

## ON THE EXCEPTIONAL SET OF TRANSCENDENTAL ENTIRE FUNCTIONS IN SEVERAL VARIABLES

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(Received 6 June 2023; accepted 3 September 2023; first published online 20 October 2023)

### Abstract

We prove that any subset of  $\overline{\mathbb{Q}}^m$  (closed under complex conjugation and which contains the origin) is the exceptional set of uncountably many transcendental entire functions over  $\mathbb{C}^m$  with rational coefficients. This result solves a several variables version of a question posed by Mahler for transcendental entire functions [*Lectures on Transcendental Numbers*, Lecture Notes in Mathematics, 546 (Springer-Verlag, Berlin, 1976)].

2020 *Mathematics subject classification*: primary 11J81; secondary 32A15.

*Keywords and phrases*: exceptional set, algebraic, transcendental, transcendental functions of several variables.

### 1. Introduction

An analytic function  $f$  over a domain  $\Omega \subseteq \mathbb{C}$  is said to be an *algebraic function* over  $\mathbb{C}(z)$  if there exists a nonzero polynomial  $P \in \mathbb{C}[X, Y]$  for which  $P(z, f(z)) = 0$ , for all  $z \in \Omega$ . A function which is not algebraic is called a *transcendental function*.

The study of the arithmetic behaviour of transcendental functions started in 1886 with a letter of Weierstrass to Strauss, proving the existence of such functions taking  $\mathbb{Q}$  into itself. Weierstrass also conjectured the existence of a transcendental entire function  $f$  for which  $f(\overline{\mathbb{Q}}) \subseteq \overline{\mathbb{Q}}$  (as usual,  $\overline{\mathbb{Q}}$  denotes the field of all algebraic numbers). Motivated by results of this kind, he defined the *exceptional set* of an analytic function  $f : \Omega \rightarrow \mathbb{C}$  as

$$S_f = \{\alpha \in \overline{\mathbb{Q}} \cap \Omega : f(\alpha) \in \overline{\mathbb{Q}}\}.$$

Thus, Weierstrass' conjecture can be rephrased as: *does there exist a transcendental entire function  $f$  such that  $S_f = \overline{\mathbb{Q}}$ ?* This conjecture was settled in 1895 by Stäckel [4],

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Diego Marques was supported by CNPq-Brazil. Pavel Trojovský was supported by the Project of Excellence, Faculty of Science, University of Hradec Králové, No. 2210/2023-2024.

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who proved, in particular, that for any  $\Sigma \subseteq \overline{\mathbb{Q}}$ , there exists a transcendental entire function  $f$  for which  $\Sigma \subseteq S_f$ .

In his classical book [1], Mahler introduced the problem of studying  $S_f$  for various classes of functions. After discussing a number of examples, Mahler posed several problems about the admissible exceptional sets for analytic functions, one of which is as follows. Here  $B(0, \rho)$  denotes the closed ball with centre 0 and radius  $\rho$  in  $\mathbb{C}$ .

**PROBLEM 1.1.** Let  $\rho \in (0, \infty]$  be a real number. Does there exist for any choice of  $S \subseteq \overline{\mathbb{Q}} \cap B(0, \rho)$  (closed under complex conjugation and such that  $0 \in S$ ) a transcendental analytic function  $f \in \mathbb{Q}[[z]]$  with radius of convergence  $\rho$  for which  $S_f = S$ ?

In 2016, Marques and Ramirez [3] proved that the answer to this question is ‘yes’ provided that  $\rho = \infty$  (that is, for entire functions). Indeed, they proved the following more general result about the arithmetic behaviour of certain entire functions.

**LEMMA 1.2 [3, Theorem 1.3].** Let  $A$  be a countable set and let  $\mathbb{K}$  be a dense subset of  $\mathbb{C}$ . For each  $\alpha \in A$ , fix a dense subset  $E_\alpha \subseteq \mathbb{C}$ . Then there exist uncountably many transcendental entire functions  $f \in \mathbb{K}[[z]]$  such that  $f(\alpha) \in E_\alpha$  for all  $\alpha \in A$ .

This result was improved by Marques and Moreira in [2] giving an affirmative answer to Mahler’s Problem 1.1 for any  $\rho \in (0, \infty]$ .

In this paper, we consider Mahler’s Problem 1.1 in the context of transcendental entire functions of several variables. Although the previous definitions extend to the context of several variables in a very natural way, we shall include them here for the sake of completeness.

An analytic function  $f$  over a domain  $\Omega \subseteq \mathbb{C}^m$  (we also say that  $f$  is *entire* if  $\Omega = \mathbb{C}^m$ ) is said to be *algebraic* over  $\mathbb{C}(z_1, \dots, z_m)$  if it is a solution of a polynomial functional equation

$$P(z_1, \dots, z_m, f(z_1, \dots, z_m)) = 0 \quad \text{for all } (z_1, \dots, z_m) \in \Omega,$$

for some nonzero polynomial  $P \in \mathbb{C}[z_1, \dots, z_m, z_{m+1}]$ . A function which is not algebraic is called a transcendental function. (We remark that an entire function in several variables is algebraic if and only if it is a polynomial function just as in the case of one variable.) Let  $\mathbb{K}$  be a subset of  $\mathbb{C}$  and let  $f$  be an analytic function on the polydisc  $\Delta(0, \rho) := B(0, \rho_1) \times \dots \times B(0, \rho_m) \subseteq \mathbb{C}^m$  for some  $\rho = (\rho_1, \dots, \rho_m) \in (0, \infty]^m$ . We say that  $f \in \mathbb{K}[[z_1, \dots, z_m]]$  if

$$f(z_1, \dots, z_m) = \sum_{(k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^m} c_{k_1, \dots, k_m} z_1^{k_1} \cdots z_m^{k_m},$$

with  $c_{k_1, \dots, k_m} \in \mathbb{K}$  for all  $(k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^m$  and for all  $(z_1, \dots, z_m) \in \Delta(0, \rho)$ .

The exceptional set  $S_f$  of an analytic function  $f : \Omega \subseteq \mathbb{C}^m \rightarrow \mathbb{C}$  is defined as

$$S_f := \{(\alpha_1, \dots, \alpha_m) \in \Omega \cap \overline{\mathbb{Q}}^m : f(\alpha_1, \dots, \alpha_m) \in \overline{\mathbb{Q}}\}.$$

For example, let  $f : \mathbb{C}^2 \rightarrow \mathbb{C}$  and  $g : \mathbb{C}^2 \rightarrow \mathbb{C}$  be the transcendental entire functions given by

$$f(w, z) = e^{w+z} \quad \text{and} \quad g(w, z) = e^{wz}.$$

By the Hermite–Lindemann theorem,

$$S_f = \{(\alpha, -\alpha) : \alpha \in \overline{\mathbb{Q}}\} \quad \text{and} \quad S_g = (\overline{\mathbb{Q}} \times \{0\}) \cup (\{0\} \times \overline{\mathbb{Q}}).$$

In general, if  $P_1(X, Y), \dots, P_n(X, Y) \in \overline{\mathbb{Q}}[X, Y]$ , then the function

$$f(w, z) = \exp\left(\prod_{k=1}^n P_k(w, z)\right)$$

has the exceptional set given by

$$S_f = \bigcup_{k=1}^n \{(\alpha, \beta) \in \overline{\mathbb{Q}}^2 : P_k(\alpha, \beta) = 0\}.$$

We refer the reader to [1, 5] (and references therein) for more about this subject.

In the main result of this paper, we shall prove that every subset  $S$  of  $\overline{\mathbb{Q}}^m$  (under some mild conditions) is the exceptional set of uncountably many transcendental entire functions of several variables with rational coefficients.

**THEOREM 1.3.** *Let  $m$  be a positive integer. Then, every subset  $S$  of  $\overline{\mathbb{Q}}^m$ , closed under complex conjugation and such that  $(0, \dots, 0) \in S$ , is the exceptional set of uncountably many transcendental entire functions  $f \in \mathbb{Q}[[z_1, \dots, z_m]]$ .*

To prove this theorem, we shall provide a more general result about the arithmetic behaviour of a transcendental entire function of several variables.

**THEOREM 1.4.** *Let  $X$  be a countable subset of  $\mathbb{C}^m$  and let  $\mathbb{K}$  be a dense subset of  $\mathbb{C}$ . For each  $u \in X$ , fix a dense subset  $E_u \subseteq \mathbb{C}$  and suppose that if  $(0, \dots, 0) \in X$ , then  $E_{(0, \dots, 0)} \cap \mathbb{K} \neq \emptyset$ . Then there exist uncountably many transcendental entire functions  $f \in \mathbb{K}[[z_1, \dots, z_m]]$  such that  $f(u) \in E_u$  for all  $u \in X$ .*

Theorem 1.4 is a several variables extension of the one-variable result due to Marques and Ramirez [3, Theorem 1.3].

## 2. Proofs

**2.1. Proof that Theorem 1.4 implies Theorem 1.3.** In the statement of Theorem 1.4, choose  $X = \overline{\mathbb{Q}}^m$  and  $\mathbb{K} = \mathbb{Q}^* + i\mathbb{Q}$ . Write  $S = \{u_1, u_2, \dots\}$  and  $\overline{\mathbb{Q}}^m/S = \{v_1, v_2, \dots\}$  (one of them may be finite) and define

$$E_u := \begin{cases} \overline{\mathbb{Q}} & \text{if } u \in S, \\ \mathbb{K} \cdot \pi^n & \text{if } u = v_n. \end{cases}$$

By Theorem 1.4, there exist uncountably many transcendental entire functions

$$f(z_1, \dots, z_m) = \sum_{k_1 \geq 0, \dots, k_m \geq 0} c_{k_1, \dots, k_m} z_1^{k_1} \cdots z_m^{k_m}$$

in  $\mathbb{K}[[z_1, \dots, z_m]]$  such that  $f(u) \in E_u$  for all  $u \in \overline{\mathbb{Q}}^m$ . Define  $\psi(z_1, \dots, z_m)$  as

$$\psi(z_1, \dots, z_m) := \frac{f(z_1, \dots, z_m) + \overline{f(\overline{z_1}, \dots, \overline{z_m})}}{2}.$$

By the properties of the conjugation of power series,

$$\psi(z_1, \dots, z_m) = \sum_{(k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^m} \operatorname{Re}(c_{k_1, \dots, k_m}) z_1^{k_1} \cdots z_m^{k_m}$$

is a transcendental entire function in  $\mathbb{Q}[[z_1, \dots, z_m]]$  since  $\operatorname{Re}(c_{k_1, \dots, k_m})$  is rational and nonzero for all  $(k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^m$  by construction. (Here, as usual,  $\operatorname{Re}(z)$  denotes the real part of the complex number  $z$ .)

Therefore, it suffices to prove that  $S_\psi = S$ . In fact, since  $S$  is closed under complex conjugation, if  $u \in S$ , then  $\overline{u} \in S$  and thus  $f(u)$  and  $\overline{f(\overline{u})}$  are algebraic numbers and so is  $\psi(u)$ . (Observe also that  $f(0, \dots, 0) = c_{0, \dots, 0} \in \mathbb{Q}$ .) In the case in which  $u = v_n$ , for some  $n$ , we can distinguish two cases. When  $v_n \in \mathbb{R}^m$ , then  $\psi(u) = \operatorname{Re}(f(v_n))$  is transcendental, since  $f(v_n) \in \mathbb{K} \cdot \pi^n$ . For  $v_n \notin \mathbb{R}^m$ , we have  $\overline{v_n} = v_l$  for some  $l \neq n$ . Thus, there exist nonzero algebraic numbers  $\gamma_1, \gamma_2$  such that

$$\psi(v_n) = \frac{\gamma_1 \pi^n + \gamma_2 \pi^l}{2},$$

which is transcendental, since  $\overline{\mathbb{Q}}$  is algebraically closed and  $\pi$  is transcendental. In conclusion,  $\psi \in \mathbb{Q}[[z_1, \dots, z_m]]$  is a transcendental entire function whose exceptional set is  $S$ .

**2.2. Proof of Theorem 1.4.** Let us proceed by induction on  $m$ . The case  $m = 1$  is covered by Lemma 1.2. Suppose that the theorem holds for all positive integers  $k \in [1, m - 1]$ . That is, if  $\mathbb{K}$  is a dense subset of  $\mathbb{C}$ ,  $X$  is a countable subset of  $\mathbb{C}^k$  and  $E_u$  is a dense subset in  $\mathbb{C}$  for each  $u \in X$ , then there exist uncountably many transcendental entire functions  $f \in \mathbb{K}[[z_1, \dots, z_k]]$  such that  $f(u) \in E_u$  for all  $u \in X$ , for any integer  $k \in [1, m - 1]$ .

Now, let  $X$  be a countable subset of  $\mathbb{C}^m$  and  $E_u$  a fixed dense subset of  $\mathbb{C}$  for all  $u \in X$ . Without loss of generality, we can assume that  $(0, \dots, 0) \in X$ . In this case, by hypothesis,  $\mathbb{K} \cap E_{(0, \dots, 0)} \neq \emptyset$ . To apply the induction hypothesis, we consider the partition of  $X$  given by

$$X = \bigcup_{S \in \mathcal{P}_m} X_S,$$

where  $\mathcal{P}_m$  denotes the powerset of  $[1, m] = \{1, \dots, m\}$  and  $X_S$  denotes the set of all  $z = (z_1, \dots, z_m)$  in  $X \subseteq \mathbb{C}^m$  such that  $z_i \neq 0$  if and only if  $i \in S$ . In particular,  $X_\emptyset = \{(0, \dots, 0)\}$  and  $X_{[1, m]} = X \cap (\mathbb{C} \setminus \{0\})^m$ .

Given  $S = \{i_1, \dots, i_k\}$  in  $\mathcal{Q}_m = \mathcal{P}_m \setminus \{\emptyset, [1, m]\}$  and  $z = (z_1, \dots, z_m)$  in  $\mathbb{C}^m$ , we denote by  $z_S$  the element  $(z_{i_1}, \dots, z_{i_k}) \in \mathbb{C}^k$ . To simplify the exposition, we will assume that  $i_1 < \dots < i_k$  for all  $S \in \mathcal{Q}_m$ . Our goal is to show that there exist uncountably many ways to construct a transcendental entire function  $f \in \mathbb{K}[[z_1, \dots, z_m]]$  given by

$$f(z_1, \dots, z_m) = a_0 + \left( \sum_{S \in \mathcal{Q}_m} \left( \prod_{i \in S} z_i \right) f_S(z_S) \right) + f^*(z_1, \dots, z_m),$$

where  $a_0 \in E_{(0, \dots, 0)} \cap \mathbb{K}$  and, for each  $S = \{i_1, \dots, i_k\} \in \mathcal{Q}_m$ , the function  $f_S : \mathbb{C}^k \rightarrow \mathbb{C}$  is a transcendental entire function such that

$$f_S(u_S) \in \frac{1}{\alpha_{i_1} \cdots \alpha_{i_k}} \cdot (E_u - \Theta_{S,u})$$

for all  $u = (\alpha_1, \dots, \alpha_m) \in X_S$  with

$$\Theta_{S,u} = a_0 + \sum_{T \in \mathcal{Q}_m, T \neq S} \left( \prod_{i \in T} \alpha_i \right) f_T(u_T) \in \mathbb{C}.$$

By the induction hypothesis,  $f_S$  exists for all  $S \in \mathcal{Q}_m$  (noting that if  $E_u$  is a dense subset of  $\mathbb{C}$ , then  $(\alpha_{i_1} \cdots \alpha_{i_k})^{-1} \cdot (E_u - \Theta_{S,u})$  is also a dense set). Moreover, we want the function  $f^*(z_1, \dots, z_m) \in \mathbb{K}[[z_1, \dots, z_m]]$  to satisfy the condition

$$f^*(u) \in \left( E_u - a_0 - \sum_{S \in \mathcal{Q}_m} \left( \prod_{i \in S} \alpha_i \right) f_S(u_S) \right) \tag{2.1}$$

for all  $u = (\alpha_1, \dots, \alpha_m) \in X_{[1,m]}$ , and  $f^*(z_1, \dots, z_m) = 0$  whenever  $z_i = 0$  for some  $i$  with  $1 \leq i \leq m$ . Under these conditions, it is easy to see that if  $S \in \mathcal{Q}_m$  and  $u \in X_S$ , then  $f^*(u) = 0$  and  $f(u) \in E_u$ .

To construct the function  $f^* : \mathbb{C}^m \rightarrow \mathbb{C}$ , let us consider an enumeration  $\{u_1, u_2, \dots\}$  of  $X_{[1,m]}$ , where we write  $u_j = (\alpha_1^{(j)}, \dots, \alpha_m^{(j)})$ . We construct a function  $f^* \in \mathbb{K}[[z_1, \dots, z_m]]$  given by

$$f^*(z_1, \dots, z_m) = \sum_{n=m}^{\infty} P_n(z_1, \dots, z_m) = \sum_{i_1 \geq 1, \dots, i_m \geq 1} c_{i_1, \dots, i_m} z_1^{i_1} \cdots z_m^{i_m},$$

where  $P_n$  is a homogeneous polynomial of degree  $n$  and the coefficients  $c_{i_1, \dots, i_m} \in \mathbb{K}$  will be chosen so that  $f^*$  will satisfy the desired conditions.

The first condition is

$$|c_{i_1, \dots, i_m}| < s_{i_1 + \dots + i_m} := \frac{1}{\binom{i_1 + \dots + i_m - 1}{m-1} (i_1 + \dots + i_m)!},$$

where  $c_{i_1, \dots, i_m} \neq 0$  for infinitely many  $m$ -tuples of integers  $i_1 \geq 1, \dots, i_m \geq 1$ . These conditions will be used to guarantee that  $f^*$  is an entire function. Let  $L(P)$  denote the length of the polynomial  $P(z_1, \dots, z_m) \in \mathbb{C}[[z_1, \dots, z_m]]$  given by the sum of the absolute values of its coefficients. Since

$$|P_n(z_1, \dots, z_m)| \leq L(P_n) \max\{1, |z_1|, \dots, |z_m|\}^n,$$

for all  $n \geq m$  and  $(z_1, \dots, z_m)$  belonging to the open ball  $B(0, R)$ ,

$$|P_n(z_1, \dots, z_m)| < \frac{\binom{n-1}{m-1}}{\binom{n-1}{m-1}n!} \max\{1, R\}^n = \frac{\max\{1, R\}^n}{n!},$$

since  $P_n(z_1, \dots, z_m)$  has at most  $\binom{n-1}{m-1}$  monomials of degree  $n$ . Hence, the series  $\sum_{n \geq m} P_n(z_1, \dots, z_m)$  converges uniformly in any of these balls. Thus,  $f^*$  is a transcendental entire function such that  $f^*(0, z_2, \dots, z_m) = f^*(z_1, 0, z_3, \dots, z_m) = f^*(z_1, z_2, \dots, 0) = 0$ .

To obtain the coefficients  $c_{i_1, \dots, i_m} \in \mathbb{K}$  such that  $f^*$  satisfies the condition (2.1), we consider a hyperplane  $\pi(n, j)$  for positive integers  $n$  and  $j$  with  $1 \leq j \leq n$ , given by

$$\pi(n, j) : \mu_{n,1}^{(j)} z_1 + \dots + \mu_{n,m}^{(j)} z_m - \lambda_n^{(j)} = 0,$$

and such that if  $u_j, u_{n+1}$  and the origin are noncollinear, then  $\pi(n, j)$  is a hyperplane containing  $u_j$  and parallel to the line passing through the origin and the point  $u_{n+1}$ , and, if  $u_j, u_{n+1}$  and the origin are collinear, then  $\pi(n, j)$  is a hyperplane containing  $u_j$  and perpendicular to the line passing through the origin and the point  $u_{n+1}$ . Note that in both cases,  $\lambda_n^{(j)} \neq 0$  and  $u_{n+1}$  does not belong to any hyperplane  $\pi(n, j)$  with  $1 \leq j \leq n$ .

Now, we define the polynomials  $A_0(z_1, \dots, z_m) := z_1 \cdots z_m$  and

$$A_n(z_1, \dots, z_m) := \prod_{j=1}^n (\mu_{n,1}^{(j)} z_1 + \dots + \mu_{n,m}^{(j)} z_m - \lambda_n^{(j)})$$

for all  $n \geq 1$ . By the definition of  $\pi(n, j)$ , we have  $A_n(u_j) = 0$  for  $1 \leq j \leq n$ . Since  $u_{n+1}$  and the origin do not belong to  $\pi(n, j)$ , we also have  $A_n(0, \dots, 0) \neq 0$  and  $A_n(u_{n+1}) \neq 0$  for all  $n \geq 1$ . Thus, we can define the function

$$f_{1,0}^*(z_1, \dots, z_m) := \delta_{1,0} A_0(z_1, \dots, z_m) = \delta_{1,0} z_1 \cdots z_m$$

such that  $\Theta_1 + f_{1,0}^*(u_1) \in E_{u_1}$  and  $0 < |\delta_{1,0}| < s_m/m$ , where

$$\Theta_j := a_0 + \sum_{S \in \mathcal{Q}_m} \left( \prod_{i \in S} \alpha_i^{(j)} \right) f_S(u_{j,S}),$$

and  $u_{j,S} = (\alpha_{i_1}^{(j)}, \dots, \alpha_{i_k}^{(j)})$  for  $S = \{i_1, \dots, i_k\}$ , for all integers  $j \geq 1$ .

Since  $\mathbb{K}$  is a dense subset of  $\mathbb{C}$ , we can choose  $\delta_{1,1}$  such that the coefficient  $c_{1,1,\dots,1}$  of  $z_1 \cdots z_m$  in the function

$$f_{1,1}^*(z_1, \dots, z_m) := f_{1,0}^*(z_1, \dots, z_m) + \delta_{1,1} z_1 \cdots z_m A_1^{(1)}(z_1, \dots, z_m)$$

belongs to  $\mathbb{K}$  with  $|c_{1,1,\dots,1}| < s_m$ . Therefore, we take

$$f_1^*(z_1, \dots, z_m) := f_{1,1}^*(z_1, \dots, z_m),$$

where  $P_1(z_1, \dots, z_m) = c_{1,1,\dots,1} z_1 \cdots z_m$ .

Recursively, we can construct a function  $f_{n,0}^*(z_1, \dots, z_m)$  given by

$$f_{n,0}^*(z_1, \dots, z_m) := f_{n-1}^*(z_1, \dots, z_m) + \delta_{n,0} z_1^n z_2 \cdots z_m A_{n-1}(z_1, \dots, z_m)$$

where we take  $\delta_{n,0} \neq 0$  in the ball  $B(0, s_{n+m-1}/(n+m-1))$  such that

$$\Theta_n + f_{n,0}^*(u_n) \in E_{u_n}.$$

This is possible since  $E_{u_n}$  is a dense subset of  $\mathbb{C}$  and all coordinates of  $u_n$  are nonzero.

Since  $\mathbb{K}$  is a dense subset of  $\mathbb{C}$ , if we consider the ordering of the monomials of degree  $n+m-1$  given by the lexicographical order of the exponents, then we can choose  $\delta_{n,l}$  such that the coefficient  $c_{j_1, \dots, j_m}$  of the  $l$ th monomial  $z_1^{j_1} \cdots z_m^{j_m}$  in

$$f_{n,l}^*(z_1, \dots, z_m) := f_{n,l-1}^*(z_1, \dots, z_m) + \delta_{n,l} z_1^{j_1} \cdots z_m^{j_m} A_n(z_1, \dots, z_m)$$

belongs to  $\mathbb{K}$  with  $|c_{j_1, \dots, j_m}| < s_{n+m-1}$ . Thus, we define

$$f_n^*(z_1, \dots, z_m) := f_{n,L}^*(z_1, \dots, z_m),$$

where  $L = \binom{n+m-2}{m-1}$  is the number of distinct monomials of degree  $n+m-1$ . Then  $f_n^*(z_1, \dots, z_m)$  is a polynomial function such that  $c_{j_1, \dots, j_m} \in \mathbb{K}$  for every  $m$ -tuple  $(j_1, \dots, j_m)$  such that  $j_1 + \dots + j_m \leq n+m-1$ .

Finally, this construction implies that the functions  $f_n^*$  converge to a transcendental entire function  $f^* \in \mathbb{K}[[z_1, \dots, z_m]]$  as  $n \rightarrow \infty$  such that

$$f^*(u_j) = f_n^*(u_j) = f_j^*(u_j)$$

for all  $n \geq j \geq 1$ . Let  $f : \mathbb{C}^m \rightarrow \mathbb{C}$  be the entire function given by

$$f(z_1, \dots, z_m) = a_0 + \left( \sum_{S \in Q_m} \left( \prod_{i \in S} z_i \right) f_S(z_S) \right) + f^*(z_1, \dots, z_m).$$

Then  $f(u) \in E_u$  for all  $u \in X \subset \mathbb{C}^m$ . Since  $f$  is an entire function that is not a polynomial, it follows that  $f$  is transcendental. Note that there are uncountably many ways to choose the constants  $\delta_{n,j}$ . This completes the proof.

### Acknowledgments

The authors are grateful to the referee for their valuable suggestions about this paper. Part of this work was done during a visit by Diego Marques to University of Hradec Králové (Czech Republic) which provided excellent working conditions.

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