INEQUALITIES RELATING DIFFERENT DEFINITIONS OF DISCREPANCY

Dedicated to the memory of Hanna Neumann

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1. Introduction

Let $p_1, p_2, \cdots, p_N$ be N points in the unit s-dimensional closed square $Q = [0, 1]^s$. For any measurable set $S \subseteq Q$, we define $\delta(S)$, the discrepancy of S, by

$$\delta(S) = V(S) - n(S)/N, \tag{1}$$

where $V(S)$ is the s-dimensional volume of S, and $n(S)$ is the number of indices i for which $p_i \in S$. Let

$$D_K = \sup |\delta(B)|,$$

where the supremum is taken over all s-balls $B \subseteq Q$, and

$$D_c = \sup |\delta(C)|,$$

the supremum in this case being taken over all convex sets $C \subseteq Q$. Clearly $D_c \geqq D_K$. In this paper we establish

THEOREM 1. *$D_K \geqq \phi_1(s) D_c^{s+1}$, where ϕ_1 is a constant depending only on s.*

This replaces the best previously known inequality in this direction,

$$D_K \geqq \exp(-\gamma_1 D_c^{-\gamma_2})$$

(γ_1, γ_2 depending only on s) due to Hlawka [3].

In the other direction, I have given an example in [3] to show that the exponent in Theorem 1 cannot be improved to less than $\frac{1}{2}(s + 1)$.

Denote by D the classical s-interval discrepancy $D = \sup |\delta(J)|$, the supremum being taken over all intervals J of the form

$$J = \{x = (x_j) \,|\, 0 \leqq \alpha_j \leqq x_j < \beta_j \leqq 1 (j = 1, \cdots, s)\}.$$

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Cassels (unpublished) has proved

$$(2) \qquad D_K \geqq \phi_2(s)D^{s+1};$$

my proof of Theorem 1 is a fairly natural generalisation, from intervals to convex bodies, of his proof of (2). Recently I [3] have improved (2) to $D_K \geqq \phi_3(s, \varepsilon)D^{s+\varepsilon}$. Hlawka [2] has proved the inequality (5) below relating D and D_c (see also [3]). Thus we have the following six inequalities connecting D_K, D_c and D:

$$(3) \qquad D_c \geqq D$$

$$(4) \qquad D_c \geqq D_K$$

$$(5) \qquad D \geqq \phi_4(s)D_c^s$$

$$(6) \qquad D \geqq \phi_4(s)D_K^s$$

$$(7) \qquad D_K \geqq \phi_1(s)D_c^{s+1}$$

(8) *for any* $\varepsilon > 0$, $D_K \geqq \phi_3(s, \varepsilon)D^{s+\varepsilon}$.

(ϕ_1, ϕ_4 depend only on s, ϕ_3 only on s and ε).

Of these inequalities (3) and (4) are, of course, trivial while (6) follows straight from (4) and (5),

We give a proof of Theorem 1 in section 3. Section 2 contains preliminaries needed for the proof,

I would like to thank Professor Cassels for suggesting this problem to me, and for making available his unpublished proof of (2).

2. Preliminaries

Before proving the theorem, we need some more notation, and two lemmas relating to convex bodies.

First of all, since Theorem 1 is trivial when $s = 1$, we always assume that $s \geqq 2$. We let $\theta_s = \pi^{s/2}/\Gamma(s/2+1)$ be the volume of the unit s-ball, and $B_r(y)$ be the closed ball with centre y, radius r. Denote by χ_y the characteristic function of $B_r(y)$ ($= 1$ if $x \in B_r(y)$, $= 0$ if not).

Let C be any convex set. For any $r > 0$ we denote by C^r the set

$$(9) \qquad C^r = \{x + y \,|\, x \in C, |y| \leqq r\}$$

and by C_r the set of points of C, distant greater than r from every point outside C. Clearly $C_r \subseteq C \subseteq C^r$. We have an expression for the volume of C^r, given by

LEMMA 1. *For any convex set* C,

$$(10) \qquad V(C^r) = V(C) + A_1(C)r + A_2(C)r^2 + \cdots + A_s(C)r^s,$$

where $A_1, A_2, \cdots, A_s$ *are constants ("mixed volumes") depending only on* C, *with* *the property that if $C \subseteq C'$ C' convex, then*

$$A_i(C) \leqq A_i(C'). \qquad (i = 1\ 2, \cdots, s) \tag{11}$$

The coefficient $A_1(C)$ is the $(s-1)$-dimensional area of the boundary of C. Furthermore, if also $C \subseteq Q$, then

$$V(C^r) - V(C) \leqq 2sr(1+2r)^{s-1}. \tag{12}$$

PROOF. Equations (10) and (11) are special cases of Theorems 41 and 42 respectively in Eggleston's Cambridge Tract [1], and the expression for $A_1(C)$ as as the surface are of C is on p. 88 of [1]. It therefore remains only to deduce the inequality (12). Now $C \subseteq Q$, so

$$A_i(C) \leqq A_i(Q) \qquad (i = 1, 2, \cdots, s)$$

by (11). Hence from (10),

$$V(C^r) - V(C) \leqq V(Q^r) - V(Q) = V(Q^r \setminus Q)$$

But

$$V(Q^r \setminus Q) \leqq (1+2r)^s - 1 \leqq 2rs(1+2r)^{s-1}$$

by the mean value theorem. This completes the proof of Lemma 1.

Also, for C_r, we have

LEMMA 2. *For any convex set $C \subseteq Q$,*

$$V(C) - V(C_r) \leqq rA_1(C) \leqq 2rs. \tag{13}$$

PROOF. For a convex polytope P, the first inequality becomes clear by mounting a cylinder of height r, facing inwards, on each face of P. We now consider the case of a general convex body C. By [1] (Theorem 33), for each $\varepsilon > 0$ there is a convex polytope P such that

$$P \subseteq C \subseteq P^\varepsilon.$$

So

$$\begin{aligned} V(C \setminus C_r) &\leqq V(P \setminus P_r) + V(P^\varepsilon \setminus P) \\ &\leqq rA_1(P) + 2s\varepsilon(1+2\varepsilon)^{s-1} \end{aligned}$$

by (13) for p and (12)

$$\leqq rA_1(C) + 2s\varepsilon(1+2\varepsilon)^{s-1}$$

by (11). Hence on letting $\varepsilon \to 0$, we obtain the first part of (13).

The second part of (13) follows from the fact that $A_1(C) \leqq A_1(Q) = 2s$. This proves the lemma.

3. Proof of Theorem 1

We are now ready to prove Theorem 1. We take a convex body $C \subseteq Q$, which we can assume has $\delta(C) \neq 0$. We must produce a ball B with

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$$|\delta(B)| \geqq \phi_1 |\delta(C)|^{s+1}.$$

The method of the proof is to use Cassels' idea of averaging the discrepancies of circles of fixed radius, as the centre varies throughout the given body C.

We first show (equation (14)) that, apart from the boundary effects, the integral of $\delta(B_r(y))$ over $y \in C$ is roughly $\theta_s r^s \delta(C)$. Ignoring boundary effects, we then have

$$|\delta(B_r(y))| V(C) \geqq \theta_s r^s |\delta(C)|$$

for some $y \in C$. It is then only a question of choosing r as large as one can, without the boundary effects becoming too significant, and we obtain Theorem 1. I now give the details.

We first show that for any $r > 0$,

$$\int_C \delta(B_r(y) \cap C_r) dy = \theta_s r^s \delta(C_r). \tag{14}$$

Now as δ is a signed measure,

$$\begin{aligned}\int_C \delta(B_r(y) \cap C_r) dy &= \int_C \int_{(B_r(y) \cap C_r)} d\delta(x) dy \\ &= \int_C \int_{C_r} \chi_y(x) d\delta(x) dy \\ &= \int_C \int_{C_r} \chi_x(y) d\delta(x) dy \\ &= \int_{C^r} \int_C \chi_x(y) dy d\delta(x) \\ &= \int_{C_r} V(B_r(x) \cap C) d\delta(x) \\ &= \theta_s r^s \int_{C_r} d\delta(x) = \theta_s r^s \delta(C_r)\end{aligned}$$

as $B_r(x) \subseteq C$ for $x \in C_r$. This proves (14).

We now separate the three cases I: $\delta(C) > 0$, II: $\delta(C) < 0$, $C^{2r} \subseteq Q$, III: $\delta(C) < 0$, $C^{2r} \nsubseteq Q$, where in II and III $r = -\delta(C)/16s$.

Case I. $\delta(C) > 0$. For $y \in C_{2r}$, we have $B_r(y) \subseteq C$, and hence

$$B_r(y) = B_r(y) \cap C.$$

Using (14), we obtain

$$\int_{C_{2r}} \delta(B_r(y)) dy = \theta_s r^s \delta(C_r) - \int_{(C \backslash C_{2r})} \delta(B_r(y) \cap C_r) dy$$

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$$= \theta_s r^s \delta(C) - \theta_s r^s \delta(C \setminus C_r) - \int_{(C \setminus C_{2r})} \delta(B_r(y) \cap C_r) dy$$

$$\geqq \theta_s r^s \delta(C) - \theta_s r^s V(C \setminus C_r) - \int_{(C \setminus C_{2r})} V(B_r(y) \cap C_r) dy$$

as $\delta(S) \leqq V(S)$ for any set S,

$$\geqq \theta_s r^s \{\delta(C) - V(C \setminus C_r) - V(C \setminus C_{2r})\}$$

$$\geqq \theta_s r^s (\delta(C) - 6rs)$$

by Lemma 2. So we choose $r = \delta(C)/12s$. Then

$$\int_{C_{2r}} \delta(B_r(y)) dy \geqq \frac{\theta_s r^s}{2} \cdot \delta(C) = \frac{\theta_s}{2} \frac{(\delta(C))^{s+1}}{(12s)^s}.$$

Hence there is some $y \in C_{2r}$ such that

$$V(C_{2r}) \delta(B_r(y)) \geqq \frac{\theta_s}{2} \frac{(\delta(C))^{s+1}}{(12s)^s}.$$

Since $V(C_{2r}) \leqq V(C) \leqq 1$, we now have Theorem 1 as required.

CASE II. $\delta(C) < 0$, $C^{2r} \subseteq Q$, where $r = (-\delta(C))/16s$. We write $\bar{\delta}(S) = -\delta(S)$.

Firstly, it is clear that the convex subset of C with $\bar{\delta}$ maximal is the convex hull of some subset of the points $p_1, p_2, \cdots, p_N$. Hence, replacing C by this set if necessary, we can assume that, both for cases II and III,

$$\bar{\delta}(C') \leqq \bar{\delta}(C) \tag{15}$$

for all convex subsets C' of C.

Now it is easy to verify that $(C^r)_r = C$, and so we have, from (14),

$$\int_{C^r} \bar{\delta}(B_r(y) \cap C) dy = \theta_s r^s \bar{\delta}(C). \tag{16}$$

Further, for $y \in C^r$ we have $B_r(y) \subseteq C^{2r} \subseteq Q$. and hence $B_r(y) = B_r(y) \cap C^{2r}$. So we can calculate

$$\int_{C^r} \bar{\delta}(B_r(y)) dy = \theta_s r^s \bar{\delta}(C) + \int_{C^r} \bar{\delta}(B_r(y) \cap (C^{2r} \setminus C)) dy$$

$$= \theta_s r^s \bar{\delta}(C) + \int_{(C^r \setminus C)} \bar{\delta}(B_r(y) \cap (C^{2r} \setminus C)) dy$$

$$\geqq \theta_s r^s \bar{\delta}(C) - V(C^r \setminus C_r) \theta_s r^s \tag{17}$$

$$= \theta_s r^s \{\bar{\delta}(C) - V(C^r \setminus C) - V(C \setminus C_r)\}$$

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$$\geqq \theta_s r^s\{\bar{\delta}(C) - 2rs(1+2r)^s - 2rs\}$$

by Lemmas 1 and 2. Now, on substituting for r, we obtain

$$\int_{C^r} \bar{\delta}(B_r(y))dy \geqq \theta_s r^s \left\{ \bar{\delta}(C) - \frac{\bar{\delta}(C)}{4}\left(1 + \frac{\bar{\delta}(C)}{8s}\right)^s \right\}$$

$$\geqq \theta_s r^s \bar{\delta}(C)\left(1 - \frac{1}{4}e^{\frac{1}{8}}\right)$$

$$\geqq \tfrac{1}{2}\theta_s r^s \bar{\delta}(C) = \frac{\theta_s}{2(16s)^s}(\bar{\delta}(C))^{s+1}.$$

So for some $y \in C^r$,

$$V(C^r)\bar{\delta}(B_r(y)) \geqq \frac{\theta_s}{2(16s)^s}(\bar{\delta}(C))^{s+1}.$$

Since $V(C^r) \leqq 1$, we therefore have the theorem with $\phi_1 = \theta_s/2(16)^s$.

Case III: $\delta(C) < 0$, $C^{2r} \nsubseteq Q$, $r = \bar{\delta}(C)/16s$. Now if

$$\delta(C \cap Q_{2r}) \geqq \tfrac{1}{2}\delta(C),$$

then by (15),

$$r' = \bar{\delta}(C \cap Q_{2r})/16s \leqq \bar{\delta}(C)/16s = r,$$

and so $(C \cap Q_{2r})^{2r'} \subseteq Q$. We could therefore apply case II to $C \cap Q_{2r}$ instead of C, and obtain

$$|\delta(B)| \geqq \phi_1|\delta(C \cap Q_{2r})|^{s+1} \geqq \phi_1 2^{-(s+1)}|\delta(C)|^{s+1}.$$

Thus the theorem would still be true, but with

$$\phi_1 = \theta_s/2^{s+2}(16s)^s.$$

We can therefore assume that

$$\bar{\delta}(C \cap (Q \setminus Q_{2r})) \geqq \tfrac{1}{2}\bar{\delta}(C).$$

But now

$$\bar{\delta}(Q \setminus Q_{2r}) = \bar{\delta}(C \cap (Q \setminus Q_{2r})) + \bar{\delta}((Q \setminus Q_{2r}) \setminus C)$$

$$\geqq \tfrac{1}{2}\bar{\delta}(C) - V(Q \setminus Q_{2r})$$

$$\geqq \tfrac{1}{2}\bar{\delta}(C) - 2s \cdot 2r$$

by Lemma 2,

$$= \bar{\delta}(C)/4.$$

Hence, as $\bar{\delta}(A) + \bar{\delta}(Q \setminus A) = 0$ for any $A \subseteq Q$, we have

$$\delta(Q_{2r}) \geqq \tfrac{1}{4}\bar{\delta}(C).$$

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Thus we are reduced to case I, with the convex set Q_{2r} replacing C. So we again obtain the required result, but with constant

$$\phi_1 = \frac{\theta_s}{2(12s)^s} \cdot \frac{1}{4^{s+1}} = \frac{\theta_s}{8(48s)^s}. \tag{18}$$

This last constant is the smallest obtained in any of the cases, so the theorem is true with ϕ_1 given by (18).

References

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