} \tag{3.9}$$

Next, define smooth functions U_ε and V_ε such that

$$U_\varepsilon(x,t) = \begin{cases} u_1, & x < ct - 20\varepsilon, \\ c+1, & ct - 10\varepsilon < x < ct + 10\varepsilon, \\ u_2, & x > ct + 20\varepsilon, \end{cases}$$

$$V_\varepsilon(x,t) = \begin{cases} v_1, & x < ct - 20\varepsilon, \\ 0, & ct - 10\varepsilon < x < ct + 10\varepsilon, \\ v_2, & x > ct + 20\varepsilon. \end{cases}$$

Note that

$$R_\varepsilon \rightharpoonup 0, \quad U_\varepsilon R_\varepsilon \rightharpoonup 0 \quad \text{and} \quad U_\varepsilon \delta_\varepsilon \rightharpoonup 2(c+1)\delta(x-ct). \tag{3.10}$$

Moreover, we have

$$V_\varepsilon R_\varepsilon \equiv 0, \quad V_\varepsilon \delta_\varepsilon \equiv 0 \quad \text{and} \quad \delta_\varepsilon R_\varepsilon \equiv 0. \tag{3.11}$$

Now make the ansatz

$$\left.\begin{aligned} u_\varepsilon(x,t) &= U_\varepsilon(x,t), \\ v_\varepsilon(x,t) &= V_\varepsilon(x,t) + \alpha(t)(\delta_\varepsilon(x,t) + R_\varepsilon(x,t)), \end{aligned}\right\} \tag{3.12}$$

and substitute it into (3.1). Note first of all that

$$v_\varepsilon^2(x,t) = V_\varepsilon^2 + \alpha^2(t)(R_\varepsilon^2 + \delta_\varepsilon^2)$$

by invoking (3.11). Focusing on the expression $R_\varepsilon^2 + \delta_\varepsilon^2$, we take $\varphi \in C_c^\infty(\mathbb{R})$ and consider the integral

$$\begin{aligned} \int_{\mathbb{R}} (R_\varepsilon^2 + \delta_\varepsilon^2)\varphi \,\mathrm{d}x &= \int_{\mathbb{R}} \frac{1}{\varepsilon^2}\biggl(-\rho^2\biggl(\frac{x-ct+2\varepsilon}{\varepsilon}\biggr) - \rho^2\biggl(\frac{x-ct-2\varepsilon}{\varepsilon}\biggr)\biggr) \\ &\qquad + \rho^2\biggl(\frac{x-ct+4\varepsilon}{\varepsilon}\biggr) + \rho\biggl(\frac{x-ct-4\varepsilon}{\varepsilon}\biggr)\varphi\,\mathrm{d}x \\ &= \mathcal{O}(\varepsilon). \end{aligned}$$

In the above reasoning, use was made of the following computation:

$$\begin{aligned} &\int_{\mathbb{R}} \frac{1}{\varepsilon^2}\biggl(\rho^2\biggl(\frac{x-ct+\alpha\varepsilon}{\varepsilon}\biggr) + \rho^2\biggl(\frac{x-ct-\beta\varepsilon}{\varepsilon}\biggr)\biggr)\varphi(x)\,\mathrm{d}z \\ &\qquad = \int_{\mathbb{R}} \frac{1}{\varepsilon}\rho^2(z)(\varphi(ct+\varepsilon(z-\alpha)) + \varphi(ct+\varepsilon(z+\beta)))\,\mathrm{d}z \\ &\qquad = \int_{\mathbb{R}} \frac{1}{\varepsilon}\rho^2(z)(2\varphi(ct) + \varepsilon z\varphi'(ct)(\beta-\alpha))\,\mathrm{d}z + \mathcal{O}(\varepsilon) \quad \text{for } \alpha,\beta \in \mathbb{R}. \end{aligned}$$

The last relation was found by making the changes of variables $(x-ct+\alpha\varepsilon)/\varepsilon = z$ and $(x-ct-\beta\varepsilon)/\varepsilon = z$ and observing that

$$\int z\rho^2(z)\,\mathrm{d}z = 0$$

since ρ is an even function. In the case at hand, we use $\alpha = \beta = 2$ for the first integral, and $\alpha = \beta = 4$ in the second integral. Finally, it becomes clear that

$$v_\varepsilon^2 = V_\varepsilon^2 + o_{\mathcal{D}'}(1). \tag{3.13}$$

Therefore, taking into account Definition 1.1, from the first equation in (3.1), we conclude that we need to check whether

$$\partial_t U_\varepsilon + \tfrac{1}{2}\partial_x(U_\varepsilon^2 + V_\varepsilon^2) = o_{\mathcal{D}'}(1),$$

and this reduces to

$$\int_0^T (-c[U] + \tfrac{1}{2}[U^2 + V^2])\varphi(ct, t)\,\mathrm{d}t = o(1), \tag{3.14}$$

where $[U] = u_2 - u_1$ and $[U^2 + V^2] = u_2^2 + v_2^2 - u_1^2 - u_1^2$.

However, this is indeed satisfied, thanks to the choice of the constant c which was found from the Rankine–Hugoniot condition for the first equation in (1.1).

Let us now consider the second equation in (3.1). First, note that

$$\begin{aligned}\partial_x(v_\varepsilon(u_\varepsilon - 1)) &= \partial_x(U_\varepsilon V_\varepsilon + (c+1)\alpha(t)\delta_\varepsilon - V_\varepsilon - \alpha(t)\delta_\varepsilon) + o_{\mathcal{D}'}(1)\\ &= (v_1(1-u_1) + v_2(u_2-1)\delta(x-ct) + c\alpha(t)\delta'(x-ct)) + o_{\mathcal{D}'}(1).\end{aligned}$$

Next, note also that

$$\partial_t v_\varepsilon = -c(v_2 - v_1)\delta(x-ct) + \alpha'(t)\delta(x-ct) - c\alpha(t)\delta'(x-ct) + o_{\mathcal{D}'}(1).$$

Adding the latter two expressions, we obtain

$$\partial_t v_\varepsilon + \partial_x(v_\varepsilon(u_\varepsilon - 1)) = -c(v_2 - v_1) + \alpha'(t) + (v_1(1-u_1) + v_2(u_2-1))\delta(x-ct) + o_{\mathcal{D}'}(1).$$

From here, we conclude that, by choosing α as given in (3.5), the first equation in (3.1) is also satisfied. This concludes the proof of part (a).

(b) In this case, an appropriate weak asymptotic solution is given by

$$\begin{aligned}u_\varepsilon(x,t) &= U_\varepsilon(x,t) + \alpha(t)(\delta_\varepsilon(x,t) + R_{1\varepsilon}(x,t)) + \sqrt{2c\alpha(t)}R_{2\varepsilon}(x,t),\\ v_\varepsilon(x,t) &= V_\varepsilon(x,t),\end{aligned}$$

where

$$c = \frac{v_1u_1 - v_2u_2}{v_1 - v_2} - 1 \quad \text{and} \quad \alpha(t) = (c(u_1 - u_2) - \tfrac{1}{2}(u_1^2 + v_1^2 - u_2^2 - v_2^2))t,$$

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and

$$U_\varepsilon(x,t) = \begin{cases} u_1, & x < ct - 20\varepsilon, \\ 0, & ct - 10\varepsilon < x < ct + 10\varepsilon, \\ u_2, & x > ct + 20\varepsilon, \end{cases}$$

$$V_\varepsilon(x,t) = \begin{cases} v_1, & x < ct - 20\varepsilon, \\ 0, & ct - 10\varepsilon < x < ct + 10\varepsilon, \\ v_2, & x > ct + 20\varepsilon, \end{cases}$$

$$R_{1\varepsilon}(x,t) = \frac{\mathrm{i}}{\varepsilon}\rho\left(\frac{x - ct - 2\varepsilon}{\varepsilon}\right) - \frac{\mathrm{i}}{\varepsilon}\rho\left(\frac{x - ct + 2\varepsilon}{\varepsilon}\right),$$

$$R_{2\varepsilon}(x,t) = \frac{1}{\sqrt{\varepsilon}}\left[\rho\left(\frac{x - ct}{\varepsilon}\right)\right]^{1/2},$$

$$\delta_\varepsilon(x,t) = \frac{1}{\varepsilon}\rho\left(\frac{x - ct - 4\varepsilon}{\varepsilon}\right) + \frac{1}{\varepsilon}\rho\left(\frac{x - ct + 4\varepsilon}{\varepsilon}\right),$$

where ρ is the same smooth non-negative even function as used in the previous examples. The proof then follows the ideas of the proof of (a). □

An important corollary (to be used in Appendix A) of the proof of the previous theorem is that it gives another interesting class of weak asymptotic solutions to (1.1) having the δ distribution as their limit.

Corollary 3.2. *If $u_1 = u_2$ and $v_1^2 = v_2^2$, then for any $c \in \mathbb{R}$ the families (u_ε) and (v_ε) given by (3.12) are the weak asymptotic solution to (1.1), (2.5).*

Proof. It is enough to note that (3.14) is satisfied independently on c since $[U] = [U^2 + V^2] = 0$. □

To close the section, we should mention that, while the extension of the weak asymptotic method to complex-valued solutions was crucial for finding a solution of the system (1.1), it might not be appropriate in other contexts, as it might lead to strong non-uniqueness. For example, using complex-valued weak asymptotic solutions of a similar form to (3.6) for the inviscid Burgers equation, one may construct a family of distinct solutions emanating from the same initial data, all of which also satisfy the Lax admissibility condition.

4. Generalized weak solutions for the Brio system and the uniqueness issue

By comparing Theorems 2.3 and 3.1, we see that the limit distributions u and v given in Theorem 3.1 represent δ-shock solutions to (1.1) with initial data $u(x,0) = U_0(x)$ and $v(x,0) = V_0(x)$. However, we want to incorporate such solutions into the Lax admissibility concept and this is the goal in this section.

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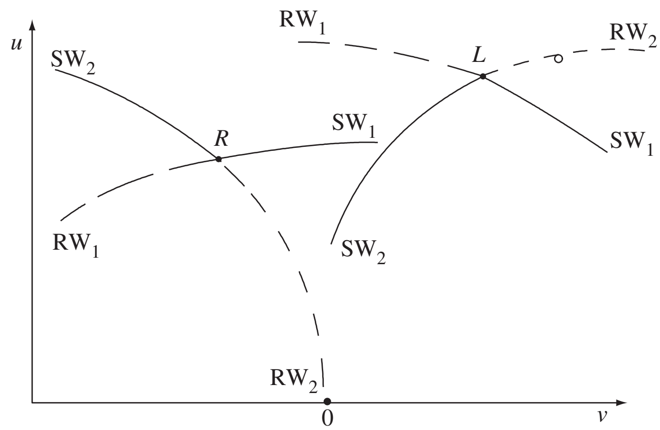

Figure 1. L is the left state and R is the right state.

We focus on Definition 2.2, where the fluxes f and g are given by the Brio system. The same can be done with Definition 2.1, but it appears that the solutions that it generates do not fit into the Lax admissibility concept. More details of this case will be provided in Appendix A.

Recall that in the case $v_1 > 0 > v_2$ there exists no Lax-admissible solution to the Riemann problem (1.1), with the Riemann initial data U_0 and V_0 given by (2.5) (see [**14**]). If v_1 and v_2 do not satisfy this relation, we have the classical Lax-admissible solution to the appropriate Riemann problem consisting of the elementary waves, i.e. shock and rarefaction waves. For L^∞-small data, such a solution is unique, since the system is genuinely nonlinear for $v > 0$ and $v < 0$. Theorem 2.3 states that we can also have δ-shock-wave solutions, but as Proposition 2.4 shows, there is strong non-uniqueness. In order to eliminate at least some of solutions which are inconsistent with the physical intuition, we shall use the Lax compressivity conditions for the δ-shock wave. In order to introduce them, let us recall that the characteristic velocities for the Brio system [**14**] are

$$\lambda_1(u,v) = u - \tfrac{1}{2} - \sqrt{\tfrac{1}{4} + v^2}, \qquad \lambda_2(u,v) = u - \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + v^2}.$$

The corresponding rarefaction waves are given by

$$u = -\tfrac{1}{2}(\sqrt{4v^2+1} - \log(1 + \sqrt{4v^2+1})) + C_1, \tag{RW$_1$}$$

$$u = \tfrac{1}{2}(\sqrt{4v^2+1} + \log(-1 + \sqrt{4v^2+1})) + C_2. \tag{RW$_2$}$$

The shock waves are given by

$$u - u_1 = \frac{v - v_1}{v + v_1}(1 \mp \sqrt{(v+v_1)^2 + 1}). \tag{SW$_{1,2}$}$$

A phase-space picture for given left and right states is shown in Figure 1.

The following definition introduces a compressivity demand on the characteristics of (1.1), meaning that the characteristics enter the δ-shock from both sides. It is standard for the classical shock waves, and they are known as Lax admissibility conditions. Note that the usual demand on the δ-shock wave is an overcompressivity condition demanding that both characteristic fields λ_1 and λ_2 satisfy (4.1) [**9**, **10**, **12**, **20**, **34**]. However, we were not able to find solutions involving overcompressive δ-shocks, and we confine ourselves on a less restrictive demand that still includes concentration effects. The definition concerning the admissible δ-shock solutions of (1.1) such as those defined in Theorem 3.1 follows.

Definition 4.1. A δ-shock solution of (1.1), connecting a left state $L = (u_1, v_1)$ and a right state $R = (u_2, v_2)$ is i-admissible if

$$\lambda_i(u_2, v_2) \leqslant c \leqslant \lambda_i(u_1, v_1) \tag{4.1}$$

for $i = 1$ or $i = 2$. For such a δ-shock wave, we say that it is compressive.

Thus, for a general Riemann problem, one may say that a solution of (1.1), (2.5) which contains a δ-shock wave is admissible if it consists of a combination of the classical Lax-admissible simple waves (shock or rarefaction) and compressive δ waves.

The following lemma will be crucial for proving the existence of admissible δ-shock solutions to Riemann problems corresponding to (1.1).

Lemma 4.2. *Assume that the initial data in (2.5) are such that* $u_1 = u_2 = \tilde{u}$, $v_1 = 0$ *and* $v_2 < 0$. *Then, the* δ*-shock solution*

$$\left.\begin{aligned} u(x,t) &= \tilde{u} + \alpha(t)\delta(x - ct), \\ v(x,t) &= 0, \end{aligned}\right\} \tag{4.2}$$

where $\alpha(t)$ *and* c *are given by (3.8), represents a* 1*-admissible* δ*-shock solution.*

Proof. The functions given by (4.2) represent δ-shock solution to (1.1), (2.5) according to Theorem 2.3 (b). In order to prove that the solution is 1-admissible, recall that

$$c = \frac{v_2(u_2 - 1) - v_1(u_1 - 1)}{v_2 - v_1}.$$

Then, due to (4.1), we need to show that

$$\begin{aligned} \lambda_1(u_2, v_2) = u_2 - \tfrac{1}{2} - \sqrt{\tfrac{1}{4} + v_2^2} &\leqslant \frac{v_2(u_2 - 1) - v_1(u_1 - 1)}{v_2 - v_1} \\ &\leqslant u_1 - \tfrac{1}{2} - \sqrt{\tfrac{1}{4} + v_1^2} = \lambda_1(u_1, v_1). \end{aligned}$$

Since $u_1 = u_2 = \tilde{u}$ and $v_1 = 0$, the latter reduces to

$$\tilde{u} - \tfrac{1}{2} - \sqrt{\tfrac{1}{4} + v_2^2} \leqslant \tilde{u} - 1 \leqslant \tilde{u} - 1 \implies \tfrac{1}{2} - \sqrt{\tfrac{1}{4} + v_2^2} \leqslant 0,$$

which is clearly true. This concludes the proof. □

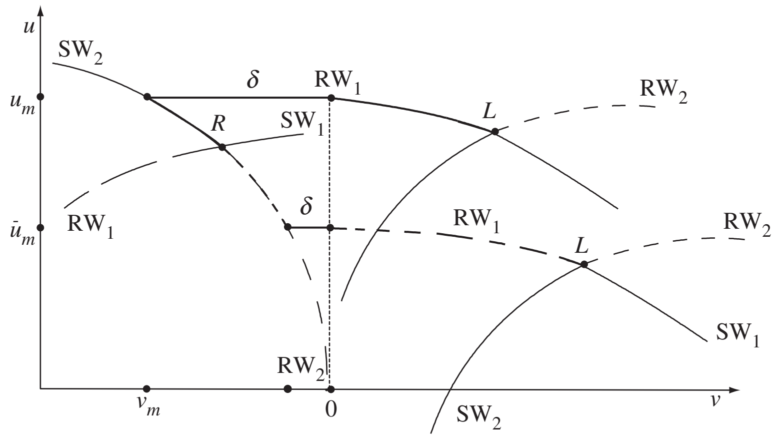

Figure 2. Thick full line: $L \xrightarrow{\mathrm{RW}_1} (u_m, 0) \xrightarrow{\delta} (u_m, v_m) \xrightarrow{\mathrm{SW}_2} R$.
Thick dashed line: $\bar{L} \xrightarrow{\mathrm{RW}_1} (\bar{u}_m, 0) \xrightarrow{\delta} (\bar{u}_m, \bar{v}_m) \xrightarrow{\mathrm{RW}_2} R$.

With this lemma established, we can attempt the proof of the following theorem.

Theorem 4.3. *Given any Riemann initial data (2.5) such that $v_2 < 0 < v_1$, there exists a solution of (1.1) in the sense of Definition 2.2 that consists of a combination of the classical Lax-admissible simple waves (shock or rarefaction) and compressive δ waves, 1-admissible in the sense of Definition 4.1.*

Proof. The solution is plotted in Figure 2. First, we have the rarefaction wave 1 (RW_1) issuing from the left state $L = (u_1, v_1)$ and connecting it to the state $(u_m, 0)$. Then, we connect the state $(u_m, 0)$ with the state (u_m, v_m) by the δ-shock wave, and finally we connect (u_m, v_m) with $R = (u_2, v_2)$ by shock wave 2 (SW_2) or rarefaction wave 2 (RW_2).

The solution is admissible, since all the simple shocks which it contains are admissible. Namely, Lemma 4.2 provides admissibility for the δ-shock wave, while the other waves are admissible according to the standard theory (see Figure 1). Furthermore, such a combination of shocks is clearly possible since the speed of the state L equals $\lambda_1(u_1, v_1)$ and is less than the speed $\lambda_1(u_m, 0)$ of the middle point $(u_m, 0)$ (since they are connected by the rarefaction wave). Furthermore, the speed of the δ shock connecting $(u_m, 0)$ and (u_m, v_m) equals $v_m(u_m - 1)/v_m = \lambda_1(u_m, 0)$ and it is slower than the speed of the state (u_m, v_m), which equals either $\lambda_2(u_m, v_m)$ (if we have RW_2 between (u_m, v_m) and (u_2, v_2)) or we have

$$c = \frac{v_2(u_2 - 1) - v_m(u_m - 1)}{v_2 - v_1} < \lambda_1(u_m, 0)$$

(if we have SW_2 between (u_m, v_m) and (u_2, v_2)). Both situations are plotted in the phase-space picture shown in Figure 3.

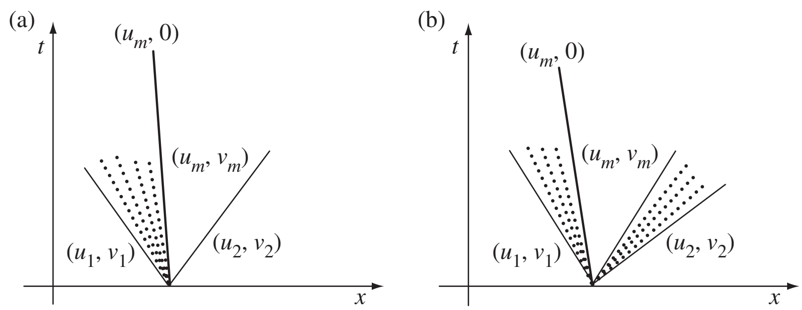

Figure 3. (a) The situation when (u_m, v_m) is connected with (u_2, v_2) by SW_2. (b) The situation when (u_m, v_m) is connected with (u_2, v_2) by RW_2.

Indeed, if (u_m, v_m) is connected to $R = (u_2, v_2)$ by RW_2, then the speed of (u_m, v_m) is $\lambda_2(u_m, v_m) > \lambda_1(u_m, 0)$. On the other hand, if (u_m, v_m) is connected to $R = (u_2, v_2)$ by SW_2, then its speed is

$$\frac{v_2(u_2 - 1) - v_m(u_m - 1)}{v_2 - v_m} = u_m - 1 + v_2 \frac{u_2 - u_m}{v_2 - v_m} > \lambda_1(u_m, 0) = u_m - 1,$$

since $v_2 < 0$, $v_2 - v_m > 0$ and $u_2 < u_m$ (see Figure 2). □

This theorem provides existence of an admissible δ-shock solution of the system (1.1) with Riemann data (2.5). However, even with the admissibility concept provided by Definition 4.1. it is not difficult to see that uniqueness may not hold. For example, a left state $L = (u_1, v_1)$ and a right state $R = (u_2, v_2)$ may be joined directly by a 1-admissible δ shock as long as

$$v_1 \frac{u_2 - u_1}{v_2 - v_1} \geqslant \tfrac{1}{2} - \sqrt{\tfrac{1}{4} + v_2^2} \quad \text{and} \quad v_2 \frac{u_2 - u_1}{v_2 - v_1} \leqslant \tfrac{1}{2} - \sqrt{\tfrac{1}{4} + v_1^2},$$

and this is true whenever $u_1 - u_2$ is large enough and $v_2 < 0 < v_1$. We could, of course, add certain conditions which would eliminate the non-uniqueness. For instance, we could announce a δ shock as admissible only if it connects states $L = (u, 0)$ and $R = (u, v)$, $v < 0$. However, we do not have any physical justification for such a condition and we shall confine ourselves to the existence statement.

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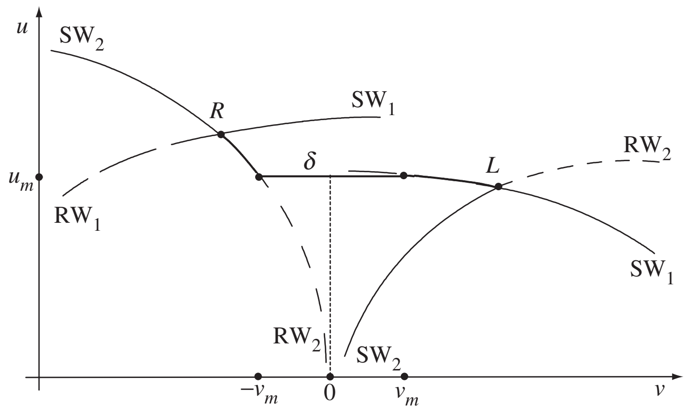

Figure 4. $L = (u_1, v_1) \xrightarrow{\mathrm{RW}_1} (u_m, v_m) \xrightarrow{\delta} (u_m, -v_m) \xrightarrow{\mathrm{RW}_2} R = (u_2, v_2)$.

Appendix A.

We conclude by considering the possibility of the δ distribution to be adjoint to the unknown v in (1.1). We start with a lemma which will help us to connect certain states by admissible δ shocks residing in the unknown v.

Lemma A 1. *Assume that in (2.5) we have* $u_1 = u_2$ *and* $v_1 = -v_2 > 0$. *Then, if* $c = \lambda_i(u_1, v_1) = \lambda_i(u_2, v_2)$, $i = 1, 2$, *the functions*

$$\left.\begin{aligned} u(x,t) &= U_0(x-ct),\\ v(x,t) &= V_0(x-ct) + \alpha(t)\delta(x-ct),\end{aligned}\right\} \tag{A 1}$$

where $\alpha(t)$ *is given by (3.5), represent the* i*-admissible* δ*-shock solution to (1.1).*

Proof. It is sufficient to rely on Corollary 3.2 and proof of Theorem 2.3. Indeed, taking data (A 1) for any $c \in \mathbb{R}$ and inserting them into Definition 4.1, we see that such u and v represent the δ-shock solution to (1.1), (2.5). To see this, one may use the same reasoning as in the proof of Theorem 2.3 and relation (3.14).

Next, we take $c = \lambda_1(u_1, v_1) = \lambda_1(u_2, v_2)$ or $c = \lambda_2(u_1, v_1) = \lambda_2(u_2, v_2)$ to conclude that the pair (u, v) is 1-admissible or 2-admissible, respectively, in the sense of Definition 4.1. □

Using Lemma A 1, we can connect the states

$$L = (u_1, v_1) \quad \text{and} \quad R = (u_2, v_2), \quad \text{where } u_2 > u_1,$$

by an admissible δ-shock solution (u, v) to (1.1), (2.5) admitting the δ shock in the function v through one of the following procedures.

1. $L \to (v_m, u_m)$ by RW_1; $(v_m, u_m) \to (-v_m, u_m)$ by the δ shock with the speed $c = \lambda_1(u_m, v_m)$; $(-v_m, u_m) \to R$ by RW_2 (see Figure 4). In this case, we can also set $c = \lambda_2(u_m, v_m)$. If we take such c, then the δ shock travels with the state $(u_m, -v_m)$. If we take $c = \lambda_1(u_m, v_m)$, then the δ shock travels with the state (u_m, v_m). Note the non-uniqueness that we have here.

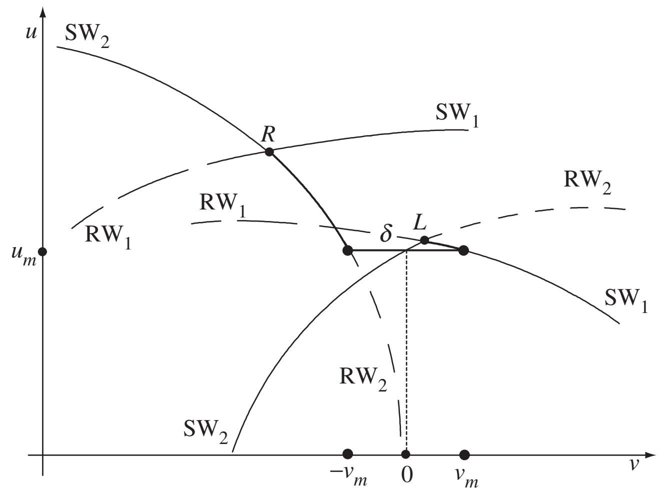

Figure 5. $L = (u_1, v_1) \xrightarrow{SW1} (u_m, v_m) \xrightarrow{\delta} (u_m, -v_m) \xrightarrow{\text{RW}_2} R = (u_2, v_2)$.

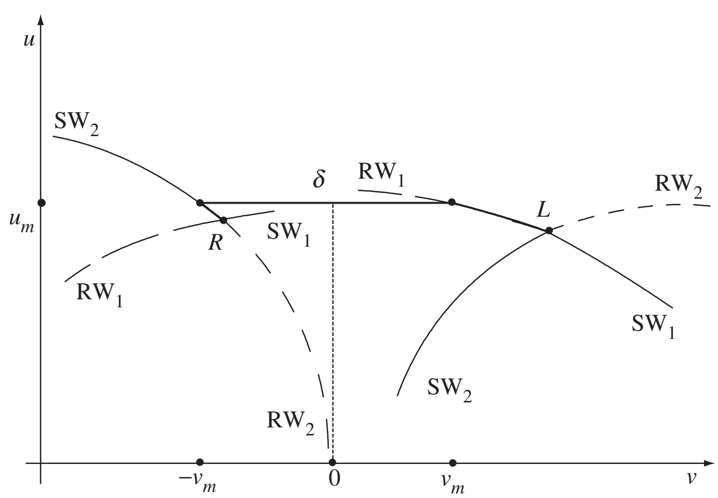

Figure 6. $L = (u_1, v_1) \xrightarrow{\text{RW}_1} (u_m, v_m) \xrightarrow{\delta} (u_m, -v_m) \xrightarrow{\text{SW}_2} R = (u_2, v_2)$.

2. $L \to (v_m, u_m)$ by SW1; $(v_m, u_m) \to (-v_m, u_m)$ by the δ shock with the speed $c = \lambda_2(u_m, v_m)$; $(-v_m, u_m) \to R$ by RW_2 (see Figure 5).

3. $L \to (v_m, u_m)$ by RW_1; $(v_m, u_m) \to (-v_m, u_m)$ by the δ shock with the speed $c = \lambda_1(u_m, v_m)$; $(-v_m, u_m) \to R$ by SW_2 (see Figure 6).

In the case when $u_2 < u_1$, we do not have a general recipe for connecting the states $L = (u_1, v_1)$ and $R = (u_2, v_2)$ by an admissible δ-shock solution with the δ function adjoined to v. Finally, observe that each of the δ shocks in this appendix is not really compressive, since characteristics from both sides of the shock are parallel to the shock. Thus, we cannot say that concentration effects are present.

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