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COMMUTATIVITY RESULTS FOR RINGS

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Let R be an associative ring. We prove that if for each finite subset F of R there exists a positive integer $n = n(F)$ such that $(xy)^n - y^n x^n$ is in the centre of R for every x, y in F, then the commutator ideal of R is nil. We also prove that if n is a fixed positive integer and R is an $n(n+1)$-torsion-free ring with identity such that $(xy)^n - y^n x^n = (yx)^n - x^n y^n$ is in the centre of R for all x, y in R, then R is commutative.

A theorem of Herstein [5] states that a ring R which satisfies the identity $(xy)^n = x^n y^n$ where n is a fixed positive integer greater than 1, must have nil commutator ideal. In [1], the author proved that if n is a fixed positive integer greater than 1, and R is an $n(n-1)$-torsion-free ring with identity such that $(xy)^n = x^n y^n$ for all x, y in R, then R is commutative. In this direction we prove the following results. Theorem 3 below generalises the above mentioned result in [1]. Throughout, let Z denote the centre of R.

We start by stating without proof the following known lemma [4].

LEMMA 1. *Let R be a prime ring and let x and y be elements of R with $x \neq 0$. If $x \in Z$ and $xy \in Z$ then $y \in Z$.*

THEOREM 1. *Let R be an associative ring such that for each finite subset F of R there exists a positive integer $n = n(F)$ such that $(xy)^n - y^n x^n$ is in the centre Z of R, $\forall x, y \in F$. Then the commutator ideal of R is nil.*

PROOF: To prove that the commutator ideal of R is nil it is enough to show that if R has no nonzero nil ideals then it is commutative. So we suppose that R has no nonzero nil ideals. Then R is a subdirect product of prime rings R_α, having no nonzero nil ideals. Each R_α being a homomorphic image of R, $R_\alpha = \phi_\alpha(R)$, satisfies the hypothesis of R. For let $F_\alpha = \{x_{1\alpha}, x_{2\alpha}, \dots, x_{k\alpha}\}$ be a finite subset of R_α and let $F = \{x_1, x_2, \dots, x_k\}$ be a finite subset of R such that $\phi_\alpha(x_i) = x_{i\alpha}$, $i = 1, \dots k$. There exists a positive integer $n = n(F)$, such that $(xy)^n - y^n x^n \in Z$ for all x, $y \in F$. Clearly $(x_\alpha y_\alpha)^n - y_\alpha^n x_\alpha^n \in Z_\alpha$ [the center of R_α] for all x_α, $y_\alpha \in F_\alpha$. So we may assume that R is a prime ring having no nonzero nil ideals. Let x and y be any two

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elements of R. By the hypothesis, there exists a positive integer $n = n(x, y, xy, yx)$ such that

(1) $$(xy)^n - y^n x^n = z \in Z$$

(2) $$(yx)^n - x^n y^n = z' \in Z$$

(3) $$((xy)x)^n - x^n (xy)^n \in Z$$

(4) $$(x(yx))^n - (yx)^n x^n \in Z.$$

Now (3) and (1) imply that

$$((xy)x)^n - x^n (y^n x^n + z) \in Z.$$

Thus,

(5) $$(xyx)^n - x^n y^n x^n - z x^n \in Z.$$

Using (4) and (2) we have

$$(x(yx))^n - (x^n y^n + z') x^n \in Z.$$

Thus,

(6) $$(xyx)^n - x^n y^n x^n - z' x^n \in Z.$$

Combining (5) and (6), we conclude that

(7) $$(z - z') x^n \in Z.$$

Since R is prime and using Lemma 1, (7) implies that

$$z = z' \text{ or } x^n \in Z.$$

We now distinguish two cases.

Case 1. $(xy)^n - y^n x^n = (yx)^n - x^n y^n = z \in Z$. Then since $y(xy)^n = (yx)^n y$ we conclude that

$$y(y^n x^n + z) = (x^n y^n + z) y,$$

and hence

(8) $$y^{n+1} x^n = x^n y^{n+1}.$$

Case 2. $x^n \in Z$. This implies that

(9) $$y^{n+1} x^n = x^n y^{n+1}.$$

Using (8) and (9), we see that, in either case, $y^{n+1} x^n = x^n y^{n+1}$, which implies that R is commutative by a well-known theorem of Herstein [**6**]. ∎

In preparation for the proof of our next theorem, we first state without proof the following known lemmas (see [**8**, p. 221] and [**10**, Lemma 2]). We use the usual notation $[x, y] = xy - yx$.

LEMMA 2. *If* $[x,y]$ *commutes with* x, *then* $[x^k,y] = kx^{k-1}[x,y]$ *for all positive integers* k.

LEMMA 3. *Let* x, $y \in R$. *Suppose that for some positive integer* k, $x^k y = 0 = (x+1)^k y$. *Then necessarily* $y = 0$.

We use Theorem 1 to prove our next theorem which deals with the case where n is a fixed positive integer.

THEOREM 2. *Let* R *be an associative ring with identity 1 and* n *is a fixed positive integer such that* $(xy)^n - y^n x^n = (yx)^n - x^n y^n \in Z$, $\forall x,y$ *in* R. *If* R *is* $n(n+1)$-*torsion-free, then* R *is commutative.*

PROOF: Let x, y be any two elements of R. From the hypothesis we have

$$(xy)^n - y^n x^n = (yx)^n - x^n y^n = z \in Z. \tag{10}$$

But $y(xy)^n = (yx)^n y$. Using (10), we get

$$y(y^n x^n + z) = (x^n y^n + z)y$$

and since $z \in Z$, we get

$$[y^{n+1}, x^n] = 0, \qquad x,y \in R. \tag{11}$$

Let N be the set of all nilpotent elements of R and let $u \in N$. There exists a minimal positive integer p such that

$$[u^k, x^n] = 0 \text{ for all integers } k \geqslant p,\ p \text{ minimal.} \tag{12}$$

Suppose $p > 1$. Combining (11) and (12), we get

$$0 = [(u^{p-1}+1)^{n+1}, x^n] = (n+1)[u^{p-1}, x^n],$$

and hence $[u^{p-1}, x^n] = 0$, since R is $(n+1)$-torsion-free. But this contradicts the minimality of p. This shows that $p = 1$ and hence by (12)

$$[u, x^n] = 0 \text{ for all } x \in R,\ u \in N. \tag{13}$$

Let S be the subring of R generated by all n-th powers of elements of R. Then by (13) we have

(14) The nilpotent elements of S are central to S.

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From Theorem 1, the commutator ideal of S is nil, and hence by (14) we get

(15) $$[a,b] \text{ is central in } S \text{ for all } a.\ b \in S.$$

Now using (11). (15) and Lemma 2 we obtain

(16) $$na^{n-1}[a,b^{n+1}] = 0 \text{ for all } a,b \in S.$$

Since R is n-torsion-free, (16) implies that $a^{n-1}[a,b^{n+1}] = 0$ for all a, $b \in S$. By replacing a by $(a+1)$ we have $(a+1)^{n-1}[a,b^{n+1}] = 0$, and hence by Lemma 3 we get

(17) $$[a,b^{n+1}] = 0 \text{ for all } a,b \in S.$$

Now using (17), (15) and Lemma 2 we obtain

(18) $$(n+1)b^n[a,b] = 0 \text{ for all } a,b \in S.$$

Since R is $(n+1)$-torsion-free, (18) implies that $b^n[a,b] = 0$ for all a, $b \in S$. By replacing b by $(b+1)$ and applying Lemma 3, we get

(19) $$[a,b] = 0 \text{ for all } a,b \in S.$$

Since S is generated by all n-th powers of elements of R, (19) implies that

(20) $$[x^n,y^n] = 0 \text{ for all } x,y \in R.$$

But $(xy)^n - y^n x^n = (yx)^n - x^n y^n$. This implies using (20) that $(xy)^n = (yx)^n$, and since R is n-torsion-free, R is commutative by a theorem of Bell [**3**]. This completes the proof of Theorem 2. ∎

The following lemma is needed for Theorem 3 below. This lemma is proved in [**7**] by applying a result of Kezlan [**9**] and Bell [**2**].

LEMMA 4. *Let R be a ring such that for each pair of elements x, y in R there exists an integer $n = n(x,y)$ such that $1 \leqslant n \leqslant N$ and $[(xy)^n - x^n y^n, x] = 0$, where N is a fixed positive integer greater than 1. Then the commutator ideal of R is nil.*

Theorem 3 is a generalisation of the theorem of [**1**] mentioned above. The proof of Theorem 3 proceeds exactly as the proof of Theorem 2 except at one point where Theorem 1 is used. Instead, Lemma 4 should be used. We omit the proof of Theorem 3 to avoid repetition.

THEOREM 3. *Let R be an associative ring with identity 1 and n is a fixed positive integer greater than 1, such that $(xy)^n - x^n y^n = (yx)^n - y^n x^n \in Z$ for all x, y in R. If R is $n(n-1)$-torsion-free, then R is commutative.*

Remark. Let $R = \left\{\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} | a,b,c,d \in GF(3)\right\}$. Then $(xy)^3 = x^3 y^3$ and $(xy)^4 = x^4 y^4$. So, with $n = 3$, R is $(n-1)$-torsion-free and $(xy)^n - x^n y^n = (yx)^n - y^n x^n = 0 \in Z$; however, R is not commutative. With $n = 4$, R is n-torsion-free and $(xy)^n - x^n y^n = (yx)^n - y^n x^n = 0 \in Z$ but R is not commutative. This shows that the condition "$n(n-1)$-torsion-free" in Theorem 3 cannot be replaced by "$(n-1)$-torsion-free" or "n-torsion-free".

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