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FINITELY-GENERATED SOLUTIONS OF CERTAIN INTEGRAL EQUATIONS

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Recent work has shown that the solutions of the second-kind integral equation arising from a difference kernel can be expressed in terms of two particular solutions of the equation. This paper establishes analogous results for a wider class of integral operators, which includes the special case of those arising from difference kernels, where the solution of the general case is generated by a finite number of particular cases. The generalisation is achieved by reducing the problem to one of finite rank. Certain non-compact operators, including those arising from Cauchy singular kernels, are amenable to this approach.

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1. Introduction

In recent years, a number of authors have investigated the integral equation

$$\mu\phi(x) = f(x) + \int_0^1 k(x-t)\phi(t)\,dt \quad (0 \leqq x \leqq 1) \tag{1.1}$$

and shown how its solution for any free term f can be expressed in terms of its solutions for certain particular free terms. Such results can lead to considerable economy when numerical techniques have to be employed.

Porter [4] considered (1.1) via the equation

$$(\mu I - K)\phi = f \tag{1.2}$$

in $L_2(0,1)$, where

$$(K\phi)(x) = \int_0^1 k(x-t)\phi(t)\,dt \quad (0 \leqq x \leqq 1), \tag{1.3}$$

and it was assumed that $k \in L_2(-1,1)$. The analysis hinged on the fact that $V_\alpha A + AV_\alpha^*$ is a rank-two operator, where $A = \mu I - K$,

$$(V_\alpha\phi)(x)=\int_0^x e^{-i\alpha(x-t)}\phi(t)\,dt \quad (0\leqq x\leqq 1,\alpha\in\mathbb{R}) \tag{1.4}$$

and V_α^* denotes the adjoint of V_α. In fact,

$$V_\alpha A\phi+AV_\alpha^*\phi=(\phi,\bar{\mu}f_\alpha-V_\alpha l)f_\alpha-(\phi,f_\alpha)V_\alpha k \quad (\phi\in L_2(0,1)), \tag{1.5}$$

in which

$$f_\alpha(x)=e^{-i\alpha x}\,(0\leqq x\leqq 1,\alpha\in\mathbb{R}) \tag{1.6}$$

and l is the kernel adjoint to k, that is,

$$l(x)=\overline{k(-x)}. \tag{1.7}$$

As Porter [4] showed, (1.5) can be used to construct the solution of (1.2) for any $f\in L_2(0,1)$ once two particular solutions are known, corresponding to certain choices of f. A number of such pairs of particular solutions serve this purpose, the pairs being related through (1.5). For instance, a knowledge of ϕ_α and ϕ_β, which satisfy $A\phi_\alpha=f_\alpha$ and $A\phi_\beta=f_\beta$, α and β being distinct, real numbers, is sufficient to solve (1.2) for any $f\in L_2(0,1)$, provided $(\phi_\alpha,f_\beta)\neq 0$. Another solution pair with this property is ψ, χ, where $A\psi=k$ and $A^*\chi=l$, A^* being the adjoint of A. The resulting formula for the solution of (1.2) in terms of ψ and χ was first derived by Gohberg and Feldman [1], by analogy with a parallel structure in matrix algebra, and has been extended to matrix-valued kernels by Mullikin and Victory [2].

Sakhnovich [6] identified another pair of solutions of (1.2), in terms of which the associated resolvent operator can be expressed, by starting from the version of (1.5) in which $\alpha=0$. This pair was generalised by Porter [4] who related it to the pairs ϕ_α, ϕ_β and ψ, χ by using (1.5). Thus, (1.5) can be regarded as a key property of (1.2), capable of producing and connecting solution formulae previously found by various other means and of generating useful new formulae.

The present work seeks to develop existing material in several ways. The pivotal relationship (1.5) is extended by considering operators $A=\mu I-K$ on $L_2(0,1)$ having the property that $V_\alpha A+V_\alpha A^*$ is a finite rank operator. We express this property by means of the notation

$$V_\alpha A\phi+AV_\alpha^*\phi=\sum_m(\phi,b_m)a_m \quad (\phi\in L_2(0,1)), \tag{1.8}$$

where the sum is over finitely many terms.

The condition (1.8) is satisfied by operators $A=\mu I-K$ other than those in which K is generated by a difference kernel. In fact, if A is an invertible operator satisfying an equation of the form (1.8) then so is its inverse. This feature dictates our strategy, which is to establish methods of solving (1.8) for A and to apply these methods to the determination of A^{-1}, thereby deriving explicit formulae for the solution of $\mu\phi = f + K\phi$.

We also broaden the theory by including the practically important class of singular integral equations. That is, we include integral operators on $L_2(0,1)$ of the form $K = aH + bJ$, where a and b are constants, J is a compact operator and H denotes the Hilbert transform defined by

$$(H\phi)(x) = \int_0^1 \frac{\phi(t)\,dt}{t-x}. \tag{1.9}$$

H is a bounded operator on $L_2(0,1)$ but it is not compact.

The generality we seek to introduce requires us to adopt a different approach from that of Porter [4]. Here we seek a structured framework for dealing with operators satisfying (1.8), which subsumes the earlier work. It is also evidently capable of being adapted to other classes of operators, by altering the V_α of (1.8).

2. Some basic results

We make considerable use of the operator V_α on $L_2(0,1)$ defined by (1.4) and note that

$$(V_\alpha + V_\alpha^*)\phi = (\phi, f_\alpha)f_\alpha, \tag{2.1}$$

using the notation of (1.6). In terms of the convolution $\psi * \phi$, where

$$(\psi * \phi)(x) = \int_0^x \psi(x-t)\phi(t)\,dt \quad (\psi, \phi \in L_2(0,1)),$$

we may write

$$V_\alpha \phi = f_\alpha * \phi \tag{2.2}$$

and it is easy to show that

$$V_\alpha(\phi * \psi) = V_\alpha\phi * \psi = V_\alpha\psi * \phi. \tag{2.3}$$

We shall also encounter operators which can be expressed in the form $(U\bar{\psi}) * \phi$, where U is the reflection operator on $L_2(0,1)$ defined by

$$(U\phi)(x) = \phi(1-x).$$

It is convenient to employ the summation convention to express finite rank operators and their consequences in a concise form. Thus we write

$$D\phi=(\phi,b_m)a_m \quad (\phi\in L_2(0,1)), \tag{2.4}$$

D being the operator on $L_2(0,1)$ generated by the kernel

$$d(x,t)=\sum_m a_m(x)\overline{b_m(t)},$$

the sum being over finitely many terms. Only the subscript m will imply summation.

In the notation of (2.4), the operators A under consideration are those which satisfy

$$V_\alpha A+AV_\alpha^*=D \tag{2.5}$$

for some D.

By using (2.1) we see that $(V_\alpha+V_\alpha^*)A\phi=(\phi,A^*f_\alpha)f_\alpha$ and $A(V_\alpha+V_\alpha^*)\phi=(\phi,f_\alpha)Af_\alpha$. Therefore

$$(V_\alpha^*A+AV_\alpha)\phi=(\phi,A^*f_\alpha)f_\alpha+(\phi,f_\alpha)Af_\alpha-(V_\alpha A+AV_\alpha^*)\phi \quad (\phi\in L_2(0,1)), \tag{2.6}$$

and it follows that if A satisfies (2.5) then $V_\alpha^*A+AV_\alpha$ is a finite rank operator and conversely. Clearly, if $V_\alpha A+AV_\alpha^*$ is an operator of rank n then $V_\alpha^*A+AV_\alpha$ has rank at most $n+2$ and we could replace (2.5) by the equivalent condition $V_\alpha^*A+AV_\alpha=D'$, where D' is a finite rank operator. It turns out to be convenient to consider $V_\alpha A+AV_\alpha^*$ and $V_\alpha^*A+AV_\alpha$ separately, although the relationship between them will prove to be useful.

We therefore consider (2.5) in conjunction with $V_\alpha^*A+AV_\alpha=D$, using the operator D again as a matter of notational convenience and temporarily setting aside (2.4). Our first objective is to solve the two relationships for A, assuming D to be given. In a practical problem, A will of course be given and it is A^{-1} that we shall seek, using the following results.

Lemma 1. *Let A be a bounded operator on $L_2(0,1)$. Then*

(i) $V_\alpha A+AV_\alpha^*=0\Leftrightarrow A=0$,

(ii) $V_\alpha^*A+AV_\alpha=0\Leftrightarrow A=0$.

Proof. Suppose for the moment that A is an operator of the form

$$(A\phi)(x)=\int_0^1 k(x,t)\phi(t)\,dt \quad (0\leqq x\leqq 1)$$

where k possesses continuous first-order partial derivatives. Then the kernel of the integral operator $V_0A+AV_0^*$ is $\int_0^x k(s,t)\,ds+\int_0^t k(x,u)\,du$. Therefore (since this function is continuous) if $V_0A+AV_0^*=0$ then

$$\int_0^x k(s,t)\,ds+\int_0^t k(x,u)\,du=0 \quad (0\leqq x,t\leqq 1).$$

Differentiating in turn with respect to x and t yields $k_t(x,t)+k_x(x,t)=0$ for $0 \leqq x, t \leqq 1$ whence $k(x,t)=f(x-t)$ for some function f. However, $k(0,s)=k(s,0)=0$ for $0 \leqq s \leqq 1$ whence $f(t)=0$ for $-1 \leqq t \leqq 1$ and $k(x,t)=0$ $(0 \leqq x, t \leqq 1)$. In this case $A=0$.

A similar argument shows that if A arises from a C_1 kernel and $V_0^*A+AV_0=0$ then $A=0$.

Now let A be a typical bounded operator and suppose that $V_\alpha A+AV_\alpha^*=0$. Then, defining U_α by $(U_\alpha\phi)(x)=e^{-i\alpha x}\phi(x)$ we have $V_\alpha=U_\alpha V_0 U_\alpha^*$, and $U_\alpha U_\alpha^*=U_\alpha^* U_\alpha=I$. Therefore

$$0=V_\alpha A+AV_\alpha^*=U_\alpha V_0 U_\alpha^* A+AU_\alpha V_0^* U_\alpha^*=U_\alpha(V_0(U_\alpha^* A U_\alpha)+(U_\alpha^* A U_\alpha)V_0^*)U_\alpha^*$$

whence $V_0B+BV_0^*=0$ where $B=U_\alpha^* A U_\alpha$. The invertibility of U_α and U_α^* guarantees that $B=0 \Rightarrow A=0$, so it is enough to show the result for $\alpha=0$.

Now if $V_0A+AV_0^*=0$ it follows that $V_0(V_0^n A V_0^{*n})+(V_0^n A V_0^{*n})V_0^*=0$. Because V_0 is a Hilbert–Schmidt operator, so is $V_0AV_0^*$, and it is therefore generated by an L_2-kernel on $[0,1]\times[0,1]$. Then $V_0^3AV_0^{*3}$ is generated by a kernel whose first order partial derivatives exist and are continuous. But $V_0(V_0^3AV_0^{*3})+(V_0^3AV_0^{*3})V_0^*=0$ so by the first part $V_0^3AV_0^{*3}=0$. Since V_0 is injective, it follows that $AV_0^{*3}=0$, $V_0^3A^*=0$ and, in turn, $A=0$.

The case $V_\alpha^*A+AV_\alpha=0$ is reduced to $V_0^*A+AV_0=0$ and thence to the situation where A arises from a continuously differentiable kernel, in a similar way, or by noticing that $V_\alpha^*A+AV_\alpha=0$ implies that $V_\alpha^*A^*+A^*V_\alpha=0$ and using (i).

Theorem 1. *Let A be a bounded operator on* $L_2(0,1)$ and let D denote the finite rank operator on $L_2(0,1)$ *defined by* $D\phi=(\phi,b_m)a_m$.

(i) $V_\alpha A+AV_\alpha^*=D \Leftrightarrow V_\alpha AV_\alpha^*=A_mB_m^*$, *where* $A_m\phi=a_m*\phi$, $B_m\phi=b_m*\phi$.

(ii) $V_\alpha^*A+AV_\alpha=D \Leftrightarrow V_\alpha^*AV_\alpha=\tilde{A}_m^*\tilde{B}_m$, *where* $\tilde{A}_m\phi=(U\bar{a}_m)*\phi$, $\tilde{B}_m\phi=(U\bar{b}_m)*\phi$.

Proof. Note first that, by (2.2), $V_\alpha a_m=f_\alpha*a_m=a_m*f_\alpha=A_mf_\alpha$ and that $V_\alpha b_m=B_mf_\alpha$. Therefore

$$\begin{aligned} V_\alpha DV_\alpha^*\phi &= (\phi, V_\alpha b_m)V_\alpha a_m \\ &= (B_m^*\phi, f_\alpha)A_mf_\alpha \\ &= A_m(V_\alpha+V_\alpha^*)B_m^*\phi, \end{aligned}$$

using (2.1). Since A_m and B_m commute with V_α (all being defined by convolutions) we thus have

$$V_\alpha DV_\alpha^*=V_\alpha A_mB_m^*+A_mB_m^*V_\alpha^*. \tag{2.7}$$

Now $V_\alpha A+AV_\alpha^*=D$ implies

$$V_\alpha(V_\alpha AV_\alpha^*)+(V_\alpha AV_\alpha^*)V_\alpha^*=V_\alpha DV_\alpha^*$$

which combines with (2.7) to imply that $V_\alpha AV_\alpha^*=A_mB_m^*$, by Lemma 1. Conversely, if $V_\alpha AV_\alpha^*=A_mB_m^*$ then

$$\begin{aligned}V_\alpha(V_\alpha AV_\alpha^*)\phi+(V_\alpha AV_\alpha^*)V_\alpha^*\phi&=A_m(V_\alpha+V_\alpha^*)B_m^*\phi\\&=V_\alpha DV_\alpha^*\phi,\end{aligned}$$

by (2.7). Therefore $V_\alpha(V_\alpha A+AV_\alpha^*-D)V_\alpha^*=0$, which implies that $V_\alpha A+AV_\alpha^*=D$.

The proof of (ii) follows in a similar way using

$$V_\alpha^*a_m=e^{i\alpha}\tilde{A}_m^*f_\alpha,\ V_\alpha^*b_m=e^{i\alpha}\tilde{B}_m^*f_\alpha,$$

which are easy to establish. □

A further step is required to solve $V_\alpha A+AV_\alpha^*=D$ for A, which, by the theorem, is equivalent to solving $V_\alpha AV_\alpha^*=A_mB_m^*$. In some cases (that is, for some operators A) this step is virtually immediate as we now show.

Theorem 2. *Let A be a bounded operator on $L_2(0,1)$ and let D denote the finite rank operator on $L_2(0,1)$ defined by $D\phi=(\phi,b_m)a_m$.*

(i) *If there exist bounded operators P_m and Q_m such that*

$$V_\alpha P_m\phi=a_m*\phi,\ V_\alpha Q_m\phi=b_m*\phi$$

for all relevant m, then

$$V_\alpha A+AV_\alpha^*=D\Leftrightarrow A=P_mQ_m^*.$$

(ii) *If there exist bounded operators $\tilde{P}_m$ and $\tilde{Q}_m$ such that*

$$V_\alpha\tilde{P}_m\phi=(U\bar{a}_m)*\phi,\ V_\alpha\tilde{Q}_m\phi=(U\bar{b}_m)*\phi$$

for all relevant m, then

$$V_\alpha^*A+AV_\alpha=D\Leftrightarrow A=\tilde{P}_m^*\tilde{Q}_m.$$

Proof. We notice that, since the operators A_m (where $A_m\phi=a_m*\phi$) and V_α commute (both being defined by convolutions), $A_m=V_\alpha P_m$ implies that P_m and V_α also commute. Similarly, Q_m, $\tilde{P}_m$ and $\tilde{Q}_m$ commute with V_α. Using $A_m=V_\alpha P_m$ and $B_m=V_\alpha Q_m$ in Theorem 1(i) gives

$$V_\alpha A + AV_\alpha^* = D \Leftrightarrow V_\alpha(A - P_m Q_m^*)V_\alpha^* = 0 \Leftrightarrow A = P_m Q_m^*.$$

Using $\tilde{A}_m = V_\alpha \tilde{P}_m = \tilde{P}_m V_\alpha$ and $\tilde{B}_m = V_\alpha \tilde{Q}_m = \tilde{Q}_m V_\alpha$ in Theorem 1(ii) gives

$$V_\alpha^* A + AV_\alpha = D \Leftrightarrow V_\alpha^*(A - \tilde{P}_m^* \tilde{Q}_m)V_\alpha = 0 \Leftrightarrow A = \tilde{P}_m^* \tilde{Q}_m. \qquad \square$$

The class of operators satisfying (2.5) therefore contains those of the form $A = P_m Q_m^*$ where P_m and Q_m are bounded operators which commute with V_α. In fact, $V_\alpha A\phi + AV_\alpha^*\phi = (\phi, Q_m f_\alpha)P_m f_\alpha$ for such A.

Theorem 2 does not apply to all of the operators having the property (2.5). Although Theorem 1 shows that the sum $A_m B_m^*$ is equal to $V_\alpha A V_\alpha^*$, it may not be the case that V_α and V_α^* can be removed from the sum on a term-by-term basis. The extraction of A from $V_\alpha A V_\alpha^* = A_m B_m^*$ may need to be carried out for the sum as a whole. For some operators A, the intermediate relationship $V_\alpha A = A_m Q_m^*$ (or $AV_\alpha^* = P_m B_m^*$) can be deduced, the removal of V_α (or V_α^*) being possible only in an overall sense.

Further deductions can of course be made if A is known to satisfy an additional condition, over and above (2.5), such as

$$A^* = \bar{U} A \bar{U}, \tag{2.8}$$

where $(\bar{U}\phi)(x) = \overline{\phi(1-x)}$ $(0 \leqq x \leqq 1)$. Notice that $\bar{U}$ is not a linear map, but conjugate-linear. If A is of the form $\mu I - K$, where K is an integral operator, (2.8) implies that the kernel k generating K satisfies $k(x,t) = k(1-t, 1-x)$ (for almost all x and t). As the important class of difference kernels falls into this category, we pursue the consequences of (2.8).

The condition (2.5) implies that $V_\alpha A^* + A^* V_\alpha^* = D^*$ and if (2.8) is used to remove A^*, we find that

$$V_\alpha^* A + AV_\alpha = \bar{U} D^* \bar{U},\ \bar{U} D^* \bar{U}\phi = (\phi, U\bar{a}_m)U\bar{b}_m, \tag{2.9}$$

because $V_\alpha \bar{U} = \bar{U} V_\alpha^*$, which is easily verified. Therefore $V_\alpha A + AV_\alpha^*$ and $V_\alpha^* A + AV_\alpha$ have the same rank in this case. Further, we find that Theorem 1 applied to (2.9) gives $V_\alpha^* A V_\alpha = B_m^* A_m$ and we therefore have two formulae for A, namely

$$V_\alpha A V_\alpha^* = A_m B_m^*,\ V_\alpha^* A V_\alpha = B_m^* A_m. \tag{2.10}$$

Similarly, if Theorem 2 applies it may be modified if (2.8) is in force to give

$$A = P_m Q_m^* = Q_m^* P_m. \tag{2.11}$$

One more useful deduction follows from (2.8). Making use of (2.5) and (2.9) in (2.6) shows that

$$(\phi, b_m)a_m + (\phi, U\bar{a}_m)U\bar{b}_m = (\phi, A^* f_\alpha)f_\alpha + (\phi, f_\alpha)Af_\alpha.$$

Setting $\phi = f_\alpha$ therefore,

$$Af_\alpha = (f_\alpha, b_m)a_m + (f_\alpha, U\bar{a}_m)U\bar{b}_m - (Af_\alpha, f_\alpha)f_\alpha$$

and, taking the inner product of both sides with f_α and using $U\bar{f}_\alpha = e^{i\alpha}f_\alpha$, we find that $(Af_\alpha, f_\alpha) = (a_m, f_\alpha)(f_\alpha, b_m)$. Hence

$$Af_\alpha = (f_\alpha, b_m)a_m + e^{-i\alpha}(a_m, f_\alpha)U\bar{b}_m - (a_m, f_\alpha)(f_\alpha, b_m)f_\alpha. \tag{2.12}$$

We give some illustrations at this stage to fix ideas.

Suppose that $A = \mu I - K$, where K is defined by (1.3) and $k \in L_1(-1,1)$. According to (1.5), $V_\alpha A + AV_\alpha^*$ is a rank-two operator and, in the notation of (2.4) and (2.5),

$$a_1 = -b_2 = f_\alpha, a_2 = V_\alpha k, b_1 = \bar{\mu}f_\alpha - V_\alpha l. \tag{2.13}$$

Therefore the operators A_m and B_m arising in this case are given by

$$A_1\phi = -B_2\phi = f_\alpha * \phi = V_\alpha\phi,\ A_2\phi = V_\alpha k * \phi = V_\alpha(k * \phi),$$

$$B_1\phi = \bar{\mu}f_\alpha * \phi - V_\alpha l * \phi = V_\alpha(\bar{\mu}\phi - l * \phi),$$

where (2.2) and (2.3) have been used. Obviously Theorem 2 applies here, with

$$P_1 = -Q_2 = I,\ P_2\phi = k * \phi,\ Q_1\phi = \bar{\mu}\phi - l * \phi,$$

using which it easily follows that $A = P_m Q_m^* = \mu I - K$ is recovered, because of (1.7). Since this A satisfies (2.8), we also have $A = Q_m^* P_m$, by (2.11), ,while (2.12) gives

$$Af_\alpha = \mu f_\alpha - V_\alpha k - e^{-i\alpha}U\overline{V^\alpha l}. \tag{2.14}$$

Now, let A be the different operator $\mu I - H$, where H is the Hilbert transform defined by (1.9) and $H\phi$ can be evaluated as a Cauchy principal value if ϕ is sufficiently smooth. Here we find that

$$V_\alpha A\phi + AV_\alpha^*\phi = (\phi, \bar{\mu}f_\alpha - p_\alpha)f_\alpha + (\phi, f_\alpha)p_\alpha \tag{2.15}$$

where

$$p_\alpha = (I - i\alpha V_\alpha)q,\ q(x) = \log x\ (0 < x \leqq 1).$$

For the purpose of this example we may take $\alpha = 0$ in which case the associated rank-two operator D has the elements

$$a_1 = b_2 = f_0,\ a_2 = q,\ b_1 = \bar{\mu}f_0 - q.$$

Writing $Q\phi = q * \phi$, we thus have

$$A_1\phi = B_2\phi = V_0\phi,\ A_2\phi = Q\phi,\ B_1\phi = \bar{\mu}V_0\phi - Q\phi$$

and Theorem 1 gives

$$V_0AV_0^* = V_0(\mu V_0^* - Q^*) + QV_0^*. \tag{2.16}$$

As there is no bounded operator P such that $Q = V_0P$, Theorem 2 does not apply and a more subtle approach is needed to extract A. Since V_0 and Q commute and $Qf_0 = q * f_0 = f_0 * q = V_0q$, (2.16) implies that

$$V_0AV_0^*\phi = \mu V_0V_0^*\phi - (V_0 + V_0^*)Q^*\phi + (Q + Q^*)V_0^*\phi$$

$$= \mu V_0V_0^*\phi - (Q^*\phi, f_0)f_0 + (Q + Q^*)V_0^*\phi$$

$$= \mu V_0V_0^*\phi - (V_0^*\phi, q)f_0 + (Q + Q^*)V_0^*\phi,$$

where (2.1) has been used. Hence

$$V_0A\phi = \mu V_0\phi - (\phi, q)f_0 + (Q + Q^*)\phi$$

and V_0 is removed by differentiation to give

$$(A\phi)(x) = \mu\phi(x) + \frac{d}{dx}\int_0^1 \log|x - t|\phi(t)\,dt,$$

almost everywhere in $[0, 1]$. The second term on the right-hand side can be shown to be equal to $-(H\phi)(x)$ (almost everywhere), as required.

The condition (2.8) is satisfied by $A = \mu I - H$ and (2.10) gives an alternative to (2.16), namely,

$$V_0^*AV_0 = (\mu V_0^* - Q^*)V_0 + V_0^*Q,$$

which may be solved for A by a similar rearrangement to that used for (2.16).

It is not surprising, of course, that singular integrals should introduce additional complication into the proceedings. The following lemma, which will prove to be useful later, gives another example in which Theorem 2 does not apply directly.

Lemma 2. *Let* $p, q \in L_2(0, 1)$. *Then*

(i) $V_\alpha A\phi + AV_\alpha^*\phi = (\phi, p)V_\alpha q + (\phi, V_\alpha p)q \Leftrightarrow A\phi = (\phi, p)q.$

(ii) $V_\alpha^*A\phi + AV_\alpha\phi = (\phi, p)V_\alpha^*q + (\phi, V_\alpha^*p)q \Leftrightarrow A\phi = (\phi, p)q.$

Proof. With $A\phi = (\phi, p)q$ it follows at once that

$$V_\alpha A\phi + AV_\alpha^*\phi = (\phi, p)V_\alpha q + (\phi, V_\alpha p)q.$$

Conversely, given the previous equation and applying Lemma 1 gives the result. □

One final illustration is required to dispel the idea that we are, in effect, only able to deal with the operator $A=\mu I-K$, where K is generated by a difference kernel. We have already noted that the class of eligible operators includes those of the form $A=P_mQ_m^*$ where P_m and Q_m are bounded operators which commute with V_α. A simple example of such an operator is $A=I+K$ where

$$(K\phi)(x)=\int_0^1 \log\left|\frac{x^{1/2}+t^{1/2}}{x^{1/2}-t^{1/2}}\right| \phi(t)\,dt.$$

This A can be written in the form $A=P_1P_1^*+P_2P_2^*$ where $P_1=I$ and

$$(P_2\phi)(x)=\int_0^x (x-t)^{-1/2}\phi(t)\,dt.$$

These operators P_1 and P_2 have the required properties and therefore $V_\alpha A+AV_\alpha^*$ has rank two by the foregoing theory. It can be confirmed that $(V_\alpha K+KV_\alpha^*)\phi=(\phi,P_2f_\alpha)P_2f_\alpha$ whence $V_\alpha KV_\alpha^*\phi=CC^*\phi$ where $C\phi=\int_0^x(P_2f_\alpha)(x-t)\phi(t)\,dt$. It is easy to check that $C=V_\alpha P_2$ whence the expression of K in the form $P_2^*P_2$ can be recovered.

3. Inverse operators

Let k: $[0,1]\times[0,1]\to\mathbb{C}$ be such that

$$(K\phi)(x)=\int_0^1 k(x,t)\phi(t)\,dt \tag{3.1}$$

defines a bounded operator on $L_2(0,1)$ and suppose that $A=\mu I-K$ is an invertible operator with the property (2.5), that is,

$$V_\alpha A+AV_\alpha^*=D,\ D\phi=(\phi,b_m)a_m, \tag{3.2}$$

for some (known) $a_m, b_m\in L_2(0,1)$.

It follows from (3.2) that

$$V_\alpha^*A^{-1}+A^{-1}V_\alpha=E,\ E\phi=(\phi,A^{*-1}b_m)A^{-1}a_m. \tag{3.3}$$

Obviously, E has the same rank as D. Adapting (2.6) we find that

$$\left.\begin{aligned}V_\alpha A^{-1}+A^{-1}V_\alpha^*=F,\\ F\phi=(\phi,A^{*-1}f_\alpha)f_\alpha+(\phi,f_\alpha)A^{-1}f_\alpha-E\phi.\end{aligned}\right\}\tag{3.4}$$

The rank of F exceeds that of E by no more than two. Rather than make use of (3.4), the counterpart of (3.2), we base the determination of A^{-1} on the apparently less cumbersome (3.3).

Let the elements c_m and d_m of $L_2(0,1)$ be defined by

$$Ac_m=a_m, A^*d_m=b_m.\tag{3.5}$$

Then (3.3) may be written as

$$V_\alpha^*A^{-1}+A^{-1}V_\alpha=E,\ E\phi=(\phi,d_m)c_m\tag{3.6}$$

and, according to Theorem 1,

$$\left.\begin{aligned}V_\alpha^*A^{-1}V_\alpha=\tilde{C}_m^*\tilde{D}_m,\\ \tilde{C}_m\phi=(U\bar{c}_m)*\phi,\ \tilde{D}_m\phi=(U\bar{d}_m)*\phi.\end{aligned}\right\}\tag{3.7}$$

Theorem 2 gives A^{-1} explicitly as

$$\left.\begin{aligned}A^{-1}=\tilde{S}_m^*\tilde{T}_m,\\ V_\alpha\tilde{S}_m=\tilde{C}_m,\ V_\alpha\tilde{T}_m=\tilde{D}_m,\end{aligned}\right\}\tag{3.8}$$

provided that the operators $\tilde{S}_m$ and $\tilde{T}_m$ so defined exist. If this is not the case, A^{-1} has to be determined from (3.7) less directly. Either way, A^{-1} can be considered known once c_m and d_m have been found.

For some operators A, an explicit formula for the resolvent kernel can be deduced from (3.8). We wish to express $\tilde{C}_m$ in the form $V_\alpha\tilde{S}_m$ and $\tilde{D}_m$ in the form $V_\alpha\tilde{T}_m$, which is possible if and only if $U\bar{c}_m$ has the form $\lambda f_\alpha-V_\alpha\psi$ for some $\psi\in L_2(0,1)$ and $\lambda\in\mathbb{C}$ or equivalently $c_m=\lambda' f_\alpha-V_\alpha^*\psi'$ for $\psi'\in L_2(0,1)$ and $\lambda'\in\mathbb{C}$, with similar conditions for d_m. This is true, in particular, if all of c_m and d_m are continuous $(0,1]$, differentiable on $(0,1)$ and their derivatives are in $L_2(0,1)$. Then if we define p_m and q_m by

$$p_m=c_m'+i\alpha c_m,\ q_m=d_m'+i\alpha d_m\tag{3.9}$$

we have $c_m=c_m(1)e^{i\alpha}f_\alpha-V_\alpha^*\,p_m$ and $d_m=d_m(1)e^{i\alpha}f_\alpha-V_\alpha^*q_m$. Defining $\tilde{S}_m$ and $\tilde{T}_m$ by

$$\tilde{S}_m\phi=\overline{c_m(1)}\phi-(U\bar{p}_m)*\phi,\ \tilde{T}_m\phi=\overline{d_m(1)}\phi-(U\bar{q}_m)*\phi\tag{3.10}$$

now yields $V_\alpha\tilde{S}_m=\tilde{C}_m$ and $V_\alpha\tilde{T}_m=\tilde{D}_m$.

Forming $A^{-1}=\tilde{S}_m^*\tilde{T}_m$ in this case leads to

$$A^{-1}=c_m(1)\overline{d_m(1)}I+R \tag{3.11}$$

where

$$(R\phi)(x)=\int_0^1 r(x,t)\phi(t)\,dt,$$

the resolvent kernel $r(x,t)$ being defined for almost all x and t in $[0,1]$ by

$$r(x,t)=\int_{\max(x,t)}^1 p_m(1-s+x)\overline{q_m(1-s+t)}\,ds$$

$$-\begin{Bmatrix} c_m(1)\overline{q_m(1-x+t)} & (t\leqq x) \\ \overline{d_m(1)}p_m(1-t+x) & (x\leqq t)\end{Bmatrix}. \tag{3.12}$$

If K is a compact operator, we can be sure that $\mu^{-1}=c_m(1)\overline{d_m(1)}$, since (3.11) must have the form

$$A^{-1}=\mu^{-1}I+R, \tag{3.13}$$

where R is compact.

Now suppose that A satisfies (2.8), in which cases so does A^{-1}; that is $\bar{U}A^{-1}\bar{U}=(A^{-1})^*$, and the foregoing formulae can be amplified. It follows, for instance, that the operator F of (3.4) is given by $F=\bar{U}E^*\bar{U}$ and application of Theorem 1 to $V_\alpha A^{-1}+A^{-1}V_\alpha^*=\bar{U}E^*\bar{U}$ results in $V_\alpha A^{-1}V_\alpha^*=\tilde{D}_m\tilde{C}_m^*$. We therefore have

$$V_\alpha^*A^{-1}V_\alpha=\tilde{C}_m^*\tilde{D}_m,\ V_\alpha A^{-1}V_\alpha^*=\tilde{D}_m\tilde{C}_m^*, \tag{3.14}$$

if (2.8) holds, $\tilde{C}_m$ and $\tilde{D}_m$ being defined in (3.7). In consequence, (3.8) can be revised to give

$$A^{-1}=\tilde{S}_m^*\tilde{T}_m=\tilde{T}_m\tilde{S}_m^*, \tag{3.15}$$

if $\tilde{S}_m$ and $\tilde{T}_m$ exist. The formulae (3.14) and (3.15) are the counterparts for A^{-1} of (2.10) and (2.11).

An alternative expression for the resolvent kernel r results from $A^{-1}=\tilde{T}_m\tilde{S}_m^*$, under the same conditions as those prevailing for the version (3.12). This is

$$r(x,t)=\int_0^{\min(x,t)} p_m(1-t+s)\overline{q_m(1-x+s)}\,ds-\begin{Bmatrix} c_m(1)\overline{q_m(1-x+t)} & (t\leqq x) \\ \overline{d_m(1)}p_m(1-t+x) & (x\leqq t)\end{Bmatrix}. \tag{3.16}$$

We note that, if $A^*\phi=f$ and if (2.8) holds, then $AU\bar{\phi}=U\bar{f}$, showing that (3.5) can be replaced in this case by

$$Ac_m=a_m, AU\bar{d}_m=U\bar{b}_m. \tag{3.17}$$

4. The determination of c_m and d_m

We have shown how A^{-1} can be expressed in terms of the solutions of

$$Ac_m=a_m, A^*d_m=b_m, \tag{4.1}$$

a_m and b_m being defined by

$$V_\alpha A\phi+AV_\alpha^*\phi=(\phi,b_m)a_m. \tag{4.2}$$

This relationship can be used to relate c_m and d_m to other elements of $L_2(0,1)$, allowing (4.1) to be replaced by other equations which may be more convenient to solve.

Let ϕ_α denote the solution of $A\phi=f_\alpha$, where $\alpha\in\mathbb{R}$ and let $\beta\in\mathbb{R}$ be distinct from α. Since

$$V_\alpha f_\beta=i(\beta-\alpha)^{-1}(f_\beta-f_\alpha)=i(\beta-\alpha)^{-1}A(\phi_\beta-\phi_\alpha),$$

setting $\phi=\phi_\beta$ in (4.2) gives

$$i(\beta-\alpha)^{-1}A(\phi_\beta-\phi_\alpha)+AV_\alpha^*\phi_\beta=(\phi_\beta,b_m)a_m.$$

Therefore, using (4.1),

$$(\phi_\beta,b_m)c_m=i(\beta-\alpha)^{-1}(\phi_\beta-\phi_\alpha)+V_\alpha^*\phi_\beta. \tag{4.3}$$

If the rank of $V_\alpha A+AV_\alpha^*$ is n, we can choose n distinct values of β, β_j say, all different from α, giving

$$(\phi_{\beta_j},b_m)c_m=i(\beta_j-\alpha)^{-1}(\phi_{\beta_j}-\phi_\alpha)+V_\alpha^*\phi_{\beta_j}\ (j=1,\dots,n).$$

Thus, if the solution of $A\phi_\alpha=f_\alpha$ is known for $n+1$ distinct values of α, this system of equations provides the required $c_1,\dots,c_n$, as long as $\det(\phi_{\beta_j},b_m)\neq 0$. In practical terms, the solution of $A\phi_\alpha=f_\alpha$ is required for $\alpha\in\mathbb{R}$.

To determine $d_1,\dots,d_n$, let $A^*\psi_\alpha=f_\alpha$, where $\alpha\in\mathbb{R}$, and note that (4.2) implies

$$V_\alpha A^*\psi_\beta+A^*V_\alpha\psi_\beta=(\psi_\beta,a_m)b_m,$$

which leads to the system of equations

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$$(\psi_{\beta_j}, a_m)d_m = i(\beta_j - \alpha)^{-1}(\psi_{\beta_j} - \psi_\alpha) + V_\alpha^* \psi_{\beta_j}\ (j=1,\ldots,n).$$

We have shown that the equations can generally be replaced by the auxiliary equations $A\phi_\alpha = f_\alpha$ and $A^*\psi_\alpha = f_\alpha$ for various values of α, whatever the rank of $V_\alpha A + AV_\alpha^*$. If A satisfies (2.8) then $\psi_\alpha = e^{-i\alpha} U\bar{\phi}_\alpha$ so only one of these sets of equations needs to be solved.

Other auxiliary equations can be devised, in which the free terms are polynomials rather than exponentials, for instance. Let $A\chi_j = g_j$ where $g_j(x) = x^{j-1}$ $(j \in \mathbb{N})$ and note that, with $\alpha = 0$, (4.2) gives

$$V_0 A\chi_j + AV_0^* \chi_j = (\chi_j, b_m)a_m.$$

Since $V_0 g_j = j^{-1} g_{j+1} = j^{-1} A\chi_{j+1}$, and using (4.1), we deduce that

$$(\chi_j, b_m)c_m = j^{-1}\chi_{j+1} + V_0^*\chi_j. \tag{4.4}$$

By taking $j = 1,\ldots,n$, we produce a system of equations for $c_1,\ldots,c_n$. Another system for $d_1,\ldots,d_n$ can be constructed in a similar way from the adjoint of (4.2).

To express this process in general terms, suppose that $A\Phi_j = h_j$ $(j=1,\ldots,n)$ for some chosen h_j. Then (4.1) and (4.2) imply that

$$V_\alpha h_j + AV_\alpha^*\Phi_j = (\Phi_j, b_m)Ac_m\ (j=1,\ldots,n).$$

This reduces to a set of equations for $c_1,\ldots,c_n$ if Ψ_j $(j=1,\ldots,n)$ are known satisfying $A\Psi_j = V_\alpha h_j$, for then

$$(\Phi_j, b_m)c_m = \Psi_j + V_\alpha^*\Phi_j\ (j=1,\ldots,n).$$

The $2n$ elements $\Phi_1,\ldots,\Phi_n$, $\Psi_1,\ldots,\Psi_n$ are thus required to determine $c_1,\ldots,c_n$. In the two specific cases considered above the number of distinct elements needed is reduced by virtue of relationships between the sets (h_j) and $(V_\alpha h_j)$. Choices of (h_j) other than those we have given have this desirable property, such as $h_j = P_{j-1}$, where P_j is the Legendre polynomial of order j.

There is evidently scope for representing (c_m) (and similarly (d_m)) in terms of other elements of $L_2(0,1)$ and hence for representing A^{-1}.

The issue of whether the elements c_m and d_m have the properties which make (3.12) available will be considered by means of examples in the next section.

5. Applications

(a) Compact operators with difference kernels

The natural starting point for examining how the theory may be put into practice is to consider $A = \mu I - K$ with K given by (1.3). We assume that the kernel k generating K

is such that $k \in L_2(-1,1)$. This is a case which has been investigated by other authors and we can show how existing and new representations of A^{-1} can be constructed by the approach developed above. (As before, we assume that $\mu I - K$ is invertible.)

For this A, $V_\alpha A + AV_\alpha^*$ is a rank-two operator, D, where $D\phi = (\phi, b_m)a_m$, where (from (2.13)) we recall that

$$a_1 = -b_2 = f_\alpha,\ a_2 = V_\alpha k \quad \text{and} \quad b_1 = \bar{\mu} f_\alpha - V_\alpha l.$$

In forming the finite-rank operator $E = (\phi, d_m)c_m$, where $Ac_m = a_m$ and $A^* d_m = b_m$ we notice first that $c_1 = \phi_\alpha$ and $d_2 = -\psi_\alpha$. Because, in this case, the condition (2.8), $A^* = \bar{U} A \bar{U}$, holds we can immediately relate ϕ_α and ψ_α since $\psi_\alpha = e^{-i\alpha} U \bar{\phi}_\alpha$. A further equation arises from (2.14) showing that $U\bar{b}_1 = e^{i\alpha}(\mu f_\alpha - e^{-i\alpha} \bar{U} V_\alpha l) = e^{i\alpha}(A f_\alpha + V_\alpha k) = e^{i\alpha}(A f_\alpha + a_2)$. In this case, therefore

$$c_1 = \phi_\alpha,\ Ac_2 = V_\alpha k \tag{5.1}$$

and

$$AU\bar{d}_1 = e^{i\alpha}(A f_\alpha + Ac_2),$$

giving

$$U\bar{d}_1 = e^{i\alpha}(f_\alpha + c_2),\ U\bar{d}_2 = -e^{i\alpha} c_1. \tag{5.2}$$

Therefore d_1 and d_2 may be obtained from c_1 and c_2 which may, in turn, be obtained from (5.1).

The direct approach, then, is to chose

$$c_1 = \phi_\alpha,\ c_2 = \omega_\alpha \tag{5.3}$$

where $A\omega_\alpha = V_\alpha k$. This leads to the representation of A^{-1} derived (in the case $\alpha = 0$) by Sakhnovich [**6**]. This is, however, less appealing than alternative formulae for E since the calculation of ω_α is not as simple as others.

At this point we notice that it is the operator E which we seek rather than the individual vectors c_m and d_m which form it. We therefore write $E = (\phi, \tilde{d}_m)\tilde{c}_m$ and seek other representations by varying $\tilde{c}_m$ and $\tilde{d}_m$. Since E is to remain fixed the subspace spanned by $\tilde{c}_1$ and $\tilde{c}_2$ will be that spanned by c_1 and c_2, with the corresponding result for $\tilde{d}_1$ and $\tilde{d}_2$. We therefore seek $\tilde{c}_m$ and $\tilde{d}_m$ satisfying (5.2) for which $E = (\phi, \tilde{d}_m)\tilde{c}_m$. It is easily checked that if we set $\tilde{c}_2 = c_2 - \lambda c_1$ for a scalar λ and define $\tilde{d}_1$ by (5.2), with $\tilde{c}_1 = c_1$, then we obtain a representation of E, irrespective of the choice of λ.

From (4.3) we have, for $\beta \neq \alpha$,

$$(\phi_\beta, b_1)c_1 + (\phi_\beta, b_2)c_2 = i(\beta - \alpha)^{-1}(\phi_\beta - \phi_\alpha) + V_\alpha^* \phi_\beta$$

whence, by adding a suitable multiple of c_1 to c_2 to obtain $\tilde{c}_2$ we may choose $\tilde{c}_2$ to satisfy

$$(\phi_\beta, f_\alpha)\tilde{c}_2 = -i(\beta-\alpha)^{-1}(\phi_\beta - \phi_\alpha) - V_\alpha^* \phi_\beta \tag{5.4}$$

(noting that $b_2 = -f_\alpha$). Since $\{f_{2p\pi}: p \in \mathbb{Z}\}$ is a complete orthonormal set and A is invertible, for each $\alpha \in \mathbb{R}$ we can certainly choose β such that $(\phi_\beta, f_\alpha) \neq 0$. In this case finding ϕ_α and ϕ_β yields the operator E.

Alternatively, we could use (4.4) with $\alpha = 0$ in this case to give (with $j=1$)

$$(\chi_1, b_1)\tilde{c}_1 + (\chi_1, b_2)\tilde{c}_2 = \chi_2 + V_0^* \chi_1.$$

Here we may choose $\tilde{c}_1 = c_1 = \phi_0 (= \chi_1)$ and

$$(\chi_1, g_1)\tilde{c}_2 = -\chi_2 - V_0^* \chi_1 \tag{5.5}$$

where $g_1(x) \equiv 1$ as in Section 4, since $b_2 = -g_1$.

Now $\bar{U} V_\alpha = V_\alpha^* \bar{U}$ (using the notation $\bar{U}$ introduced with (2.8)) and $\bar{U} f_\alpha = e^{i\alpha} f_\alpha$, so, using $(V_\alpha + V_\alpha^*)\phi = (\phi, f_\alpha) f_\alpha$, we obtain $\bar{U} V_\alpha l = V_\alpha^* U \bar{l} = e^{i\alpha}(f_\alpha, l) f_\alpha - V_\alpha U \bar{l}$. Using this in (2.14) yields

$$A f_\alpha = \{\mu - (f_\alpha, l)\} f_\alpha - V_\alpha \{k - e^{-i\alpha} U \bar{l}\}. \tag{5.6}$$

From (1.5) we have, for $w_\alpha \in L_2(0,1)$

$$A V_\alpha^* w_\alpha = (w_\alpha, \bar{\mu} f_\alpha - V_\alpha\, l) f_\alpha - V_\alpha \{(w_\alpha, f_\alpha) k + A w_\alpha\},$$

and adding this to (5.6) gives

$$A(V_\alpha^* w_\alpha + f_\alpha) = C f_\alpha$$

where

$$C = \mu\{1 + (w_\alpha, f_\alpha)\} - (V_\alpha^* w_\alpha + f_\alpha, l),$$

provided we can choose w_α to satisfy

$$A w_\alpha = e^{-i\alpha} U \bar{l} - \{1 + (w_\alpha, f_\alpha)\} k. \tag{5.7}$$

If w_α is so chosen

$$V_\alpha^* w_\alpha + f_\alpha = C \phi_\alpha \tag{5.8}$$

and

$$C\{1 + (\phi_\alpha, l)\} = \mu\{1 + (w_\alpha, f_\alpha)\}. \tag{5.9}$$

Now since A is of the form $\mu I +$ compact operator, for $\mu \neq 0$, the Fredholm Alternative shows that (5.7) will have a unique solution w_α provided that

$$Aw+(w,f_\alpha)k=0$$

has only the trivial solution, which will be the case if $\psi=A^{-1}k$ has the property that $(\psi,f_\alpha)\neq -1$. If $(\psi,f_\alpha)=-1$ then (5.7) will have a solution provided that $e^{-i\alpha}U\bar{l}-k$ is orthogonal to all solutions of $A^*v+(v,k)f_\alpha=0$, that is, to all scalar multiples of ψ_α, since $(\psi_\alpha,k)=(f_\alpha,\psi)=-1$ in this case. Now

$$(e^{-i\alpha}U\bar{l}-k,\psi_\alpha)=(\bar{l},e^{i\alpha}U\psi_\alpha)+1=(\phi_\alpha,l)+1$$

and this cannot be zero, since $AV_\alpha^*\psi=(\psi,\bar{\mu}f_\alpha-V_\alpha l)f_\alpha$, giving

$$V_\alpha^*\psi=\{\mu(\psi,f_\alpha)-(V_\alpha^*\psi,l)\}\phi_\alpha \tag{5.10}$$

hence $(V_\alpha^*\psi,l)=-(\phi_\alpha,l)\{\mu+(V_\alpha^*\psi,l)\}$ showing that $(\phi_\alpha,l)\neq -1$ because $\mu\neq 0$. So a necessary and sufficient condition for (5.7) to be satisfied is $(\psi_\alpha,k)\neq -1$ or, equivalently, $(\psi,f_\alpha)\neq -1$. From the given properties of f_α, (ψ,f_α) can be equal to -1 only for an isolated set of values of α, so we may choose α for which $(\psi,f_\alpha)\neq -1$.

Assuming that $(\psi,f_\alpha)\neq -1$ then we can discount the possibility that the constant C in (5.8) is zero, since f_α does not belong to the image of V_α^*. From this we see that ϕ_α is of the form $\lambda f_\alpha-V_\alpha^*g$ for $g\in L_2(0,1)$ and $\lambda\in\mathbb{C}$, the condition required to implement the factorisation in (3.8). Even in the case when $(\psi,f_\alpha)=-1$, (5.10) shows us that ϕ_α is of this form (with $\lambda=0$) since the line following that equation shows us that $\mu+(V_\alpha^*\psi,l)\neq 0$.

Because ϕ_α is of the form $\lambda f_\alpha-V_\alpha^*g$ $(g\in L_2(0,1))$, (5.4) shows that both $\tilde{c}_1$ and $\tilde{c}_2$ in this formulation have the same form. By the remarks following (5.3), it follows that the $\tilde{c}_1$ and $\tilde{c}_2$ of (5.3) and (5.5) also have the same form, and the factorisation of (3.8) may be carried out.

This information is sufficient to show that A^{-1} is given by (3.13) in this case, where the kernel generating R has the two alternative forms (3.12) and (since (2.8) applies) (3.16). Each of these forms has three versions, given by using (5.3), (5.4) or (5.5) in (3.9).

To give an explicit form for $A^{-1}=\mu^{-1}I+R$, and one which is evidently new, we use $\tilde{c}_1$ and $\tilde{c}_2$ in the forms given by (5.5). As $\alpha=0$ in this representation, (3.9) reduces to $p_m=\tilde{c}_m'$, $q_m=\tilde{d}_m'$ (for $m=1,2$) and it follows from (5.2) that $\bar{q}_1=-Up_2$ and $\bar{q}_2=Up_1$. Constructing the kernel r of R according to (3.12) and (3.16) we find after some simplification that

$$(\chi_1,g_1)r(x,t)=\chi_1(x)\chi_1(1-t)+r_1(x,t)+r_2(x,t),$$

for almost all x and t in $[0,1]$, where

$$r_1(x,t)=\begin{Bmatrix}\chi_2(1)\chi_1'(x-t)-\chi_1(1)\chi_2'(x-t) & (t\leqq x)\\ \chi_2(0)\chi_1'(1-t+x)-\chi_1(0)\chi_2'(1-t+x) & (x\leqq t)\end{Bmatrix},$$

and

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$$r_2(x,t)=\int_{\max(x,t)}^{1}\{\chi_1'(1-s+x)\chi_2'(s-t)-\chi_1'(s-t)\chi_2'(1-s+x)\}\,ds$$

$$=\int_{0}^{\min(x,t)}\{\chi_1'(1-t+s)\chi_2'(x-s)-\chi_1'(x-s)\chi_2'(1-t+s)\}\,ds.$$

It is easy to check that the alternative expressions for r_2 are indeed equal and that $r(x,t)=r(1-t,1-x)$, in line with the remarks following (2.8).

A further representation of A^{-1} is provided by pursuing (5.8). Guided by (5.7), and assuming that $(\psi,f_\alpha)\neq -1$, let

$$A\psi=k,\ AU\bar{\chi}=U\bar{l},$$

the latter being equivalent to $A^*\chi=l$. Then

$$w_\alpha=e^{-i\alpha}U\bar{\chi}-\{1+(w_\alpha,f_\alpha)\}\psi$$

from which we deduce that

$$\{1+(w_\alpha,f_\alpha)\}\{1+(\psi,f_\alpha)\}=1+(f_\alpha,\chi)=1+(\phi_\alpha,l),$$

the last equality holding because $(f_\alpha,\chi)=(A\phi_\alpha,\chi)=(\phi_\alpha,A^*\chi)=(\phi_\alpha,l)$. Thus, (5.9) simplifies to $C\{1+(\psi,f_\alpha)\}=\mu$ and we deduce from (5.8) that

$$\left.\begin{aligned}\mu\phi_\alpha&=\{1+(\psi,f_\alpha)\}f_\alpha+V_\alpha^*[e^{-i\alpha}\{1+(\psi,f_\alpha)\}U\bar{\chi}-\{1+(f_\alpha,\chi)\}\psi]\\&=\{1+(f_\alpha,\chi)\}f_\alpha+V_\alpha[\{1+(f_\alpha,\chi)\}\psi-e^{-i\alpha}\{1+(\psi,f_\alpha)\}U\bar{\chi}]\end{aligned}\right\},\tag{5.11}$$

using (2.1) to derive the second version from the first.

Either of the equivalent expressions (5.11) for $\mu\phi_\alpha$ may be used to replace ϕ_α and ϕ_β in (5.4), giving c_1 and c_2 in terms of ψ and χ. The manipulation leading to A^{-1} is straightforward since ϕ_α and ϕ_β satisfy the conditions which allow the factorisation (3.8) and (5.11) gives

$$\mu p_1=\{1+(f_\alpha,\chi)\}\psi-e^{-i\alpha}\{1+(\psi,f_\alpha)\}U\bar{\chi}.$$

Then, using (5.4) we obtain a similar expression for $(\phi_\beta,f_\alpha)p_2$. The expression for A^{-1} which results from this approach was given by Gohberg and Feldman [1] and Porter [4].

(b) Singular integral equations

We first consider the prototype singular equation

$$(\mu I - H)\phi = f, \tag{5.12}$$

where H is the Hilbert operator defined by (1.9) and $\mu \in \mathbb{R}$. With $A = \mu I - H$, $V_\alpha A + A V_\alpha^*$ is the rank-two operator given in (2.15). The condition (2.8) applies again, allowing us to use (2.12), which gives $A f_\alpha = \mu f_\alpha + p_\alpha - e^{-i\alpha} U \bar{p}_\alpha$. Following the procedure in (a), we find that $Ac_1 = f_\alpha$, $Ac_2 = p_\alpha$ and that (5.2) supplies d_1 and d_2.

Restricting attention to the case $\alpha = 0$, we use (5.5) to provide $\tilde{c}_1$ and $\tilde{c}_2$, requiring the solutions of $(\mu I - H)\chi_j = g_j$ to be determined for $j = 1, 2$ where $g_j(x) = x^{j-1}$. Since c_1 and c_2 are real-valued, (5.2) leads to $d_1 = g_1 + Uc_2$ and $d_2 = -Uc_1$. Substituting (5.5) and these expressions for d_1 and d_2 into $E\phi = (\phi, d_m)c_m$ we find, after a minor rearrangement, that

$$(\chi_1, g_1)E\phi = (\phi, U\chi_1)V_0^*\chi_1 + (\phi, V_0^* U\chi_1)\chi_1 + (\phi, U\chi_1)\chi_2 - (\phi, U\chi_2)\chi_1.$$

Using Lemma 2 and (3.6) we see that

$$(\chi_1, g_1)A^{-1} = P + B, \tag{5.13}$$

where

$$P\phi = (\phi, U\chi_1)\chi_1 \tag{5.14}$$

and B satisfies

$$(V_0^* B + B V_0)\phi = (\phi, U\chi_1)\chi_2 - (\phi, U\chi_2)\chi_1. \tag{5.15}$$

(A similar decomposition to that in (5.13) could have been employed in (a), but is less significant there).

It remains to determine B from (5.15), employing our established techniques.

Reference to [5, pp. 309, 314] reveals that

$$\chi_1(x) = \nu x^{-\gamma}(1-x)^\gamma,\ \chi_2(x) = (x+\gamma)\chi_1(x) \quad (0 < x < 1),$$

where

$$\nu = \pi^{-1}\sin(\pi\gamma),\ \mu = \pi\cot(\pi\gamma) \quad (0 < |\gamma| < \tfrac{1}{2}),$$

for each $\mu \in \mathbb{R}$ except $\mu = 0$. The result of using χ_1 and χ_2 in (5.15) is that B satisfies

$$(V_0^* B + B V_0)\phi = (\phi, \tilde{d}_m)\tilde{c}_m, \tag{5.16}$$

where we have redefined $\tilde{c}_m$ and $\tilde{d}_m$ $(m = 1, 2)$ by

$$\tilde{c}_1(x) = \nu x^{-\gamma}(1-x)^\gamma, \tilde{c}_2(x) = \nu x^{1-\gamma}(1-x)^\gamma (0 < x < 1) \tag{5.17}$$

and

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$$\tilde{d}_1 = -U\tilde{c}_2,\ \tilde{d}_2 = U\tilde{c}_1.$$

Since $A = \mu I - H$ and P satisfy (2.8), so do A^{-1} and B. Therefore (5.16) implies

$$V_0^* B V_0 = \tilde{C}_m^* \tilde{D}_m,\ V_0 B V_0^* = \tilde{D}_m \tilde{C}_m^*, \tag{5.18}$$

with $\tilde{C}_m$ and $\tilde{D}_m$ defined in the usual way.

Unfortunately, it is not possible to factorise each of the terms $\tilde{C}_m$ and $\tilde{D}_m$ in the form (3.8) and then reassemble the results to give a simple formula for the solution of (5.12). This may, however, be carried through if we make additional assumptions about f. If we assume that f is differentiable and $f' \in L_2(0,1)$ then $f(x) = f(0) + (V_0 f')(x) = f(1) - (V_0^* f')(x)$. Since (5.13), (5.14) and the fact that $(\chi_1, g_1) = \gamma$ show that $Bg_1 = 0$ we deduce from (5.18) that

$$V_0^* B f = V_0^* B V_0 f' = \tilde{C}_m^* \tilde{D}_m f'$$

and

$$V_0 B f = -V_0 B V_0^* f' = \tilde{D}_m \tilde{C}_m^* f'.$$

These yield

$$(Bf)(x) = \frac{d}{dx}\left\{\int_x^1 \tilde{c}_1(1-s+x)\,dx \int_0^s \tilde{c}_2(s-t) f'(t)\,dt - \int_x^1 \tilde{c}_2(1-s+x)\,ds \int_0^s \tilde{c}_1(s-t) f'(t)\,dt\right\}$$

$$= \frac{d}{dx}\int_0^1 f'(t)\,dt \int_{\max(x,t)}^1 \{\tilde{c}_1(1-s+x)\tilde{c}_2(s-t) - \tilde{c}_2(1-s+x)\tilde{c}_1(s-t)\}\,ds \tag{5.19}$$

and

$$(Bf)(x) = \frac{d}{dx}\left\{-\int_0^x \tilde{c}_2(x-s)\,ds \int_s^1 \tilde{c}_1(1-t+s) f'(t)\,dt + \int_0^x \tilde{c}_1(x-s)\,ds \int_s^1 \tilde{c}_2(1-t+s) f'(t)\,dt\right\}$$

$$= \frac{d}{dx}\left\{\int_0^1 f'(t)\,dt \int_0^{\min(x,t)} \{\tilde{c}_1(x-s)\tilde{c}_2(1-t+s) - \tilde{c}_2(x-s)\tilde{c}_1(1-t+s)\}\right\} ds, \tag{5.20}$$

all versions being valid for almost all $x \in [0,1]$.

Finally, we note that $\chi_1 = \tilde{c}_1$ and therefore

$$(\mu I - H)^{-1} f = \gamma^{-1}\{(f, U\tilde{c}_1)\tilde{c}_1 + Bf\},$$

where B is defined above.

This form of the solution of $\mu\phi = f + H\phi$ appears to be new. A one-term solution consisting of repeated indefinite integrals was given by Peters [3] using a 'simplifying operator' idea, which was exploited by Porter and Stirling [5] to generate other forms of the solution. These previous solutions do not exhibit the symmetric structure of (5.19) and (5.20).

More general singular integral equations, of the form $\mu\phi = f + (H+K)\phi$, where K is compact and generated by a difference kernel, for instance, can be dealt with using the method described above. The modification required is virtually immediate, and we find that $(\chi_1, g_1)A^{-1} = P + B$, where $P\phi = (\phi, U\bar{\chi}_1)\chi_1$ and

$$(V_0^* B + B V_0)\phi = (\phi, U\bar{\chi}_1)\chi_2 - (\phi, U\bar{\chi}_2)\chi_1. \tag{5.21}$$

Here χ_1 and χ_2 are defined by $(\mu I - \mathrm{H} - K)\chi_j = g_j$ where $g_j(x) = x^{j-1}$ $(0 \leqq x \leqq 1)$. The extraction of B from (5.21) follows once the behaviour of χ_1 and χ_2 is established.

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