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RESEARCH ARTICLE

A new weighted means of failure rate and associated quantile versions

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Abstract
In this paper, we define weighted failure rate and their means from the stand point of an application. We begin by emphasizing that the formation of n independent component series system having weighted failure rates with sum of weight functions being unity is same as a mixture of n distributions. We derive some parametric and non-parametric characterization results. We discuss on the form invariance property of baseline failure rate for a specific choice of weight function. Some bounds on means of aging functions are obtained. Here, we establish that weighted increasing failure rate average (IFRA) class is not closed under formation of coherent systems unlike the IFRA class. An interesting application of the present work is credited to the fact that the quantile version of means of failure rate is obtained as a special case of weighted means of failure rate.

1. Introduction

The notion of aging plays an important role in reliability theory and in the study of lifetime data analysis. Aging of a mechanical or biological component based on a lifetime distributions is generally studied using the residual lifetime of the unit that is affected its age. Abundant literature is available on various aging concepts and their patterns of aging, comparison of life distributions and to explain their data generating mechanism. Reliability aging classes based on the monotonicity of the failure rate, such as increasing (decreasing) failure rate (IFR (DFR)) and its average, increasing (decreasing) failure rate average (IFRA (DFRA)) have been found great interest among researchers as it easily give an indication on the manner in which aging can be described, life distributions can be classified and distinguished, and appropriate models can be chosen when observations are available (cf. Barlow and Proschan [3]).

Let X be a non-negative random variable representing the lifetime of an event or living mechanism with absolutely continuous cumulative distribution function (CDF), $F(\cdot)$ and probability density function (PDF), $f(\cdot)$. Then F is said to be IFR (DFR), if the conditional survival function $\overline{F}(x|t) = \frac{\overline{F}(x+t)}{\overline{F}(t)}$ is decreasing (increasing) in $0 \le t < \infty$, $x > 0$, where $\overline{F} = 1 - F$ is the survival (reliability) function; or equivalently the failure rate $h(t) = \frac{f(t)}{\overline{F}(t)}$ is increasing (decreasing) in $t \ge 0$, provided $f(t)$ exists. Further, F is said to IFRA (DFRA), if $-\left(\frac{1}{t}\right) \log \overline{F}(t)$ is increasing (decreasing) in $t \ge 0$. However, in many real situations, $h(t)$ is not always monotonic. In such cases, the monotonicity of IFRA class condition in terms of the failure rate, $\frac{1}{t}\int_0^t h(u)du$, known as the arithmetic mean failure rate (AFR) is a useful measure (Roy and Mukherjee [17]) in identifying the monotonicity of classes of life distributions. Along with arithmetic mean failure rate, Roy and Mukherjee [17] have also studied classes of distributions through the monotonic behavior of geometric failure rate (GFR) and harmonic failure rate (HFR), and the characterizations and aging classes based on it. They pointed out until then no work has been done on GFR and HFR. The following definition is cited from Roy and Mukherjee [17].

Definition 1.1. *Let X be a non-negative random variable with absolutely continuous CDF $F(\cdot)$, PDF $f(\cdot)$, and failure rate $h(\cdot)$. Then the arithmetic mean failure rate (AFR), geometric mean failure rate (GFR), and harmonic failure rate (HFR), denoted by $A(\cdot), G(\cdot)$, and $H(\cdot)$ respectively are defined as* $A(x) = \frac{1}{x}\int_0^x h(u)du$; $G(x) = \exp\left(\frac{1}{x}\int_0^x \ln h(u)du\right)$; $H(x) = \left(\frac{1}{x}\int_0^x \frac{1}{h(u)}du\right)^{-1}$, $x > 0$.

Recently, Bhattacharjee *et al.* [4] further studied the usefulness of these measures based on the notion of aging intensity function proposed by Jiang *et al.* [8].

When sample observations are not equally likely, we use the weighted measures to capture the significance of their relative importance. Choosing appropriate weights, we can then compute various measures in a better way based on the sample observations. Such biased sampling schemes are usually employed in observational studies either due to its convenience or its cost-effectiveness. Based on this, Rao [16] identified the concept of weighted distributions in connection with the modeling statistical data, in situations where the usual practice of employing standard distributions for the purpose was not found appropriate. These distributions occur frequently in the studies related to reliability, analysis of family data, meta analysis and analysis of intervention data, biomedicine, ecology, etc, for more details, see Patil and Rao [15], Gupta and Kirmani [7], and the references therein. If X is a non-negative random variable with a PDF $f(x)$, then the PDF of the weighted random variable X^w is given by, $f_w(x) = \frac{w(x)f(x)}{Ew(X)}, x > 0$, where $w(\cdot)$ is a non-negative weight function (cf. Rao [16]). There are many weight functions used by different authors, however, the weight functions $w(x) = x$ and $w(x) = x^c, c > 0$ are found to be more popular due to its adaptability in terms of identifying the observed distribution in various applied problems wherein the probability of selecting the sample units are proportional to the length or size of the population units, the respective random variables are known as the length-biased and size-biased random variables. Motivated by these, in the present study, we propose weighted mean failure rates based on the measures of AFR, GFR, and HFR.

The paper is organized as follows. In Section 2, we introduce a new definition of weighted means of failure rate resulting into a new weighted concept of reliability functions. The proposed weighted concept differs remarkably from the existing version of weighted distribution once proposed by Rao [16]. As an application, we find that formation of an n component series system having complementary weight functions is actually a mixture of n distributions and vice-versa. We give a note on form invariance property of the baseline failure rate and its transformation from one aging class to another depending upon the choice of the weight function. In Section 3, we derive some characterization results followed by bounds and limiting behavior of means of aging functions. We define some new non-parametric aging classes based on means of failure rate and discuss their inclusive properties in Section 4. Section 5 gives some equivalent conditions of aging classes based on geometric and harmonic means. We prove our claim that weighted IFRA class is not closed under formation of coherent systems unlike IFRA class through a counterexample. In Section 6, we derive the quantile version of means of failure rate independently and also as a special case from weighted means of failure rate. We focus on the proportional quantile hazards model and compare it with conventional proportional hazards model. In Section 7, we showcase an application of the proposed weighted concept on a real-life data. Concluding remarks are given in Section 8.

2. A new weighted means of failure rate

In this paper, we introduce a generalized versions of AFR, GFR, and HFR involving a suitable choice of a non-negative weight function as defined below.

Definition 2.1. *Let X be a non-negative random variable with absolutely continuous distribution function $F(\cdot)$, probability density function $f(\cdot)$, and failure rate $h(\cdot)$. The weighted arithmetic mean failure rate (w-AFR), weighted geometric mean failure rate (w-GFR), and weighted harmonic failure rate (w-HFR) denoted by $A^w(\cdot), G^w(\cdot)$, and $H^w(\cdot)$ respectively, with a suitable non-negative weight function $w(\cdot)$, are defined as*

(i) $A^w(x) = \frac{\int_0^x w(u)h(u)du}{\int_0^x w(u)du}, x > 0;$

(ii) $G^w(x) = \exp\left(\frac{\int_0^x w(u)\ln h(u)du}{\int_0^x w(u)du}\right), x > 0;$

(iii) $H^w(x) = \left(\int_0^x w(u)du\right)\left(\int_0^x \frac{w(u)}{h(u)}du\right)^{-1}, x > 0.$

Clearly, if $w(x) = 1$ for all $x > 0$ then above definition reduces to that of AFR, GFR, and HFR given in Definition 1.1 due to Roy and Mukherjee [17].

In the pretext, of above, we shall define the other reliability functions of the weighted random variable as given in the following definition.

Definition 2.2. *The weighted survival function of X, or survival function of weighted random variable X^w, denoted by $\overline{F}^w(\cdot)$ is defined as $\overline{F}^w(x) = \exp\left(-\int_0^x w(u)h(u)du\right), x > 0$. The density and failure rate function of X^w are $f^w(x) = w(x)h(x)\exp\left(-\int_0^x w(u)h(u)du\right)$ and $h^w(x) = w(x)h(x)$ for all $x > 0$, respectively.*

The fact that $h^w(x) = w(x)h(x)$ reminds us of proportional hazard rate (PHR) models where $h(x)$ is the baseline failure rate and $w(x)$ is the proportionality function giving rise to a new hazard rate $h^w(x)$ for $x > 0$.

Referring to the related literature, one can note that corresponding to the baseline survival function $\overline{G}$ having failure rate r_G, Marshall and Olkin [10] proposed a cumulative distribution function F such that its hazard rate $h_F(\cdot)$ is given by $h_F(x, \alpha) = \frac{1}{1-\overline{\alpha}\overline{G}(x)} h_G(x)$ where $x, \alpha \in R^+$ and $\overline{\alpha} = 1-\alpha$, (the parameter α termed as tilt parameter by Marshall and Olkin [11]) and this is a special case of Definition 2.2 if one assumes $w(x) = \frac{1}{1-\overline{\alpha}\overline{G}(x)}$. Furthermore, Balakrishnan *et al.* [2] defined modified proportional hazard rates (MPHRs) of n independent components having lifetimes $X_1, X_2, \ldots, X_n$ with respective survival functions $\overline{F}_i$ if

$$F_i(x, \lambda_i) = \frac{1-\left(\overline{F}(x)\right)^{\lambda_i}}{1-\overline{\alpha}\left(\overline{F}(x)\right)^{\lambda_i}}, \alpha > 0, \overline{\alpha} = 1-\alpha, \lambda_i > 0$$

for $i = 1, 2, \ldots, n$ where $\overline{F}$ is the corresponding baseline survival function. They considered it (MPHR model) to be the generalization of PHR model because if $\alpha = 1$ then PHR is a special case of MPHR. However, one shall observe that this is based on the notion that X_i's with survival functions $\overline{F}_i(x)$ follow PHR model if there exits positive constants λ_i's such that $\overline{F}_i(x) = \left(\overline{F}(x)\right)^{\lambda_i}$. It is worthwhile to note that the definition of MPHR proposed by Balakrishnan *et al.* [2] reduces to PHR model $h_i(x) = \lambda_i h(x)$ if $\alpha = 1$. We use the notation h_i as mentioned in Balakrishnan *et al.* [2]. For better clarity, readers may note that h_i and h^w are equivalent except that h_i corresponds to weighted failure rate of ith component as discussed earlier. The present work, in other words, is an attempt to define PHR model in a more general sense.

The next proposition gives a necessary and sufficient condition that weight function $w(x)$, and hazard rate $h(x)$ must satisfy so that $\overline{F}^w(x)$ represents a (weighted) survival function. One can refer to Marshall and Olkin [11] to look into the postulates for hazard rate (non-weighted).

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Proposition 2.3. *A non-negative random variable X with failure rate h(x) gives rise to a weighted random variable X^w having failure rate $h^w(x)$ such that $h^w(x) = w(x)h(x)$ for all $x > 0$, if and only if the function $w(\cdot)$ satisfies the following conditions.*

(i) $w(x) \geq 0, h(x) \geq 0$
(ii) *For* $x > 0$, $\int_0^x w(u)h(u)du < \infty$
(iii) $\int_0^\infty w(u)h(u)du = \infty$
(iv) *If* $\int_0^x w(u)h(u)du = \infty$ *for some x then* $h(y) = \infty$ *for every* $y > x$.

Now, we look into some uses of weighted failure rate arising in practical field. Let us consider a series system formed by n components having failure rates $h_i(x)$ with respective weights $w_i(x)$, for $i = 1, 2, \ldots, n$ and $x > 0$ such that $\sum_{i=1}^n w_i(x) = 1$. The failure rate $h(x)$ of the resultant n component series system is $h(x) = \sum_{i=1}^n h_i(x)w_i(x)$. This form of $h(x)$ is similar to the failure rate of the mixture of n distributions, with cumulative distributions, $F_i(\cdot)$ having failure rates, $h_i(\cdot)$ for $i = 1, 2, \ldots, n$. The failure rate of mixture of n distributions, given by $F(x) = \sum_{i=1}^n \pi_i F_i(x)$, satisfying $\sum_{i=1}^n \pi_i = 1$ is

$$\begin{aligned} h(x) &= \frac{\sum_{i1}^n \pi_i f_i(x)}{1 - \sum_{i=1}^n \pi_i F_i(x)} \\ &= \frac{\sum_{i=1}^n \pi_i h_i(x)\overline{F}_i(x)}{\sum_{i=1}^n \pi_i \overline{F}_i(x)} = \sum_{i=1}^n p_i(x)h_i(x) \end{aligned}$$

where $p_i(x) = \frac{\pi_i \overline{F}_i(x)}{\sum_{i=1}^n \pi_i \overline{F}_i(x)}$ which in turn satisfies $\sum_{i=1}^n p_i(x) = 1$.

2.1. Invariance property of new weighted distributions

Unlike the log-exponential family (cf. Patil and Ord [14]) possessing the form-invariance property among the weighted distributions defined by Rao [16] under size biased sampling of order $c > 0$ that is, $w(x) = x^c$, in the present work we find that IFR Weibull distribution bears the said property as discussed in the following example.

Example 2.4. If a two-parameter Weibull distribution with failure rate $h(x) = \alpha\beta x^{\beta-1}$ belongs to positive aging classes, namely IFR, then the resultant size biased distribution also fall in the same aging class.

However, in some cases, a particular weight function can shift a distribution from positive aging class to its dual counterpart or vice-versa. Even, monotonic nature of hazard rate may be considerably affected for certain choice of weight function. The following counterexamples give some light on this study.

Counterexample 2.5. *Let us consider a two-parameter Weibull distribution having decreasing failure rate failure rate h(x) as mentioned in Example 2.4. We note that if $\beta + c > 1$ then the baseline decreasing failure rate Weibull distribution is shifted to IFR positive aging class under size biased sampling.*

Counterexample 2.6. *Additive Weibull distribution having failure rate $h(x) = \alpha\theta x^{\theta-1} + \beta\gamma x^{\gamma-1}$ with $\alpha, \beta, \theta, \gamma > 0$ has form-invariance property under size biased sampling. However, the weight function $w(x) = x^c$ drags the additive Weibull distribution from decreasing failure rate class to IFR class if $c + \theta > 1$ and $c + \gamma > 1$.*

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Counterexample 2.7. *If X follows four-parameter Weibull distribution (cf. Kies* [9]*) with survival function*

$$\overline{F}(x) = \exp\left(-\lambda\left(\frac{x-a}{b-x}\right)^{\beta}\right), 0 \le a < x < b, \lambda, \beta > 0$$

then $w(x) = \left(\frac{x-a}{b-x}\right)$ *is a form-invariance weight function for the distribution provided* $\beta > 1$*. We know that X has bathtub failure rate if* $0 < \beta < 1$*, and IFR if* $\beta > 1$ *but the weighted random variable* X^w *is always IFR independent of the value of* β *under the aforementioned weighted transformation.*

3. Main results

Now, we prove that the equality of any two of w-AFR, w-GFR, and w-HFR characterize exponential distribution. The following proposition gives some light on this. We continue with the same notations as discussed in previous sections of the paper.

Proposition 3.1. *A non-negative random variable* X*, follows exponential distribution if and only if for* $x > 0$*, any one of the following hold*

(i) $A^w(x) = G^w(x)$
(ii) $G^w(x) = H^w(x)$
(iii) $A^w(x) = H^w(x)$.

Proof. If X follows exponential distribution then it is easy to prove that (i), (ii), and (iii) hold. Conversely, if (i) holds then

$$\frac{\int_0^x w(u)h(u)du}{\int_0^x w(u)du} = \exp\left(\frac{\int_0^x w(u)\ln h(u)du}{\int_0^x w(u)du}\right),$$

gives

$$\left(\int_0^x w(u)du\right)\left\{\ln\left(\int_0^x h(u)w(u)du\right)\right\} = \left(\int_0^x w(u)du\right)\left(\ln\int_0^x w(u)du\right) + \int_0^x w(u)\ln h(u)du. \quad (1)$$

Differentiating (1) with respect to x, we get $\ln(ez_1(x)) = z_1(x)$ where

$$z_1(x) = \frac{h(x)\int_0^x w(u)du}{\int_0^x w(u)\ln w(u)h(u)du}.$$

Thus, $z_1(x) = 1$, for all $x \ge 0$, gives $\frac{d}{dx}h(x)\left(\int_0^x w(u)du\right) = 0$, and since $\int_0^x w(u)du \ne 0$, we conclude that $h(x)$ is constant for all $x \ge 0$. This proves that X has exponential distribution. Similarly, if (ii) holds then

$$\exp\left(\frac{\int_0^x w(u)\ln h(u)du}{\int_0^x w(u)du}\right) = \left(\int_0^x w(u)du\right)\left(\int_0^x \frac{w(u)}{h(u)}du\right)^{-1},$$

or equivalently

$$\int_0^x w(u)h(u)du + \left(\int_0^x w(u)du\right)\ln\left(\int_0^x \frac{w(u)}{r(u)}du\right) = \left(\int_0^x w(u)du\right)\ln\left(\int_0^x w(u)du\right). \quad (2)$$

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After differentiating (2), we get $\ln(ez_2(x)) = z_2(x)$ where

$$z_2(x) = \frac{\int_0^x w(u)du}{h(x)\Big(\int_0^x \frac{w(u)}{h(u)}du\Big)}.$$

Hence, $z_2(x) = 1$ which in turn gives $\frac{d}{dx}r(x)\Big(\int_0^x \frac{w(u)}{h(u)}du\Big) = 0$. Since $\Big(\frac{w(u)}{\int_0^x h(u)}du\Big) \neq 0$, it follows that $h(x) =$ constant. This proves that if (ii) holds then X has exponential distribution.

Note that if (iii) holds then it is equivalent to the fact that

$$\Big(\int_0^x w(u)h(u)du\Big)\Big(\int_0^x \frac{w(v)}{h(v)}dv\Big) = \Big(\int_0^x w(u)du\Big)^2. \tag{3}$$

Taking logarithm on both sides and then differentiating both sides with respect to x, we get

$$\frac{h(x)\int_0^x w(u)du}{\int_0^x w(u)h(u)du} + \Big(\frac{1}{h(x)}\Big)\Big(\frac{\int_0^x w(u)du}{\int_0^x \frac{w(v)}{h(v)}dv}\Big) = 2. \tag{4}$$

Since w-HFR = w-AFR, replacing w-HFR by w-AFR in the second term of (4), we get

$$\frac{h(x)\int_0^x w(u)du}{\int_0^x w(u)h(u)du} + \Big(\frac{1}{h(x)}\Big)\Big(\frac{\int_0^x w(u)h(u)du}{\int_0^x w(u)du}\Big) = 2.$$

Hence,

$$\Big(h(x)\int_0^x w(u)du - \int_0^x w(u)h(u)du\Big)^2 = 0,$$

and this gives $\frac{d}{dx}h(x) = 0$ as $\int_0^x w(u)du \neq 0$. This completes the proof. □

Note that $A^w(x) = c$ for all $x > 0$ characterizes exponential distribution, and so is true for $G^w(\cdot)$ and $H^w(\cdot)$. If we simultaneously peep into the lines in the proof of Proposition 3.1, we conclude that (i), (ii), and (iii) get reduced to $A^w(x) = G^w(x) = H^w(x) = c$ for all $x > 0$.

In the next proposition we obtain simple relationships between w-AFR, w-GFR, and w-HFR functions and hazard rate, that characterize the underlying distributions through their hazard rates. The proof is omitted.

Proposition 3.2. *Let h(x) be differentiable for all $x \geq 0$. Then for any non-negative weight function $w(x)$, and for suitable positive values of constants, a, b, c, k we have*

(i) $A^w(x) = ah(x)$ *for all x if and only if* $h(x) = k\Big(\int_0^x w(u)du\Big)^{(1-a)/a}$

(ii) $G^w(x) = bh(x)$ *for all x if and only if* $h(x) = k\Big(\int_0^x w(u)du\Big)^{\ln(e/b)-1}$

(iii) $H^w(x) = ch(x)$ *for all x if and only if* $h(x) = \Big(\frac{1}{kc}\int_0^x w(u)du\Big)^{1-c}$, *where k is an arbitrary constant.*

One can wonder whether for any particular class of well known probability distribution, weighted means are proportional to their respective hazard rates. If we choose weight function as $w(x) = x^c$, then the corresponding failure rate is that of two-parameter Weibull distribution with shape parameter

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$(c - ac + 1)/a$ and scale parameter $ka/\left((c - ac + 1)(c + 1)^{\frac{1-a}{a}}\right)$, provided $\left(\frac{1+c-ac}{a}\right) > 0$ satisfying (i). Intuitively, it follows that (ii) and (iii) are also satisfied for two-parameter Weibull distribution having a different set of scale and shape parameters. We summarize this discussion by claiming that $w(x) = x^c$ for suitable c, and $x > 0$ is a proper choice of weight function as it results in a legitimate probability distribution. A little work out will show that if we choose $w(x) = e^{nx}$, then the resultant failure rate function $h(x)$ does not correspond to a well defined probability distribution, underlying the fact that proportionality of weighted means and hazard rate do not hold good.

We end this subsection by stating some crucial observations in the upcoming remark.

Remark 3.3. An essence of introducing the weighted version of means of failure rate lies in the aforementioned Proposition 3.2, where a suitable choice of weight function characterizes some well known distributions. The readers may also note that, proportionality of each of $A^w(\cdot), G^w(\cdot)$, and $H^w(\cdot)$ with $h(\cdot)$ imply that $h(x)$ is increasing (decreasing) in x if and only if $a \le (\ge)1$, $b \le (\ge)1$, and $c \le (\ge)1$ respectively. It is clear that, under the aforementioned conditions, monotonicity of $h(\cdot)$ is independent of the choice of weight function.

3.1. Bounds and limiting behavior of aging means

We state a result from Wijsman [19] in the form of a lemma.

Lemma 3.4. *Let f_i, g_i are non-negative functions, such that the integrals $\int f_i g_i$ are positive for $i, j = 1, 2$. Then*

$$\frac{\int f_1 g_1 d\mu}{\int f_1 g_2 d\mu} \ge \frac{\int f_2 g_1 d\mu}{\int f_2 g_2 d\mu}, \tag{5}$$

provided f_1/f_2 and g_1/g_2 are monotonic in same direction. The inequality in (5) is reversed if f_1/f_2 and g_1/g_2 are monotonic in opposite direction. Equality holds if and only if either f_1/f_2 or g_1/g_2 is a constant. Here μ is Lebesgue measure on a subset of the real line or counting measure on a subset of the integers.

The following proposition gives bounds of the aging means on the basis of monotonicity of weight function and hazard rate (as the case may be).

Proposition 3.5.

(i) $A^w(x) \ge (\le) A(x), x > 0,$ *according as w(x) and h(x) are monotonic in same (opposite) direction.*
(ii) If the hazard rate function $h(x) \ge 1$ *for all* $x > 0$ *then* $G^w(x) \ge (\le) G(x), x > 0,$ *according as w(x) and h(x) are monotonic in same (opposite) direction.*
(iii) $H^w(x) \ge (\le) H(x), x > 0,$ *according as w(x) and h(x) are monotonic in same (opposite) direction.*

Proof. We choose $f_1(x) = w(x), g_1(x) = h(x), f_2(x) = g_2(x) = 1$, to prove (i). By choosing $f_1(x) = w(x), g_1(x) = \ln h(x), f_2(x) = g_2(x) = 1$, and assuming $\ln h(x) \ge 0$ (since $h(x) \ge 1$ for all $x > 0$), Lemma 3.4 gives $\left(\frac{\int_0^x w(u) \ln h(u) du}{\int_0^x w(u) du}\right) \ge (\le) \left(\frac{1}{x} \int_0^x \ln h(u) du\right)$ according as $w(x)$ and $h(x)$ are monotonic in same (opposite) direction. This proves (ii). Similarly, taking $f_1(x) = w(x), g_1(x) = 1, f_2(x) = 1, g_2(x) = 1/h(x)$, we prove (iii). □

The readers may arrive at the following remark by looking at the Proposition 3.5 and the fact that $A^w(x) \ge G^w(x) \ge H^w(x)$ for all $x > 0$.

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Remark 3.6. If $w(x)$ and $h(x)$ are monotonic in same direction then the lower and upper bounds of the aging means of failure rate are $H(x)$ and $A^w(x)$ respectively. On the other hand, $w(x)$ and $h(x)$ are monotonic in opposite direction then the lower and upper bounds of the aging means of failure rate are $H^w(x)$ and $A(x)$ respectively. The lower and upper bounds of the aging means of failure rate discussed in this article are $\min(H(x), H^w(x))$ and $\max(A(x), A^w(x))$ respectively.

In the following theorem, we obtain bounds for the ratio of the weighted hazard means by associating weights in sequence.

Theorem 3.7. *Let* $h_k(x) = w(x)h_{k-1}(x) = (w(x))^k h(x), k \geq 1, h_0(x) = h(x)$, *for* $x > 0$. *We define* $A^w_{h_k}(x) = \left(\frac{\int_0^x w(u)h_k(u)du}{\int_0^x w(u)du}\right)$, $G^w_{h_k}(x) = \exp\left(\frac{\int_0^x w(u)\ln h_k(u)du}{\int_0^x w(u)du}\right)$, *and* $H^w_{h_k}(x) = \left(\frac{\int_0^x w(u)du}{\int_0^x \frac{w(u)}{h_k(u)}du}\right)$. *For* $x > 0$, *the following statements hold.*

(i) If h(x) and w(x) are monotonic in opposite (same) direction then

$$\frac{A^w_{h_k}(x)}{A^w_h(x)} \geq (\leq) \frac{\int_0^x w(u)du}{\int_0^x (w(u))^{n+1}du}.$$

(ii) If $w(x) > 1$, *then*

$$\frac{G^w_{h_k}(x)}{G^w_h(x)} \geq \exp\left(\frac{k}{x}\int_0^x \ln w(u)du\right).$$

(iii) If h(x) and w(x) are monotonic in same (opposite) direction then

$$\frac{H^w_{h_k}(x)}{H^w_h(x)} \geq (\leq) \frac{\int_0^x w(u)du}{\int_0^x \frac{1}{(w(u))^{k-1}}du}.$$

(iv) If w(x) and h(x) are monotonic in same (opposite) direction then $A^w_{h_k}(x) \geq (\leq) A(x)$, *and* $H^w_{h_k}(x) \geq (\leq) H(x)$, *according as* $w(x) \leq (\geq) 1$.

(v) If $h(x) \geq 1$, $w(x) \geq 1$ *then* $G^w_{h_k}(x) \geq G(x)$, *provided w(x) and h(x) are monotonic in same direction.*

Proof. The proofs of (i), (ii), and (iii) follow by applying Lemma 3.4 on the ratios, viz.,

$$\frac{A^w_{h_k}(x)}{A^w_h(x)} = \frac{\int_0^x (w(u))^{k+1}h(u)du}{\int_0^x w(u)h(u)du}, \frac{G^w_{h_k}(x)}{G^w_h(x)} = \exp\left(\frac{k\int_0^x w(u)\ln w(u)du}{\int_0^x w(u)du}\right),$$

and

$$\frac{H^w_{h_k}(x)}{H^w_h(x)} = \left(\frac{\int_0^x \frac{w(u)}{h(u)}du}{\int_0^x \frac{1}{w^{k-1}(u)h(u)}du}\right), x > 0.$$

The proof of (iv) follows from (i) and (iii) of Proposition 3.5. The proof of (v) follows from (ii) of Proposition 3.5. If $w(x)$ and $h(x)$ are monotonic in same (opposite) direction then $A^w_{h_k}(x) \geq (\leq) A^w_h(x)$, and $H^w_{h_k}(x) \geq (\leq) H^w_h(x)$, according as $w(x) \geq (\leq) 1$. □

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The above theorem can be interpreted by saying that one can keep minimizing the means of failure rate (AFR and HFR) of a component, having IFR by associating weights in sequence which are monotonically decreasing with time. However, GFR increases rapidly with the increase in number of weight functions and is independent of the nature of monotonicity of weight and hazard rate. Theorem 3.7 (*ii*) reveals that if $k \to \infty$ and $w(x) > 1$ then $G^w_{h_k}(x) \to \infty$.

4. Non-parametric classes of distributions based on weighted means of failure rate

We define non-parametric classes of distributions on the basis of monotonicity of w-AFR, w-GFR, and w-HFR.

Definition 4.1. *A random variable X is said to belong to the class of*

(i) Increasing (resp. Decreasing) weighted arithmetic mean failure rate Iw-AFR (resp. Dw-AFR)) distributions if $A^w(x)$ is increasing (resp. decreasing) in $x > 0$.
(ii) Increasing (resp. Decreasing) weighted geometric mean failure rate Iw-GFR (resp. (Dw-GFR)) distributions if $G^w(x)$ is increasing (resp. decreasing) in $x > 0$.
(iii) Increasing (resp. Decreasing) weighted harmonic mean failure rate Iw-HFR (resp. Dw-HFR)) distributions if $H^w(x)$ is increasing (resp. decreasing) in $x > 0$.

4.1. Monotonicity of weighted means of failure rate

The next theorem emphasizes on the fact that the monotonic behavior of $h(x)$ is possessed by $A^w(x), G^w(x)$, and $H^w(x)$.

Theorem 4.2. *If h(x) is increasing (decreasing) in $x > 0$ then*

(i) $A^w(x)$ is increasing (decreasing) in $x > 0$;
(ii) $G^w(x)$ is increasing (decreasing) in $x > 0$;
(iii) $H^w(x)$ is increasing (decreasing) in $x > 0$.

Proof. To prove (*i*), we note that $\left(\int_0^x w(u)du\right)\left(\frac{d}{du}A^w(x)\right) = w(x)(h(x) - A^w(x))$, and thus $\left(\frac{d}{dx}A^w(x)\right) \geq (\leq)\ 0$ according as $h(x) \geq (\leq) A^w(x)$ for all $x > 0$. If $h(x)$ is increasing (decreasing) in x then $h(x) \geq (\leq)\ A^w(x)$ for $x > 0$. This proves (*i*). Similarly, to prove (*ii*), we first note that

$$\left(\frac{d}{dx}G^w(x)\right) = \frac{G^w(x)}{\left(\int_0^x w(u)du\right)} w(x) \ln\left(\frac{h(x)}{G^w(x)}\right),$$

and this implies that $\frac{d}{dx}G^w(x) \geq (\leq)\ 0$ according as $h(x) \geq (\leq) G^w(x)$. One can note that if $h(x)$ is increasing (decreasing) in x then $h(x) \geq (\leq)\ G^w(x)$ for all $x > 0$, thus proving (*ii*). To prove (*iii*), we first note that

$$\left(\frac{d}{dx}H^w(x)\right)\left(\int_0^x \frac{w(p)}{h(p)}dp\right) = w(x)\left\{1 - \frac{H^w(x)}{h(x)}\right\},$$

and hence we find that $\left(\frac{d}{dx}H^w(x)\right) \geq (\leq)\ 0$ according as $h(x) \geq (\leq) H^w(x)$ for all $x > 0$. Also, if $h(x)$ is increasing (decreasing) in x then $h(x) \geq (\leq) H^w(x)$ for all $x > 0$. This completes the proof. □

Below, we state two theorems highlighting the inclusion property of the non-parametric aging classes given in Definition 4.1. The proof follows due to Theorem 4.2, line of the proof therein and the fact that $A^w(x) \geq G^w(x) \geq H^w(x)$ for all $x > 0$.

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Theorem 4.3. *IFR ⊆ Iw-AFR ⊆ Iw-GFR ⊆ Iw-HFR.*

Theorem 4.4. *DFR ⊆ Dw-HFR ⊆ Dw-GFR ⊆ Dw-AFR.*

The next example highlights the importance of choosing weight functions in generating new distributions. We also take the help of this example in upcoming counterexample 5.4 for establishing that $w-IFRA$ class is not closed under formation of coherent systems.

Example 4.5. Let X follows two parameter Weibull distribution with scale and shape parameter α and β respectively. If we take $w(x) = e^{nx}$ for all $x > 0$, then the weighted random variable X^w has failure rate $h^w(x) = \alpha\beta e^{nx}x^{\beta-1}$. Here, taking $n = -m$ with $m > 0$,

$$\begin{aligned}\int_0^x w(u)h(u)u &= \alpha\beta\int_0^x e^{-mt}t^{\beta-1}dt\\ &= \alpha\beta(-n)^{-\beta}\gamma(\beta,-nx)\\ &= \alpha\beta(-n)^{-\beta}\Big(\Gamma(\beta)-\Gamma(\beta,-nx)\Big), \end{aligned}\tag{6}$$

where the incomplete Gamma function $\gamma(z, a)$ and its complement $\Gamma(z, \alpha)$ (also known as Prym's function) are

$$\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt, \Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt,\ \text{Real}(a) > 0),$$

satisfying $\gamma(a, x) + \Gamma(a, x) = \Gamma(a)$. If $n < 0$ we have real values for $\bar{F}^w(t)$, as

$$\bar{F}^w(x) = \exp\Big\{-\alpha\beta(-n)^{-\beta}\Big(\Gamma(\beta)-\Gamma(\beta,-nx)\Big)\Big\}, x > 0, \beta > 0.$$

Also, considering $m = -n$, we get

$$\frac{d}{dx}h^w(x) = \frac{d}{dx}\Big(\alpha\beta e^{nx}x^{\beta-1}\Big) = \alpha\beta e^{nx}x^{\beta-2}(\beta-1-mx) \le 0$$

if $(\beta - 1 - mx) \le 0$, that is, $\frac{d}{dx}h^w(x) \le 0$ if $x \ge \frac{\beta-1}{m}$. If $\beta < 1$ then $\frac{d}{dx}h^w(x) \le 0$ for all $x > 0$. Thus, X^w is decreasing failure rate if $\beta < 1$. On the other hand if $\beta > 1$, then $\frac{d}{dx}h^w(x) \ge 0$ for $x \in (0, \frac{\beta-1}{m})$ and $\frac{d}{dx}h^w(x) \le 0$ for $x \ge \frac{\beta-1}{n}$. Thus X^w is decreasing failure rate if $\beta < 1$, whereas X^w has upside-down bathtub shaped failure rate if $\beta > 1$. Using (6) and the fact that $\int_0^x w(u)du = \frac{1}{n}\Big(e^{nx}-1\Big)$ we get

$$A^w(x) = \frac{n(-n)^{-\beta}\alpha\beta\Big(\Gamma[\beta]-\Gamma[\beta,-nx]\Big)}{\Big(e^{nx}-1\Big)}.$$

Here, for $\beta < 1$, $h^w(x)$ is decreasing in x, and so is $A^w(x)$ as evident from Theorem 4.2. Similarly, for $\beta > 1$, $A^w(x)$ is upside-down bathtub. We note that,

$$\begin{aligned}\frac{d}{dx}A^w(x) &= n(-n)^{-\beta}\alpha\beta\frac{d}{dx}\Big(\frac{\Big(\Gamma[\beta]-\Gamma[\beta,-nx]\Big)}{e^{nx}-1}\Big)\\ &= n(-n)^{-\beta}\alpha\beta\left\{\frac{(e^{nx}-1)e^{nx}(-nx)^{\beta-1}(-n)-\Big(\Gamma[\beta]-\Gamma[\beta,-nx]\Big)e^{nx}n}{(e^{nx}-1)^2}\right\}\end{aligned}$$

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$$
\begin{aligned}
&= n(-n)^{-\beta}\alpha\beta\left\{\frac{(e^{nx}-1)e^{nx}(-nx)^{\beta}\left(\frac{(-n)}{-(nx)}\right)-\left(\Gamma[\beta]-\Gamma[\beta,-nx]\right)e^{nx}n}{(e^{nx}-1)^2}\right\}\\
&= \frac{n(-n)^{-\beta}\alpha\beta e^{nx}}{x(e^{nx}-1)^2}\left\{(e^{nx}-1)(-nx)^{\beta}+nx\left(\Gamma[\beta,-nx]-\Gamma[\beta]\right)\right\}\\
&= \frac{n^2x(-n)^{-\beta}\alpha\beta e^{nx}}{x(e^{nx}-1)^2}\left\{\Gamma[\beta,-nx]-\Gamma[\beta]-(e^{nx}-1)(-nx)^{\beta-1}\right\}\\
&= \frac{n^2x(-n)^{-\beta}\alpha\beta e^{nx}}{x(e^{nx}-1)^2}\left\{-\gamma[\beta,-nx]-P(x)\right\}, \text{(say)},
\end{aligned}
$$

where

$$P(x) = (e^{nx}-1)(-nx)^{\beta-1} \leq 0, x > 0.$$

The change point of monotonicity of $A^w(x)$ is determined by the root of equation $\gamma[\beta,-nx]+P(x)=0$. Similarly, we obtain

$$G^w(x) = \alpha\beta t^{\beta-1}(-nt)^{\frac{\beta-1}{e^{nt}-1}}e^{\frac{(\beta-1)(E_1(-nt)+\gamma)}{e^{nt}-1}}$$

$$H^w(x) = \frac{\alpha\beta(e^{nt}-1)nt^{\beta}(-nt)^{-\beta}}{\Gamma[2-\beta]-\Gamma[2-\beta,-nt]},$$

where $\gamma \sim 0.577216$ is Euler's constant and $E_n(z)$ is the exponential integral function.

5. Characterization results of w-AFR and w-GFR classes of distributions

We introduce the concept of weighted star-shaped (anti-star) function which is a generalization of star-shaped (anti-star) function to give an equivalent condition of increasing weighted AFR (*Iw-AFR*) and decreasing weighted AFR (*Dw-AFR*) classes of distributions.

Definition 5.1. *A function g(x) defined on* $[0,\infty)$ *is said to be a weighted star-shaped function (weighted anti-star shaped) with respect to a non-negative weight function w(x) if* $\left(-\frac{1}{\int_0^x w(u)du}\right)g(x)$ *is decreasing (increasing) in x > 0. Equivalently, for* $0<\alpha\leq 1$ *and* $x>0$,

$$g(\alpha x) \leq (\geq)\left(\frac{\int_0^{\alpha x} w(u)du}{\int_0^x w(u)du}\right)g(x).$$

The next theorem gives a necessary and sufficient condition of a increasing (decreasing) weighted arithmetic failure rate or weighted failure rate average class of distributions, denoted by w-AFR. We omit the proof for the sake of brevity.

Theorem 5.2. *Let X has Iw-AFR* (*Dw-AFR*)*. Then the following conditions are equivalent.*

(i) $\left(-\frac{1}{\int_0^x w(u)du}\right)\ln\overline{F}^w(x)$ *is increasing (decreasing) in x > 0.*

(ii) $-\ln\overline{F}^w(x)$ *is weighted star-shaped (weighted anti-star shaped) with respect to* $w(\cdot)$.

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(iii) $\left(\overline{F}^w(x)\right)^{\frac{1}{\int_0^x w(u)du}}$ *is decreasing (increasing) in* $x > 0$.

(iv) *For* $\alpha \in [0,1]$, *and* $x > 0$, $\overline{F}^w(\alpha x) \geq (\leq) \left(\overline{F}^w(x)\right)^{\frac{\int_0^{\alpha x} w(u)du}{\int_0^x w(u)du}}$.

The following theorem gives equivalent conditions for $w - GFR$ aging class.

Theorem 5.3. *Let X has Iw-GFR (Dw-GFR). Then the following conditions are equivalent.*

(i) $\left(\frac{1}{\int_0^x w(u)du}\right)\left(\int_0^x w(u)\ln h(u)du\right)$ *is increasing (decreasing) in* $x > 0$.

(ii) $\left(\int_0^x w(u)\ln h(u)du\right)$ *is weighted star-shaped (weighted anti-star shaped) with respect to* $w(\cdot)$.

(iii) *For* $\alpha \in [0,1]$, *and* $x > 0$, $\int_0^{\alpha x} w(u)\ln h(u)du \leq (\geq) \frac{\int_0^{\alpha x} w(u)du}{\int_0^x w(u)du}\left(\int_0^x w(u)\ln h(u)du\right)$.

We note that, $0 \leq \frac{\int_0^{\alpha x} w(u)du}{\int_0^x w(u)du} \leq 1$ since $w(x) \geq 0$ for all $x \geq 0$ and $0 \leq \alpha \leq 1$.

5.1. Results on coherent system

In this section, we primarily focus on Iw-AFR class and its closure properties. We know that IFRA class is closed under the formation of coherent system. Naturally, a question ponders, whether the same result is true for Iw-AFR class.

Let us consider a coherent system with n components having weighted survival functions $\overline{F}_i^w(x)$ for $i = 1, 2, \ldots, n$. The survival function $\overline{F}^w(x)$ of the resultant coherent system satisfies

$$\overline{F}^w(\alpha x) = h(\overline{F}_1^w(\alpha x), \overline{F}_2^w(\alpha x), \ldots, \overline{F}_n^w(\alpha x)), \tag{7}$$

where h represents the survival function of the coherent system. Further, if we assume that each X_i has increasing w-AFR, then we explore what would be the survival function of the resultant coherent system. Since $\overline{F}_i^w(\alpha x) \geq \left(\overline{F}_i^w(x)\right)^{\frac{\int_0^{\alpha x} w(u)du}{\int_0^x w(u)du}}$ for $i = 1, 2, \ldots, n$, $\alpha \in [0,1], x > 0$, and h is increasing in each argument, (7) reduces to

$$\overline{F}^w(\alpha x) \geq h\left(\left(\overline{F}_1^w(x)\right)^{\frac{\int_0^{\alpha x} w(u)du}{\int_0^x w(u)du}}, \left(\overline{F}_2^w(x)\right)^{\frac{\int_0^{\alpha x} w(u)du}{\int_0^x w(u)du}}, \ldots, \left(\overline{F}_n^w(x)\right)^{\frac{\int_0^{\alpha x} w(u)du}{\int_0^x w(u)du}}\right).$$

The following counter example shows that Iw-AFR is not closed under the formation of coherent system.

Counterexample 5.4. *Let us consider a series system with lifetime X formed by two components with lifetimes* ${X^w}_1$ *and* ${X^w}_2$ *respectively. Let the failure rates be* $h_1(x)$, *and* $h_2(x)$ *with corresponding weights* $w_1(x)$ *and* $w_2(x)$ *respectively. Let* $h_1(x) = \alpha\beta x^{\beta-1}, w_1(x) = e^{nx}$, *and* $h_2(x) = abx^{b-1}, w_2(x) = (1 - e^{nx})$ *where* $\alpha, a > 0; \beta, b > 1; n < 0$. *Since,* $\beta, b > 1$; $h_1(x)$ *and* $h_2(x)$ *are increasing in* x. *By Theorem 4.2, it follows that* $A_1^w(x)$ *and* $A_2^w(x)$ *are increasing in* x *as* $h_1(x)$ *and* $h_2(x)$ *are increasing in* x. *Then the hazard rate of the series system is given by* $h_X(x) = h_1(x)w_1(x) + h_2(x)w_2(x)$ *for all* $x > 0$. *From Example 4.5, it follows that each of* $h_1^w(x) = h_1(x)w_1(x)$ *and* $h_2^w(x) = h_2(x)w_2(x)$ *are non-monotonic in* $x > 0$ *(upside-down bathtub curve). Thus,* ${X^w}_1$ *and* ${X^w}_2$ *are Iw-AFR but not IFR. Here, X is not Iw-AFR since h(x) is non-monotonic (as noted in Example 4.5) and non-monotonicity of h(x) is transmitted to A(x) (by Theorem 4.2).*

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6. Quantile version of AFR, GFR, and HFR

Recently, there is a greater interest among researchers in modeling and analysis of lifetime data using quantile functions (QFs) as an efficient alternative to the traditional distribution function approach. Accordingly, we examine here some properties of the quantile versions of AFR, GFR, and HFR, denoted respectively by $QA(\cdot), QG(\cdot), QH(\cdot)$. The formulae of $QA(\cdot), QG(\cdot), QH(\cdot)$ can also be independently derived from Definition 2.2 by replacing the failure rate $h(\cdot)$ by hazard QF $h_q(\cdot)$ and $w(\cdot)$ by density QF $q(\cdot)$ respectively, with support restricted to $[0, 1]$.

We first begin with a few preliminaries on quantile based reliability concepts. For a random variable X, QF is defined as

$$Q(u) = F^{-1}(u) = \inf\Big\{x : F(x) \geq u\Big\}, 0 \leq u \leq 1 \tag{8}$$

gives $FQ(u) = u$. Differentiating with respect to u, we get $f(Q(u))q(u) = 1$ or $f(Q(u)) = \frac{1}{q(u)}$, where $f(Q(u))$ and $q(u) = \frac{d}{du}Q(u)$ are respectively known as the density QF and quantile density function of X. From the definition of hazard rate, the corresponding hazard QF is given by

$$h_q(u) = h(Q(u)) = \frac{f(Q(u))}{\bar{F}(Q(u))} = \frac{1}{(1-u)q(u)}.$$

This implies $q(u) = \frac{1}{(1-u)h_q(u)}$. Integrating, we get $Q(u) = \int_0^u \frac{1}{(1-p)h_q(p)}dp$. The quantile approach is an alternative to the traditional distribution function method as it can also used to specify a probability distribution. As the quantile approach possess some interesting properties not shared by its distribution function counterpart and in many situations, quantile measures provide simple expressions that are easily amenable to many computational analysis. Abundant literature are now available on various properties of QFs and different measures based on it and their applications, for details see Gilchrist [6], Nair *et al.* [12, 13], Aswin *et al.* [1], and references therein.

$$\begin{aligned} QA(u) = QA(Q(u)) &= \frac{-\ln(1-F(Q(u)))}{Q(u)} \\ &= \frac{-\ln(1-u)}{Q(u)} = -\Big(\ln(1-u)\Big)\Big\{\int_0^u \frac{1}{(1-p)h_q(p)}dp\Big\}^{-1}. \end{aligned} \tag{9}$$

$$\begin{aligned} QG(u) = QG(Q(u)) &= \exp\Big(\frac{1}{Q(u)}\int_0^u \ln\Big(\frac{1}{(1-p)q(p)}\Big)dQ(p)\Big) \\ &= \exp\Big(-\frac{1}{Q(u)}\int_0^u q(p)\ln\Big((1-p)q(p)dp\Big)\Big), \end{aligned}$$

or equivalently,

$$\begin{aligned} Q(u)\ln QG(u) &= -\int_0^u (\ln(1-p))q(p)dp - \int_0^u (\ln q(p))q(p)dp \\ &= -\int_0^u q(p)\ln\Big\{(1-p)q(p)\Big\}dp = \int_0^u q(p)\ln h_q(p)dp \end{aligned} \tag{10}$$

and

$$\begin{aligned} QH(u) = QH(Q(u)) &= \Big(\frac{1}{Q(u)}\int_0^u \frac{1}{h(Q(p))}dQ(p)\Big)^{-1} \\ &= Q(u)\Big(\int_0^u (1-p)(q(p))^2 dp\Big)^{-1} \end{aligned} \tag{11}$$

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$$= Q(u)\Big(\int_0^u \frac{q(p)}{h_q(p)}dp\Big)^{-1}. \tag{12}$$

Differentiating (9) with respect to u, we obtain

$$QA'(u)Q(u) + QA(u)q(u) = \frac{1}{1-u}.$$

When quantile AFR is increasing (decreasing), we get

$$QA(u) \leq (\geq)\ h_q(u).$$

From (11), we have

$$\frac{Q(u)}{QH(u)} = \int_0^u (1-p)(q(p))^2 dp.$$

Differentiating with respect to u, we get

$$QH(u)q(u) - Q(u)QH'(u) = (1-u)(q(u))^2\,(QH(u))^2\,.$$

Now when the quantile HFR is increasing (decreasing), yield

$$QH(u) \leq (\geq) h_q(u).$$

A similar argument as given in Theorem 4.2 depicts that monotonicity of hazard quantile function $h_q(\cdot)$ is transmitted to quantile version of AFR, GFR, and HFR, that is, $QA(\cdot), QG(\cdot)$, and $QH(\cdot)$. In continuation to the Proposition 3.2 of previous section, if we replace $w(\cdot)$ by $q(\cdot)$ and $h(\cdot)$ by $h_q(\cdot)$, we find that proportionality of weighted means of quantile hazard functions with quantile hazard function characterizes some QF. To the best of our knowledge, $Q(x)$ as obtained in Proposition 6.2 represents a new generalized version of QF where $Q(0) \neq 0$.

The next example gives the QF of AFR, GFR, and HFR of Pareto-I distribution.

Example 6.1. For Pareto I distribution, with quantile function $Q(u) = \alpha(1-u)^{-1/\alpha}$, we have

$$QA(u) = -\frac{(1-u)^{1/\alpha}\log(1-u)}{\alpha}, QG(u) = e^{1-(1-u)^{1/\alpha}}(1-u)^{1/\alpha}, \text{ and } QH(u) = -\frac{2(1-u)^{1/\alpha}}{(1-u)^{2/\alpha}-1},$$

for $0 < u < 1$.

Proposition 6.2. *Let hazard quantile function $h_q(u)$ be differentiable for all $u \in [0,1]$. Then for the non-negative weight function $q(u)$, called as density quantile function and for $a,b,c > 0$ we have*

(i) $QA(u) = a\, h_q(u)$ *for all* $u \in [0,1]$ *if and only if* $Q(u) = \left(\frac{1}{ak}\right)^a \left\{\ln(\frac{A}{1-u})\right\}^a$.

(ii) $QG(u) = b\, h_q(u)$ *for all* u *if and only if* $Q(u) = \left(\frac{\ln(e/b)}{k}\right)^{\frac{1}{\ln(e/b)}} \left\{\ln(\frac{A}{1-u})\right\}^{\frac{1}{\ln(e/b)}}$.

(iii) $QH(u) = c\ h_q(u)$ *for all* u *if and only if* $Q(x) = \left(\frac{\ln(e/b)}{k}\right)^{\frac{1}{\ln(e/b)}} \left\{\ln(\frac{A}{1-u})\right\}^{\frac{1}{\ln(e/b)}}$ *where k is an arbitrary constant.*

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Proof. From Theorem 3.2 (i), it follows that $QA(u) = a\, h_q(u)$ is equivalent to

$$h_q(u) = k\Big(\int_0^u q(p)dp\Big)^{(1-a)/a}$$

and since $h_q(u) = \frac{1}{(1-u)q(u)}$, we prove ($i$). Proofs of ($ii$) and ($iii$) are similar. □

Transformation on a random variable is generally employed to find the best model for a given set of observations. A simple alternative method to this is to keep the original data as it is and transform the QF to find the best model, using the following property of QFs which is not shared by the distribution function. If $T_X(x)$ is a continuous non-decreasing function then $T_X\,(Q_X(u))$ is the QF of $T_X(X)$ or in symbols

$$Q_{T(X)}(u) = T\,(Q_X(u))\,.$$

Theorem 6.3. *Let $T(\cdot)$ be a continuous non-decreasing and invertible transformation. Then the quantile versions of AFR, GFR, and HFR takes the form*

(i) $QA_{T(X)}(u) = \frac{-\log(1-u)}{T(Q_X(u))}$,
(ii) $QG_{T(X)}(u) = \exp\Big(-\frac{1}{T(Q_X(u))}\int_0^u T'\,(Q_X(p))\,q(p)\,[\log(1-p)T'\,(Q_X(p))\,q(p)]\,dp\Big)$, *and*
(iii) $QH_{T(X)}(u) = T\,(Q_X(u))\Big(\int_0^u (1-p)\,(T'\,(Q_X(p))\,q(p))^2\,dp\Big)^{-1}$.

Theorem 6.4. *The following statements are equivalent: (i) X follows Exponential distribution with shape parameter c, (ii) $QA(u) = c$, (iii) $QG(u) = c$, and (iv) $QH(u) = c$.*

Remark 6.5. The quantile version is not always equivalent to its distribution function approach.

Theorem 6.6. $QA(u) = (Q(u))^{C-1}$, *where $C > 0$ holds if and only X follows Weibull distribution with quantile function* $Q(u) = (-\log(1-u))^{\frac{1}{\lambda}}$, $0 < u < 1$, $\lambda > 0$.

For many models, the distribution function and quantile approaches yield similar properties as we have seen in Theorem 4.2, while for certain other cases, it gives different results. For example, when X and Y satisfy PHR model, we have $h_{q_Y}(u) = \theta h_{q_X}(u)$, or equivalently, we have $\bar{F}_Y(u) = \big(\bar{F}_X(u)\big)^{\theta}$. We look at the corresponding AFR, GFR, and HFR of Y. It is easy to note that $A_Y(x) = \theta\, A_X(x)$, $G_Y(x) = \theta\, G_X(x)$, and $H_Y(x) = \theta\, H_X(x)$. To obtain the quantile version of AFR, GFR, and HFR under PHR, it is easy to obtain the QF of Y, as

$$Q_Y(u) = \inf\Big\{x : F_Y(x) \geq u\Big\} = Q_X(1-(1-u)^{1/\theta}),$$

which in turn obtain the quantile version of AFR for PHR as

$$QA_Y(u) = -\frac{\ln(1-u)}{Q_Y(u)} = \frac{-\ln(1-u)}{Q_X(1-(1-u)^{1/\theta})} = \theta QA_X(1-(1-u)^{1/\theta}) \neq \quad \theta QA_X(u),$$

since

$$QA_X(1-(1-u)^{1/\theta}) = \frac{-\ln\Big\{\Big\{1-(1-(1-u)^{1/\theta}\Big\}\Big\}}{Q_X(1-(1-u)^{1/\theta})} = -\frac{1}{\theta}\frac{\ln(1-u)}{Q_X(1-(1-u)^{1/\theta})}. \tag{13}$$

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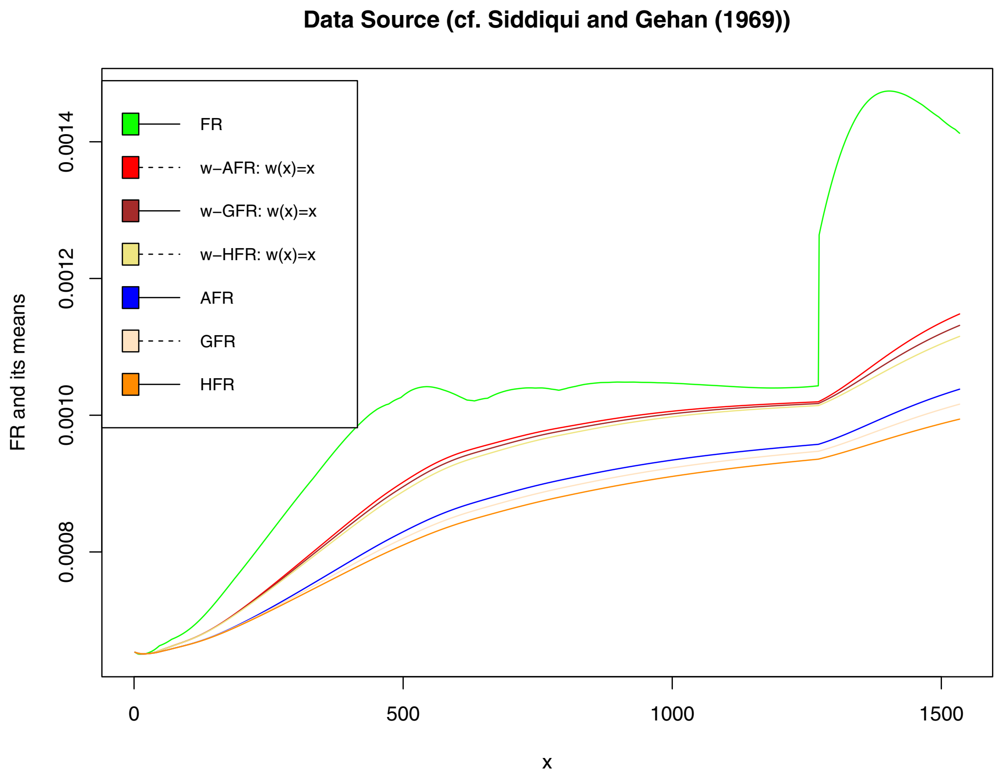

Figure 1. *Comparative analysis through proposed (termed as new) weighted means, non-weighted means, and failure rate of the data given in Example 7.1.*

The quantile GFR of PHR will be

$$QG_Y(u) = \exp\left[-\frac{1}{Q_X\left(1-(1-u)^{\frac{1}{\theta}}\right)} \int_0^u \frac{1}{\theta} q_X\left(1-(1-p)^{\frac{1}{\theta}}\right)(1-p)^{\frac{1}{\theta}-1} \ln\left(\frac{1}{\theta} q_X\left(1-(1-p)^{\frac{1}{\theta}}\right)(1-p)^{\frac{1}{\theta}}\right) dp\right],$$

or equivalently

$$QG_Y(u) = \exp\left[-\frac{1}{Q_X(u)} \int_0^{1-(1-u)^{\frac{1}{\theta}}} \frac{1}{\theta} q_X(p)(1-p)^{1-\theta} \ln\left((1-p)\frac{1}{\theta} q_X(p)\right) dp\right] \neq \theta\, \mathcal{QG}_X(u).$$

Also, the quantile version of HFR becomes

$$QH_Y(u) = \left(\frac{1}{Q_X\left(1-(1-u)^{\frac{1}{\theta}}\right)} \int_0^u \frac{1}{\theta}(1-p)^{\frac{2}{\theta}-1} q_X\left(1-(1-p)^{\frac{1}{\theta}}\right)^2 dp\right)^{-1} \neq \theta\, QH_X(u).$$

This clearly illustrates that quantile version of the AFR, GFR, and HFR for the PHR do not satisfy the properties which hold in the distribution function approach.

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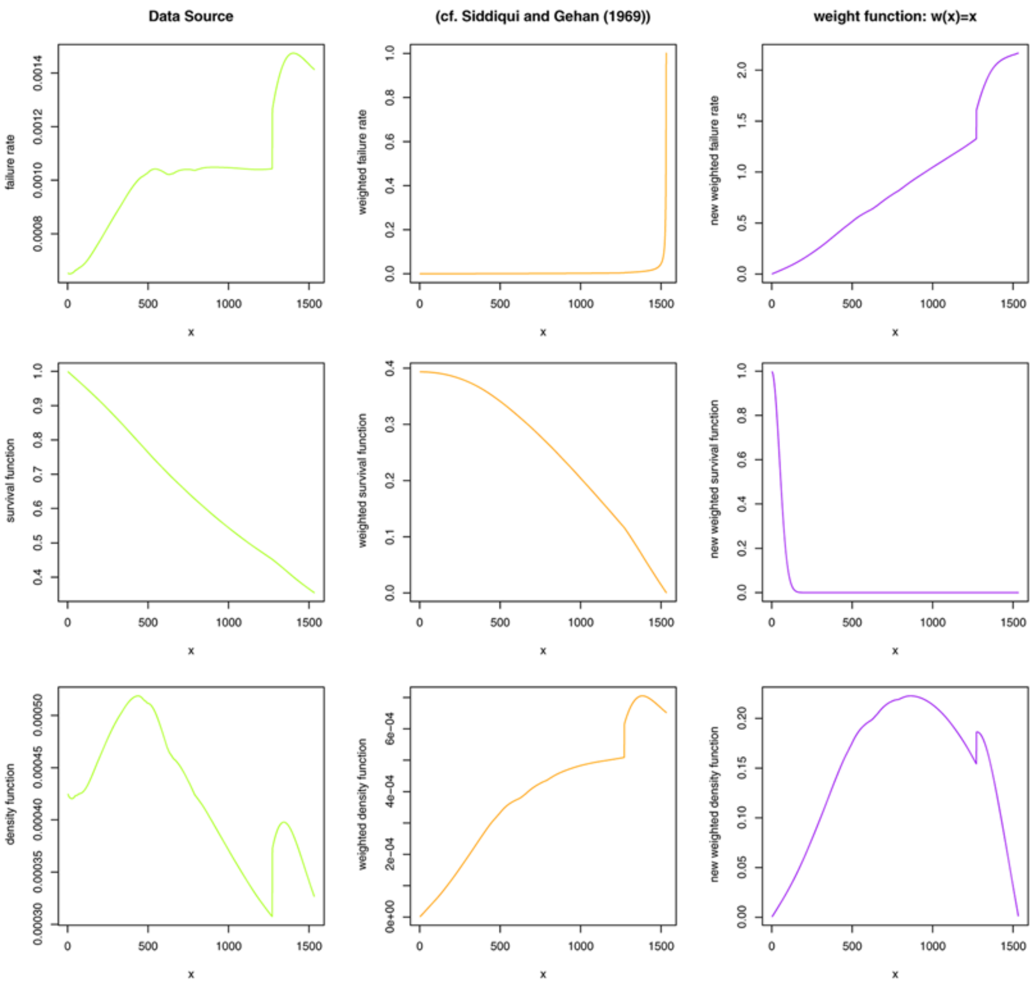

Figure 2. Reliability analysis of the data given in Example 7.1 using proposed (termed as new) weighted functions and existing weighted concept (cf. Rao [16]).

7. Applications to real-life data

In this section, we demonstrate an application of the proposed weighted concept in the following example to examine a survival time data (cf. Bryson and Siddiqui [5]) and the intrinsic aging phenomenon among patients suffering from chronic granulocytic leukemia.

Example 7.1. The ordered survival times (in days from diagnosis) of patients suffering from chronic granulocytic leukemia from the very beginning of their diagnosis are collected from National Cancer Institute (cf. Siddiqui and Gehen [18], Bryson and Siddiqui [5]). The values of survival times are given as: 747, 58,74, 177, 232, 273, 285, 317, 429, 440, 445, 455, 468, 495, 497, 532, 571, 579, 581, 650, 702, 715, 779, 881, 900, 930, 968, 1,077, 1,109, 1,314, 1,334, 1,367, 1,534, 1,712, 1,784, 1,877, 1,886, 2,045, 2,056, 2,260, 2,429, 2,509.

At the given survival times points, we apply muhaz package available in RStudio 2024.04 Build 748. Particularly, we choose Epanechnikov kernel and assigned 1,000 grid (time) points in muhaz package to obtain estimated values of hazard rate at the grid points. Subsequently, the estimated value of other reliability functions, viz., failure rate (FR), w-AFR, w-GFR, w-HFR, AFR, GFR, and HFR at estimated

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grid (time) points are obtained for further analyses by taking length biased weight function, that is, $w(x) = x$ as shown in Figure 1.

In Figure 2, we do a reliability analysis of the data using proposed (termed as new) weighted functions and existing weighted concept (cf. Rao [16]).

It is revealed that the definition of weighted concept proposed in this paper corroborates with the actual functions, namely failure rate, survival function, density function in a better way than the existing weighted concepts proposed by Rao [16]. It allows us to conclude that the proposed weighted failure and its means can be used as more flexible functions for the measurement of failure rate in the analysis of lifetime data, especially in cases where sample observations do not have equal probability of selection.

8. Conclusion

At the long last, for readers we reiterate that mixture of n distributions is a special case of formation of n independent component series system having weighted failure rates with the sum of weight functions being unity. However, the latter system having arbitrary weights is also not a generalization of the former. The idea of relating the said concepts deserves some credit because the existing literature on mixture of distributions can be extended to the formation of coherent systems (in particular, series system) so far as non-preservation properties of reliability operations are concerned. One can generate new distributions using weighted version of arithmetic, geometric and harmonic means of failure rate. Since, the quantile version of means of hazard rate is a special case of weighted means of failure rate, the properties studied for weighted means is put forth for the prior.

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