Relative Discrete Series Representations for Two Quotients of p-adic $\mathbf{GL}_n$

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Abstract. We provide an explicit construction of representations in the discrete spectrum of two p-adic symmetric spaces. We consider $\mathbf{GL}_n(F) \times \mathbf{GL}_n(F) \backslash \mathbf{GL}_{2n}(F)$ and $\mathbf{GL}_n(F) \backslash \mathbf{GL}_n(E)$, where E is a quadratic Galois extension of a nonarchimedean local field F of characteristic zero and odd residual characteristic. The proof of the main result involves an application of a symmetric space version of Casselman's Criterion for square integrability due to Kato and Takano.

1 Introduction

Let F be a nonarchimedean local field of characteristic zero and odd residual characteristic. Let $G = \mathbf{G}(F)$ be the F-points of a connected reductive group defined over F. Let θ be an F-involution (order two F-automorphism) of $\mathbf{G}$ and let $H = \mathbf{G}^\theta(F)$ be the group of θ-fixed points in G. The quotient $H \backslash G$ is a p-adic symmetric space. An irreducible representation of G that occurs in the discrete spectrum of $H \backslash G$ (an irreducible subrepresentation of $L^2(Z_G H \backslash G)$, where Z_G is the centre of G) is called a relative discrete series (RDS) representation. In this paper, we construct an infinite family of RDS representations for $H \backslash G$ that do not appear in the discrete spectrum of G. We consider the following two cases. (1) The linear case: $G = \mathbf{GL}_{2n}(F)$ and $H = \mathbf{GL}_n(F) \times \mathbf{GL}_n(F)$. (2) The Galois case: $G = \mathbf{GL}_n(E)$ and $H = \mathbf{GL}_n(F)$, where E/F is a quadratic extension. In a more general setting, Murnaghan has constructed relatively supercuspidal (Definition 4.4) representations that are not supercuspidal [32, 33]. Her construction is also via parabolic induction from representations of θ-elliptic Levi subgroups (Definition 3.6) and provided the initial motivation for this work. We obtain a special case of Murnaghan's results in our setting (Corollary 6.7); however, we apply completely different methods.

The study of harmonic analysis on $H \backslash G$ is of interest due to connections with the non-vanishing of global period integrals, functoriality, and poles of L-functions. See, for instance, the works [9,22,24]. For example, often $\mathbf{GL}_n(F) \times \mathbf{GL}_n(F)$-distinction of a representation of $\mathbf{GL}_{2n}(F)$ is equivalent to the existence of a nonzero Shalika model (Remark 7.4). In addition, $H \backslash G$ is a spherical variety and its study fits into the general framework of [39]. In a broad sense, the work of Sakellaridis–Venkatesh lays the

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formalism and foundations of a relative Langlands program [37]. The aim of such a program is to fully understand the link between global automorphic period integrals and local harmonic analysis. For certain p-adic spherical varieties X, Sakellaridis and Venkatesh give an explicit Plancherel formula describing $L^2(X)$ up to a description of the discrete spectrum [39, Theorem 6.2.1]. In addition, their results include a description of the discrete spectrum of p-adic symmetric spaces in terms of toric families of RDS. In fact, they showed that [39, Conjecture 9.4.6] is true for strongly factorizable spherical varieties. The decomposition of the norm on the discrete spectrum $L^2_{disc}(X)$ provided by this result is not necessarily a direct integral; the images of certain intertwining operators packaged with the toric families of RDS may be non-orthogonal. On the other hand, Sakellaridis and Venkatesh believe that their conjectures on **X**-distinguished Arthur parameters may give a canonical choice of mutually orthogonal toric families of RDS that span $L^2_{disc}(X)$ [39, Conjectures 1.3.1, 16.2.2]. However, they have not given an explicit description of the RDS required to build the toric families. For this reason, an explicit construction of RDS for p-adic symmetric spaces is a step towards completing the picture of the discrete spectrum of $H\backslash G$. Kato and Takano showed that any H-distinguished discrete series representation of G is an RDS [26]. Thus, we are interested in constructing RDS representations of G that are in the complement of the discrete spectrum of G.

We state our main theorem below, after giving the necessary definitions. A θ-stable Levi subgroup L of G is θ-elliptic if it is not contained in any proper θ-split parabolic subgroup, where a parabolic subgroup P is θ-split if $\theta(P)$ is opposite to P. An element $g \in G$ is said to be θ-split if $\theta(g) = g^{-1}$. An F-torus S is (θ, F)-split if it is F-split and every element $s \in S$ is θ-split. A representation τ of a Levi subgroup L of G is regular if for every non-trivial element $w \in N_G(L)/L$ we have that the twist ${}^w\tau = \tau(w^{-1}(\cdot)w)$ is not equivalent to τ.

Let $n \geq 2$ (respectively, $n \geq 4$) and let G be equal to $\mathbf{GL}_{2n}(F)$ (respectively, $\mathbf{GL}_n(E)$) and let H be equal to $\mathbf{GL}_n(F) \times \mathbf{GL}_n(F)$ (respectively, $\mathbf{GL}_n(F)$). Let A_0 be a θ-stable maximal F-split torus of G containing a fixed maximal (θ, F)-split torus S_0. Let $L_0 = C_G((A_0^\theta)^\circ)$ be a minimal θ-elliptic Levi subgroup of G containing A_0. We make a particular choice Δ^{ell} of simple roots for the root system $\Phi(G, A_0)$ (Section 5.2). The F-split component of the centre of L_0 is determined by a proper nonempty subset Δ^{ell}_{min} of Δ^{ell}. A Levi subgroup L is a standard-θ-elliptic Levi subgroup if L is standard with respect to Δ^{ell} and contains L_0. The main result of this paper is the following.

Theorem (Theorem 6.3) *Let $\Omega^{ell} \subset \Delta^{ell}$ be a proper subset such that Ω^{ell} contains Δ^{ell}_{min}. Let $Q = Q_{\Omega^{ell}}$ be the proper Δ^{ell}-standard parabolic subgroup associated with the subset Ω^{ell}. The parabolic subgroup Q is θ-stable and has θ-stable standard θ-elliptic Levi subgroup $L = L_{\Omega^{ell}}$. Let τ be a regular L^θ-distinguished discrete series representation of L. The parabolically induced representation $\pi = \iota_Q^G \tau$ is an irreducible H-distinguished relative discrete series representation of G. Moreover, π is in the complement of the discrete series of G.*

Remark 1.1 The representations constructed in Theorem 6.3 are induced from discrete series, and are therefore tempered (generic) representations of G. In particular,

one observes that $L^2(Z_G H \backslash G)$ contains the H-distinguished discrete series representations of G [26], as well as certain tempered representations that do not appear in $L^2(Z_G \backslash G)$.

Outside of two low-rank examples considered by Kato and Takano [26, §5.1–5.2], Theorem 6.3 provides the first construction of a family of non-discrete relative discrete series representations. In Corollary 7.17, we show that there are infinitely many equivalence classes of representations of the form constructed in Theorem 6.3. However, we do not prove that our construction exhausts the discrete spectrum in these two cases (Remark 6.6). It appears that the major obstruction to showing that an irreducible H-distinguished representation of G is not relatively discrete is establishing non-vanishing of the invariant forms $r_P \lambda$ defined by Kato and Takano [25] and by Lagier [28] (Section 4.5).

The author carried out a similar, more restricted, construction for the case $G = $ **GL**$_{2n}(E)$ and $H = $ **U**$_{E/F}$, a quasi-split unitary group [41, Theorem 5.2.22]. It is a work in progress to extend the construction to arbitrary symmetric quotients of the general linear group. Some modification will be required for this generalization; the representations constructed in Theorem 6.3 are generic and no such representation can be distinguished by the symplectic group [20]. It is expected that the Speh representations form the discrete spectrum of **Sp**$_{2n}(F) \backslash $**GL**$_{2n}(F)$ [34, 39].

We now give an outline of the content of the paper. In Section 2, we establish notation and our conventions in the linear and Galois cases. In Section 3 we review basic results on tori and parabolic subgroups relevant to the study of harmonic analysis on $H \backslash G$. Here we introduce the notion of a θ-elliptic Levi subgroup. Section 4 contains a review of Kato and Takano's generalization of Casselman's Criterion, preliminaries on distinguished representations, and some results on the exponents of induced representations. The most important results in this section are Proposition 4.22, Lemma 4.16, and Proposition 4.23. In Section 5 we give explicit descriptions of the tori, parabolic subgroups, and simple roots needed for our work in the linear and Galois cases (Propositions 5.4 and 5.12). The main result, Theorem 6.3, is stated and proved in Section 6; however, several preliminary results required for the proof are deferred until Section 8. In Section 7 we briefly survey the literature on distinguished discrete series representations in the linear and Galois cases. In addition, we establish the existence of infinite families of inducing representations in Lemma 7.15 from which we can deduce Corollary 7.17. Finally, in Section 8 we assemble the technical results on the exponents and distinction of Jacquet modules required to prove Theorem 6.3. The main results of the final section are Propositions 8.5 and 8.7.

2 Notation and Conventions

Let F be a nonarchimedean local field of characteristic zero and odd residual characteristic. Let $\mathcal{O}_F$ be the ring of integers of F with prime ideal $\mathfrak{p}_F$. Let E be a quadratic Galois extension of F. Fix a generator ε of the extension E/F such that $E = F(\varepsilon)$. Let $\sigma \in \mathrm{Gal}(E/F)$ be a generator of the Galois group of E over F.

Let **G** be a connected reductive group defined over F and let $G = $ **G**(F) denote the group of F-points. Let e be the identity element of G. We let Z_G denote the centre of

G while A_G denotes the F-split component of the centre of G. As is the custom, we will often abuse notation and identify an algebraic group defined over F with its group of F-points. When the distinction is to be made, we will use boldface to denote the algebraic group and regular typeface to denote the group of F-points. For any F-torus $\mathbf{A}$ of $\mathbf{G}$, we let A^1 denote the group of $\mathcal{O}_F$-points $A^1 = \mathbf{A}(\mathcal{O}_F)$.

Let P be a parabolic subgroup of G and let N be the unipotent radical of P. The modular character of P is given by $\delta_P(p) = |\det \mathrm{Ad}_{\mathfrak{n}}(p)|$, for all $p \in P$, where $\mathrm{Ad}_{\mathfrak{n}}$ denotes the adjoint action of P on the Lie algebra $\mathfrak{n}$ of N [7].

Let θ be an F-involution of $\mathbf{G}$, that is, an order-two automorphism of $\mathbf{G}$ defined over F. Define $\mathbf{H} = \mathbf{G}^\theta$ to be the closed subgroup of θ-fixed points of $\mathbf{G}$. The quotient $H\backslash G$ is a p-adic symmetric space.

Definition 2.1   We say that an involution θ_1 of G is G-conjugate (or G-equivalent) to another involution θ_2 if there exists $g \in G$ such that $\theta_1 = \mathrm{Int}\, g^{-1} \circ \theta_2 \circ \mathrm{Int}\, g$, where $\mathrm{Int}\, g$ denotes the inner F-automorphism of $\mathbf{G}$ given by $\mathrm{Int}\, g(x) = gxg^{-1}$, for all $x \in \mathbf{G}$. We write $g \cdot \theta$ to denote the involution $\mathrm{Int}\, g^{-1} \circ \theta \circ \mathrm{Int}\, g$.

Let $\mathbf{GL}_n$ denote the general linear group of $n \times n$ invertible matrices. As is customary, we denote the block-upper-triangular parabolic subgroup of $\mathbf{GL}_n$, corresponding to a partition $(\underline{m}) = (m_1, \ldots, m_k)$ of n, by $\mathbf{P}_{(\underline{m})}$, with block-diagonal Levi subgroup $\mathbf{M}_{(\underline{m})} \cong \prod_{i=1}^k \mathbf{GL}_{m_i}$ and unipotent radical $\mathbf{N}_{(\underline{m})}$. We use $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ to denote an $n \times n$ diagonal matrix with entries $a_1, \ldots, a_n$.

For any $g, x \in G$, we write $^g x = gxg^{-1}$. For any subset Y of G, we write $^g Y = \{^g y : y \in Y\}$. Let $C_G(Y)$ denote the centralizer of Y in G and let $N_G(Y)$ be the normalizer of Y in G. Given a real number r, we let $\lfloor r \rfloor$ denote the greatest integer that is less than or equal to r. We use $\widehat{(\cdot)}$ to denote that a symbol is omitted. For instance, $\mathrm{diag}(\widehat{a_1}, a_2, \ldots, a_n)$ may be used to denote the diagonal matrix $\mathrm{diag}(a_2, \ldots, a_n)$.

2.1 The Linear Case

In the linear case, we set $G = \mathbf{GL}_n(F)$, where $n \geq 4$ is an even integer. Let θ denote the inner involution of G given by conjugation by the matrix

$$w_\ell = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix},$$

that is, for any $g \in \mathbf{G}$, we have $\theta(g) = \mathrm{Int}\, w_\ell(g) = w_\ell g w_\ell^{-1}$. The element w_ℓ is diagonalizable over F; in particular, there exists $x_\ell \in \mathbf{GL}_n(F)$ such that

$$x_\ell w_\ell x_\ell^{-1} = \mathrm{diag}(1_{n/2}, -1_{n/2}),$$

where $1_{n/2}$ denotes the $n/2 \times n/2$ identity matrix. It follows that $H = x_\ell^{-1} M_{(n/2, n/2)} x_\ell$, where $M_{(n/2,n/2)}$ is the standard Levi subgroup of G of type $(n/2, n/2)$. Thus, in the linear case, $H = \mathbf{G}^\theta(F)$ is isomorphic to $\mathbf{GL}_{n/2}(F) \times \mathbf{GL}_{n/2}(F)$.

2.2 The Galois Case

In the Galois case, for $n \geq 4$, we let $\mathbf{G} = R_{E/F}\mathbf{GL}_n$ be the restriction of scalars of $\mathbf{GL}_n$ with respect to E/F. We identify the group G of F-points with $\mathbf{GL}_n(E)$. The non-trivial element σ of the Galois group of E over F gives rise to an F-involution θ of G given by coordinate-wise Galois conjugation $\theta((a_{ij})) = (\sigma(a_{ij}))$, where $(a_{ij}) \in G$. In the Galois case, we have that $H = \mathbf{G}^{\theta}(F)$ is equal to $\mathbf{GL}_n(F)$.

2.3 Choices of Particular Group Elements and Supplementary Involutions

For a positive integer r, we will write $\mathbf{G}_r$ for $\mathbf{GL}_r$ with F-points G_r in the linear case, and similarly for $R_{E/F}\mathbf{GL}_r$ with F-points $G_r \simeq \mathbf{GL}_r(E)$ in the Galois case. Write J_r for the $r \times r$-matrix in G_r with unit anti-diagonal

$$J_r = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}.$$

Note that $w_\ell = J_n$. In the linear case, θ_r will denote the inner involution $\mathrm{Int}\, J_r$ of $\mathbf{G}_r$ with fixed points $\mathbf{H}_r$. In the Galois case, we let θ_r denote the F-involution of $\mathbf{G}_r$ given by coordinate-wise Galois conjugation. Then $H_r = \mathbf{GL}_r(F)$ is the group of F-points of the θ_r-fixed subgroup of $\mathbf{G}_r$. In the Galois case, for any positive integer r, there exists $\gamma_r \in G_r$ such that $\gamma_r^{-1}\theta_r(\gamma_r) = J_r \in H_r$. For instance, if r is even, then we may take

$$\gamma_r = \begin{pmatrix} 1 & & & & & & 1 \\ & \ddots & & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & 1 & & & \\ & & -\varepsilon & \varepsilon & & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & & \\ -\varepsilon & & & & & \varepsilon \end{pmatrix},$$

where $E = F(\varepsilon)$, and if r is odd, then we set

$$\gamma_r = \begin{pmatrix} 1 & & & & & & 1 \\ & \ddots & & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & 0 & 1 & & \\ & & 0 & 1 & 0 & & \\ & & -\varepsilon & 0 & \varepsilon & & \\ & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\ -\varepsilon & & & & & & \varepsilon \end{pmatrix}.$$

Define $\gamma = \gamma_n \in G$ and note that $w_\ell = J_n = \gamma^{-1}\theta(\gamma)$ is an order-two element of H.

In the Galois case, we define a second involution ϑ of $\mathbf{G}$ that is G-conjugate to θ by declaring that $\vartheta = \gamma \cdot \theta$ (Definition 2.1). Explicitly,

$$(2.1) \qquad\qquad \vartheta(g) = \gamma^{-1}\theta(\gamma g \gamma^{-1})\gamma,$$

for any $g \in \mathbf{G}$. Since $w_\ell = \gamma^{-1}\theta(\gamma)$ is θ-fixed, we have that

$$(2.2) \qquad\qquad \vartheta = \operatorname{Int} w_\ell \circ \theta = \theta \circ \operatorname{Int} w_\ell.$$

Similarly, for any positive integer r, we define

$$(2.3) \qquad\qquad \vartheta_r = \gamma_r \cdot \theta_r = \operatorname{Int} J_r \circ \theta_r = \theta_r \circ \operatorname{Int} J_r.$$

In both cases, define $w_+ \in \mathbf{GL}_n(F) \subset \mathbf{GL}_n(E)$ to be the permutation matrix corresponding to the permutation of $\{1, \dots, n\}$ given by

$$\begin{aligned}
2i - 1 &\mapsto i && \text{for } 1 \le i \le \lfloor n/2 \rfloor + 1, \\
2i &\mapsto n + 1 - i && \text{for } 1 \le i \le \lfloor n/2 \rfloor,
\end{aligned}$$

when n is odd, and by

$$\begin{aligned}
2i - 1 &\mapsto i && \text{for } 1 \le i \le n/2, \\
2i &\mapsto n + 1 - i && \text{for } 1 \le i \le n/2
\end{aligned}$$

when n is even. Remember that in the linear case we will always assume that n is even. Finally, define

$$(2.4) \qquad w_0 = \begin{cases} w_+ & \text{in the linear case: } G = \mathbf{GL}_n(F),\ n \ge 4 \text{ even}, \\ {}^{\gamma}w_+ = \gamma w_+ \gamma^{-1} & \text{in the Galois case: } G = \mathbf{GL}_n(E),\ \text{any } n \ge 4. \end{cases}$$

3 Symmetric Spaces and Associated Parabolic Subgroups

For now, we work in general and let G be an arbitrary connected reductive group over F, with θ and H as in Section 2. An element $g \in G$ is said to be θ-split if $\theta(g) = g^{-1}$. A subtorus S of G is θ-split if every element of S is θ-split.

3.1 Tori and Root Systems Relative to Involutions

An F-torus S contained in G is (θ, F)-split if S is both F-split and θ-split. Let S_0 be a maximal (θ, F)-split torus of G. By [18, Lemma 4.5(iii)], there exists a θ-stable maximal F-split torus A_0 of G that contains S_0. Let $\Phi_0 = \Phi(G, A_0)$ be the root system of G with respect to A_0. Let W_0 be the Weyl group of G with respect to A_0. Since A_0 is θ-stable, there is an action of θ on the F-rational characters $X^*(A_0)$ of A_0. Explicitly, given $\chi \in X^*(A_0)$, we have $(\theta\chi)(a) = \chi(\theta(a))$, for all $a \in A_0$. Moreover, Φ_0 is stable under the action of θ on $X^*(A_0)$. Let Φ_0^θ denote the subset of θ-fixed roots in Φ_0.

Definition 3.1 A base Δ_0 of Φ_0 is called a θ-base if, for every positive root $\alpha \in \Phi_0^+$ with respect to Δ_0 that is not fixed by θ, we have that $\theta(\alpha) \in \Phi_0^-$.

As shown in [15], a θ-base of Φ_0 exists. Let Δ_0 be a θ-base of Φ_0. Let $p: X^*(A_0) \to X^*(S_0)$ be the surjective homomorphism defined by restricting the F-rational characters on A_0 to the subtorus S_0. The kernel of the map p is the submodule $X^*(A_0)^\theta$ of $X^*(A_0)$ consisting of θ-fixed F-rational characters. The restricted root system of $H\backslash G$ (relative to our choice of (A_0, S_0, Δ_0)) is defined to be

$$\overline{\Phi}_0 = p(\Phi_0) \smallsetminus \{0\} = p(\Phi_0 \smallsetminus \Phi_0^\theta).$$

The set $\overline{\Phi}_0$ coincides with the set $\Phi(G, S_0)$ of roots with respect to S_0 and this is a (not necessarily reduced) root system by [18, Proposition 5.9]. The set

$$\overline{\Delta}_0 = p(\Delta_0) \smallsetminus \{0\} = p(\Delta_0 \smallsetminus \Delta_0^\theta)$$

is a base for the restricted root system $\overline{\Phi}_0$. Indeed, the linear independence of $\overline{\Delta}_0$ follows from the fact that Δ_0 is a θ-base and that $\ker p = X^*(A_0)^\theta$. Given a subset $\overline{\Theta} \subset \overline{\Delta}_0$, define the subset $[\overline{\Theta}] = p^{-1}(\overline{\Theta}) \cup \Delta_0^\theta$ of Δ_0. Subsets of Δ_0 of the form $[\overline{\Theta}]$, for $\overline{\Theta} \subset \overline{\Delta}_0$, are said to be θ-split. The maximal θ-split subsets of Δ_0 are of the form $[\overline{\Delta}_0 \smallsetminus \{\bar{\alpha}\}]$, where $\bar{\alpha} \in \overline{\Delta}_0$.

3.2 Parabolic Subgroups Relative to Involutions

Given a subset Θ of Δ_0, one may canonically associate a Δ_0-standard parabolic subgroup P_Θ of G and a standard choice of Levi subgroup. Let Φ_Θ be the subsystem of Φ_0 generated by the simple roots Θ. Let Φ_Θ^+ be the set of positive roots determined by Θ. The unipotent radical N_Θ of P_Θ is generated by the root groups N_α, where $\alpha \in \Phi_0^+ \smallsetminus \Phi_\Theta^+$. The parabolic subgroup P_Θ admits a Levi factorization $P_\Theta = M_\Theta N_\Theta$, where M_Θ is the centralizer in G of the F-split torus $A_\Theta = (\bigcap_{\alpha \in \Theta} \ker \alpha)^\circ$. Here $(\cdot)^\circ$ indicates the Zariski-connected component of the identity. In fact, A_Θ is the F-split component of the centre of M_Θ, and Φ_Θ is the root system $\Phi(M_\Theta, A_0)$ of A_0 in M_Θ.

Remark 3.2 When considering standard parabolic subgroups P_Θ associated with $\Theta \subset \Delta_0$, we will always work with the Levi factorization $P_\Theta = M_\Theta N_\Theta$, where $M_\Theta = C_G(A_\Theta)$ is the standard Levi subgroup of P_Θ.

Let M be any Levi subgroup of G. The (θ, F)-split component of M is the largest (θ, F)-split torus S_M contained in the centre of M. In fact, we have that S_M is the connected component (of the identity) of the subgroup of θ-split elements in the F-split component A_M, that is,

$$(3.1) \qquad S_M = (\{x \in A_M : \theta(x) = x^{-1}\})^\circ.$$

A parabolic subgroup P of G is called θ-split if $\theta(P)$ is opposite to P. In this case, $M = P \cap \theta(P)$ is a θ-stable Levi subgroup of both P and $\theta(P) = P^{\mathrm{op}}$. Given a θ-split subset $\Theta \subset \Delta_0$, the Δ_0-standard parabolic subgroup $P_\Theta = M_\Theta N_\Theta$ is θ-split. Any Δ_0-standard θ-split parabolic subgroup arises from a θ-split subset of Δ_0 [25, Lemma 2.5(1)]. Let S_Θ denote the (θ, F)-split component of M_Θ. We have that

$$(3.2) \qquad S_\Theta = (\{s \in A_\Theta : \theta(s) = s^{-1}\})^\circ = (\bigcap_{\bar{\alpha} \in p(\Theta)} \ker(\bar{\alpha}: S_0 \to F^\times))^\circ.$$

For the second equality in (3.2), see [26, §1.5]. For any $0 < \epsilon \leq 1$, define

$$(3.3) \qquad S_\Theta^-(\epsilon) = \{s \in S_\Theta : |\alpha(s)|_F \leq \epsilon, \text{ for all } \alpha \in \Delta_0 \smallsetminus \Theta\}.$$

We write S_Θ^- for $S_\Theta^-(1)$ and refer to S_Θ^- as the dominant part of S_Θ.

By [17, Theorem 2.9], the subset Δ_0^θ of θ-fixed roots in Δ_0 determines the Δ_0-standard minimal θ-split parabolic subgroup $P_0 = P_{\Delta_0^\theta}$. By [18, Proposition 4.7(iv)], the minimal θ-split parabolic subgroup P_0 has standard θ-stable Levi $M_0 = C_G(S_0)$. Let $P_0 = M_0 N_0$ be the standard Levi factorization of P_0.

Lemma 3.3   *Let P be a θ-split parabolic subgroup with θ-stable Levi $M = P \cap \theta(P)$. The Levi subgroup M is equal to the centralizer in G of its (θ, F)-split component S_M.*

Proof   The lemma follows immediately from [18, Lemma 4.6].   ∎

Lemma 3.4   *If M is the centralizer in G of a non-central (θ, F)-split torus S, then M is the Levi subgroup of a proper θ-split parabolic subgroup of G.*

Proof   Let S_0 be a maximal (θ, F)-split torus of G containing S and A_0 a θ-stable maximal F-split torus of G containing S_0. Let $P_0 = M_0 N_0$ be a minimal θ-split parabolic subgroup containing S_0. The subgroup $M = C_G(S)$ is a θ-stable Levi subgroup of G since S is a θ-stable F-split torus. Since S is not central in G, M is a proper Levi subgroup. Moreover, since S is contained in S_0, we have that M_0 is contained in M. Let $P = MN_0$. Note that P is a closed subgroup containing P_0; therefore, P is a proper parabolic subgroup of G with Levi subgroup M. It remains to show that P is θ-split. Since P_0 is θ-split, we have that $\theta(N_0) = N_0^{op}$ is the opposite unipotent radical of N_0. Since M is θ-stable, it follows that $\theta(P) = MN_0^{op}$ and this is the parabolic opposite to P.   ∎

The minimal θ-split parabolic subgroups of G are not always H-conjugate [18, Example 4.12]. On the other hand, the following result holds.

Lemma 3.5 ([25, Lemma 2.5])   *Let $S_0 \subset A_0$, Δ_0, and $P_0 = M_0 N_0$ be as above.*
(i)   *Any θ-split parabolic subgroup P of G is conjugate to a Δ_0-standard θ-split parabolic subgroup by an element $g \in (\mathbf{HM}_0)(F)$.*
(ii)   *If the group of F-points of the product $(\mathbf{HM}_0)(F)$ is equal to HM_0, then any θ-split parabolic subgroup of G is H-conjugate to a Δ_0-standard θ-split parabolic subgroup.*

Let $P = MN$ be a θ-split parabolic subgroup and choose $g \in (\mathbf{HM}_0)(F)$ such that $P = gP_\Theta g^{-1}$ for some Δ_0-standard θ-split parabolic subgroup P_Θ. Since $g \in (\mathbf{HM}_0)(F)$, we have that $g^{-1}\theta(g) \in \mathbf{M}_0(F)$. Thus, we may take $S_M = gS_\Theta g^{-1}$. For a given $\epsilon > 0$, one may extend the definition (3.3) of S_Θ^- to the non-Δ_0-standard torus S_M. Indeed, we may set $S_M^-(\epsilon) = gS_\Theta^-(\epsilon)g^{-1}$ and we define $S_M^- = S_M^-(1)$ with $S_M^1 = S_M(\mathcal{O}_F)$, as above.

We have the following definition, following the terminology of Murnaghan [33], in analogy with the notion of an elliptic Levi subgroup.

Definition 3.6   A θ-stable Levi subgroup L of G is θ-elliptic if and only if L is not contained in any proper θ-split parabolic subgroup of G.

We note the following simple lemma, which follows immediately from Definition 3.6.

Lemma 3.7   *If a θ-stable Levi subgroup L of G contains a θ-elliptic Levi subgroup, then L is θ-elliptic.*

The following characterization of the θ-elliptic property is also useful.

Lemma 3.8 *A θ-stable Levi subgroup L is θ-elliptic if and only if $S_L = S_G$.*

Proof If $L = G$, then there is nothing to do. Without loss of generality, L is a proper subgroup of G. Suppose that L is θ-elliptic. We have that A_G is contained in A_L and it follows that S_G is contained in S_L. If S_L properly contains S_G, then L is contained in the Levi subgroup $M = C_G(S_L)$. By Lemma 3.4, M is a Levi subgroup of a proper θ-split parabolic subgroup. It follows that $L \subset M$ is contained in a proper θ-split parabolic subgroup. This contradicts the fact that L is θ-elliptic, so we must have that $S_L = S_G$.

On the other hand, suppose that S_L is equal to S_G. Argue by contradiction and suppose that L is contained in a proper θ-split parabolic $P = MN$ with θ-stable Levi subgroup $M = P \cap \theta(P)$. We have that $L = \theta(L)$ is contained in M and $S_M \subset S_L$. By Lemma 3.3, M is the centralizer of its (θ, F)-split component S_M. Since M is a proper Levi subgroup of G, we have that S_M properly contains S_G. However, by assumption $S_L = S_G$, which implies that $M = G$, and this is impossible. We conclude that L must be θ-elliptic. ∎

The next proposition first appeared in [33].

Proposition 3.9 *Let Q be a parabolic subgroup of G. If Q admits a θ-elliptic Levi factor L, then Q is θ-stable.*

Proof The subgroup L is θ-stable by definition. For any root α of A_L in G, one can show that $\theta\alpha = \alpha$. It follows that the unipotent radical of Q, hence Q, must be θ-stable. ∎

4 Distinguished Representations and the Relative Casselman Criterion

4.1 Distinguished Representations

All of the representations that we consider are on complex vector spaces. A representation (π, V) of G is smooth if for every $v \in V$ the stabilizer of v in G is an open subgroup. A smooth representation (π, V) of G is admissible if, for every compact open subgroup K of G, the subspace of K-invariant vectors V^K is finite-dimensional. A smooth one-dimensional representation of G is a quasi-character of G. A character of G is a unitary quasi-character. Let (π, V) be a smooth representation of G. If ω is a quasi-character of Z_G, then (π, V) is an ω-representation if π has central character ω.

Let χ be a quasi-character of H. We also let π denote its restriction to H.

Definition 4.1 If the space $\mathrm{Hom}_H(\pi, \chi)$ is nonzero, then the representation π is called *(H, χ)-distinguished*. If χ is the trivial character of H, then we refer to $(H, 1)$-distinguished representations simply as *H-distinguished*.

Of course, in Definition 4.1, the subgroup $H = G^\theta$ may be replaced by any closed subgroup of G; however, we are only concerned with the symmetric subgroup setting. As a first observation, we record the following lemma.

Lemma 4.2   *Let (π, V) be a finitely generated admissible representation of G. If (π, V) is H-distinguished, then there exists an (irreducible) H-distinguished sub-quotient of (π, V).*

The next lemma shows that distinction, relative to an involution θ, depends only on the equivalence class of θ under the right action of G on the set of involutions (Definition 2.1).

Lemma 4.3   *The subgroup $G^{g \cdot \theta}$ of $g \cdot \theta$-fixed points in G is G-conjugate to G^θ. Precisely, we have $G^{g \cdot \theta} = g^{-1}(G^\theta)g$. Moreover, a smooth representation (π, V) of G is G^θ-distinguished if and only if π is $G^{g \cdot \theta}$-distinguished.*

Proof   Let $h \in G^\theta$. Then we have that

$$g \cdot \theta(g^{-1}hg) = g^{-1}\theta(gg^{-1}hgg^{-1})g = g^{-1}\theta(h)g = g^{-1}hg,$$

so $g^{-1}hg$ is $g \cdot \theta$-fixed. It follows that $g^{-1}(G^\theta)g \subset G^{g \cdot \theta}$. Since conjugation by g is an automorphism of G, it follows that $G^{g \cdot \theta} = g^{-1}(G^\theta)g$.

Let λ be a nonzero element of $\mathrm{Hom}_{G^\theta}(\pi, 1)$. Define $\lambda' = \lambda \circ \pi(g)$ and observe that λ' is a nonzero $G^{g \cdot \theta}$-invariant linear functional on V. It follows that the map $\lambda \mapsto \lambda \circ \pi(g)$ is a bijection from $\mathrm{Hom}_{G^\theta}(\pi, 1)$ to $\mathrm{Hom}_{G^{g \cdot \theta}}(\pi, 1)$ with inverse $\lambda' \mapsto \lambda' \circ \pi(g^{-1})$. In particular, π is G^θ-distinguished if and only if π is $G^{g \cdot \theta}$-distinguished.   ∎

4.2   Relative Matrix Coefficients

Let (π, V) be a smooth H-distinguished representation of G. Let $\lambda \in \mathrm{Hom}_H(\pi, 1)$ be a nonzero H-invariant linear form on V and let v be a nonzero vector in V. In analogy with the usual matrix coefficients, define a complex-valued function $\varphi_{\lambda, v}$ on G by $\varphi_{\lambda, v}(g) = \langle \lambda, \pi(g)v \rangle$. We refer to the functions $\varphi_{\lambda, v}$ as relative matrix coefficients (with respect to λ) or as λ-relative matrix coefficients. Since π is a smooth representation, the relative matrix coefficient $\varphi_{\lambda, v}$ lies in $C^\infty(G)$, for every $v \in V$. In addition, since λ is H-invariant, the functions $\varphi_{\lambda, v}$ descend to well-defined functions on the quotient $H\backslash G$. In analogy with the classical case, one makes the following definitions.

Definition 4.4   The representation (π, V) is said to be

(i) (H, λ)-*relatively supercuspidal*, or *relatively supercuspidal with respect to λ*, if and only if all of the λ-relative matrix coefficients are compactly supported modulo $Z_G H$;

(ii) H-*relatively supercuspidal* if and only if π is (H, λ)-relatively supercuspidal for every $\lambda \in \mathrm{Hom}_H(\pi, 1)$.

Let ω be a unitary character of Z_G and further suppose that π is an ω-representation.

Definition 4.5   The representation (π, V) is said to be

(i) (H, λ)-*relatively square integrable*, or *relatively square integrable with respect to* λ, if and only if all of the λ-relative matrix coefficients are square integrable modulo $Z_G H$;

(ii) H-*relatively square integrable* if and only if π is (H, λ)-relatively square integrable for every $\lambda \in \mathrm{Hom}_H(\pi, 1)$.

Notice that we must also take the quotient of G by the (noncompact) centre Z_G in order to make sense of compactly supported (respectively, square integrable) functions on $H\backslash G$. Moreover, to integrate relative matrix coefficients over $Z_G H\backslash G$ we need a G-invariant measure on the quotient $Z_G H\backslash G$. The centre Z_G of G is unimodular since it is abelian. The fixed point subgroup H is also reductive [8, Theorem 1.8] and thus unimodular. It follows that there exists a G-invariant measure on the quotient $Z_G H\backslash G$ by [38, Proposition 12.8].

Note   When H is understood, we refer to H-relatively supercuspidal (respectively, H-relatively square integrable) representations simply as relatively supercuspidal (respectively, relatively square integrable).

Definition 4.6   If (π, V) is an irreducible subrepresentation of $L^2(Z_G H\backslash G)$, then we say that (π, V) occurs in the discrete spectrum of $H\backslash G$. In this case, we say that (π, V) is a relative discrete series (RDS) representation.

4.3   Parabolic Induction and Jacquet Restriction

Let P be a parabolic subgroup of G with Levi subgroup M and unipotent radical N. Given a smooth representation (ρ, V_ρ) of M we may inflate ρ to a representation of P, also denoted ρ, by declaring that N acts trivially. We define the representation $\iota_P^G \rho$ of G to be the induced representation $\mathrm{Ind}_P^G(\delta_P^{1/2} \otimes \rho)$. We refer to the functor $\rho \mapsto \iota_P^G \rho$ as (normalized) parabolic induction. When it is more convenient (see Observation 6.4 and Sections 7–8), we also use the Bernstein–Zelevinsky notation for parabolic induction on general linear groups [4, 42].

Let (π, V) be a smooth representation of G. Let (π_N, V_N) denote the Jacquet module of π along P, normalized by $\delta_P^{-1/2}$. Explicitly, if

$$V(N) = \mathrm{span}\{\pi(n)v - v : n \in N, v \in V\},$$

then $V_N = V/V(N)$ and $\pi_N(p)(v + V(N)) = \delta_P^{-1/2}(p)\pi(p)v + V(N)$, for all $p \in P$, $v + V(N) \in V_N$. Since δ_P is trivial on N, we see that (π_N, V_N) is a representation of P on which N acts trivially. We will regard (π_N, V_N) as a representation of the Levi factor $M \cong P/N$ of P.

We will now give a statement of the Geometric Lemma [4, Lemma 2.12], which is a fundamental tool in our work and the study of induced representations in general. First, we recall two results on double-coset representatives.

Lemma 4.7 ([7, Proposition 1.3.1])   *Let* Θ *and* Ω *be subsets of* Δ_0. *The set*

$$[W_\Theta \backslash W_0 / W_\Omega] = \{w \in W_0 : w\Omega, w^{-1}\Theta \subset \Phi^+\}$$

provides a choice of Weyl group representatives for the double-coset space $P_\Theta \backslash G / P_\Omega$.

Proposition 4.8 ([7, Proposition 1.3.3])   *Let $\Theta, \Omega \subset \Delta_0$ and let $w \in [W_\Theta \backslash W_0 / W_\Omega]$.*

(i)   *The standard parabolic subgroup $P_{\Theta \cap w\Omega}$ is equal to $(P_\Theta \cap {}^w P_\Omega) {}^w N_\Omega$.*

(ii)   *The unipotent radical of $P_{\Theta \cap w\Omega}$ is generated by ${}^w N_\Omega$ and $N_\Theta \cap {}^w N_\varnothing$, where $N_\varnothing$ is the unipotent radical of the minimal parabolic subgroup corresponding to $\varnothing \subset \Delta_0$.*

(iii)   *The standard Levi subgroup of $P_{\Theta \cap w\Omega}$ is $M_{\Theta \cap w\Omega} = M_\Theta \cap w M_\Omega w^{-1}$.*

(iv)   *The subgroup $P_\Theta \cap {}^w M_\Omega$ is a $w\Omega$-standard parabolic in $M_{w\Omega} = {}^w M_\Omega$ with unipotent radical $N_\Theta \cap {}^w M_\Omega$ and standard Levi subgroup $M_{\Theta \cap w\Omega} = M_\Theta \cap {}^w M_\Omega$.*

When applying the Geometric Lemma along two standard parabolic subgroups P_Θ and P_Ω, associated with $\Theta, \Omega \subset \Delta_0$, we will always use the choice of "nice" representatives $[W_\Theta \backslash W_0 / W_\Omega]$ for the double-coset space $P_\Theta \backslash G / P_\Omega$.

Lemma 4.9 (The Geometric Lemma)   *Let P_Ω and P_Θ be two Δ_0-standard parabolic subgroups of G. Let ρ be a smooth representation of M_Ω. There is a filtration of the space of the representation $(\iota_{P_\Omega}^G \rho)_{N_\Theta}$ such that the associated graded object is isomorphic to the direct sum*

$$\bigoplus_{w \in [W_\Theta \backslash W_0 / W_\Omega]} \iota_{M_\Theta \cap {}^w P_\Omega}^{M_\Theta} (({}^w \rho)_{N_\Theta \cap {}^w M_\Omega}).$$

We write $\mathcal{F}_\Theta^w(\rho) = \mathcal{F}_{N_\Theta}^w(\rho)$ to denote the representation $\iota_{M_\Theta \cap {}^w P_\Omega}^{M_\Theta} (({}^w \rho)_{N_\Theta \cap {}^w M_\Omega})$ of M_Θ.

4.4   Distinction of Induced Representations

Lemma 4.10 is well known and follows from an explicit version of Frobenius Reciprocity due to Bernstein and Zelevinsky [3, Proposition 2.29]. Let $Q = LU$ be a θ-stable parabolic subgroup with θ-stable Levi factor L and unipotent radical U. Note that the identity component of $Q^\theta = L^\theta U^\theta$ is a parabolic subgroup of H°, with the expected Levi decomposition [18], [13, Lemma 3.1]. Let μ be a positive quasi-invariant measure on the (compact) quotient $Q^\theta \backslash H$ [3, Theorem 1.21].

Lemma 4.10   *Let ρ be a smooth representation of L and let $\pi = \iota_Q^G \rho$. The map $\lambda \mapsto \lambda^G$ is an injection of $\mathrm{Hom}_{L^\theta}(\delta_Q^{1/2} \rho, \delta_{Q^\theta})$ into $\mathrm{Hom}_H(\pi, 1)$, where λ^G is given explicitly by*

$$\langle \lambda^G, \phi \rangle = \int_{Q^\theta \backslash H} \langle \lambda, \phi(h) \rangle \, d\mu(h)$$

for any function ϕ in the space of π.

Corollary 4.11   *If $\delta_Q^{1/2}$ restricted to L^θ is equal to δ_{Q^θ}, then the map $\lambda \mapsto \lambda^G$ is an injection of $\mathrm{Hom}_{L^\theta}(\rho, 1)$ into $\mathrm{Hom}_H(\pi, 1)$. In particular, if ρ is L^θ-distinguished, then π is H-distinguished.*

Proof   Observe that $\mathrm{Hom}_{L^\theta}(\delta_Q^{1/2} \rho, \delta_{Q^\theta}) = \mathrm{Hom}_{L^\theta}(\rho, \delta_Q^{-1/2}|_{L^\theta} \delta_{Q^\theta})$. ∎

Alternatively, the H-invariant linear form on $\pi = \iota_Q^G \rho$ may be understood to arise from the closed orbit in $Q \backslash G / H$ via the Mackey theory.

4.5 Invariant Linear Forms on Jacquet Modules

Let (π, V) be an admissible H-distinguished representation of G. Let λ be a nonzero element of $\operatorname{Hom}_H(\pi, 1)$. Let P be a θ-split parabolic subgroup of G with unipotent radical N and θ-stable Levi component $M = P \cap \theta(P)$. One may associate with λ a canonical M^θ-invariant linear form $r_P \lambda$ on the Jacquet module (π_N, V_N). The construction of $r_P \lambda$, via Casselman's Canonical Lifting [7, Proposition 4.1.4], was discovered independently by Kato–Takano and Lagier. We refer the reader to [25,28] for the details of the construction. We record the following result [25, Proposition 5.6].

Proposition 4.12   *Let (π, V) be an admissible H-distinguished representation of G. Let $\lambda \in \operatorname{Hom}_H(\pi, 1)$ be nonzero and let P be a θ-split parabolic subgroup of G with unipotent radical N and θ-stable Levi component $M = P \cap \theta(P)$.*

(i)   *The linear functional $r_P \lambda \colon V_N \to \mathbb{C}$ is M^θ-invariant.*
(ii)   *The mapping $r_P \colon \operatorname{Hom}_H(\pi, 1) \to \operatorname{Hom}_{M^\theta}(\pi_N, 1)$, sending λ to $r_P \lambda$, is linear.*

Kato and Takano used the invariant forms $r_P \lambda$ to provide the following characterization of relatively supercuspidal representations [25, Theorem 6.2].

Theorem 4.13   *Let (π, V) be an admissible H-distinguished representation of G and let λ be a nonzero H-invariant linear form on V. Then (π, V) is (H, λ)-relatively supercuspidal if and only if $r_P \lambda = 0$ for every proper θ-split parabolic subgroup P of G.*

4.6 Exponents and the Relative Casselman Criterion

Let (π, V) be a finitely generated admissible representation of G. Recall that A_G denotes the F-split component of the centre of G. Let χ be a quasi-character of A_G. For $n \in \mathbb{N}$, $n \geq 1$, define the subspace

$$V_{\chi,n} = \{ v \in V : (\pi(z) - \chi(z))^n v = 0 \text{ for all } z \in A_G \},$$

and set $V_{\chi,\infty} = \bigcup_{n=1}^\infty V_{\chi,n}$. Each $V_{\chi,n}$ is a G-stable subspace of V and $V_{\chi,\infty}$ is the generalized eigenspace in V for the A_G-action on V by the eigencharacter χ. By [7, Proposition 2.1.9], we have that

- V is a direct sum $V = \bigoplus_\chi V_{\chi,\infty}$, where χ ranges over quasi-characters of A_G;
- since V is finitely generated, there are only finitely many χ such that $V_{\chi,\infty} \neq 0$. Moreover, there exists $n \in \mathbb{N}$ such that $V_{\chi,\infty} = V_{\chi,n}$, for each χ.

Let $\mathcal{E}xp_{A_G}(\pi)$ be the (finite) set of quasi-characters of A_G such that $V_{\chi,\infty} \neq 0$. The quasi-characters that appear in $\mathcal{E}xp_{A_G}(\pi)$ are called the exponents of π. The second item above implies that V has a finite filtration such that the quotients are χ-representations, for $\chi \in \mathcal{E}xp_{A_G}(\pi)$. From this last observation, we obtain the following lemma.

Lemma 4.14   *The characters χ of A_G that appear in $\mathcal{E}xp_{A_G}(\pi)$ are precisely the central quasi-characters of the irreducible subquotients of π.*

Note that the same analysis as above can be carried out for any closed subgroup Z of Z_G, *i.e.*, we can consider the generalized Z-eigenspaces in V. Moreover, we have the following.

Lemma 4.15   *Let $Z_1 \supset Z_2$ be two closed subgroups of the centre Z_G of G. The map of exponents $\mathcal{E}xp_{Z_1}(\pi) \to \mathcal{E}xp_{Z_2}(\pi)$ defined by restriction of quasi-characters is surjective.*

Proof   Let $\chi \in \mathcal{E}xp_{Z_2}(\pi)$. By assumption, there exists a nonzero vector $v \in V_{\chi,\infty}$. In particular, there is an irreducible subquotient of $V_{\chi,\infty}$, hence of (π, V), where Z_2 acts by the character χ. On this irreducible subquotient, by Schur's Lemma, the subgroup Z_1 must act by some extension $\widehat{\chi}$ of χ. By Lemma 4.14, $\widehat{\chi}$ must occur in $\mathcal{E}xp_{Z_1}(\pi)$. ∎

For our purposes, we are interested in the exponents of parabolically induced representations.

Lemma 4.16   *Let $P = MN$ be a parabolic subgroup of G, let (ρ, V_ρ) be a finitely generated admissible representation of M, and let $\pi = \iota_P^G \rho$. The quasi-characters $\chi \in \mathcal{E}xp_{A_G}(\pi)$ are the restriction to A_G of characters μ of A_M appearing in $\mathcal{E}xp_{A_M}(\rho)$.*

Proof   Without loss of generality, assume that $P = MN$ is a proper parabolic subgroup of G. Given $a \in A_G$, we have that $\delta_P(a) = 1$, since a is central in G. It follows that for any f in the space V of π we have

$$(4.1) \qquad (\pi(a)f)(g) = f(ga) = f(ag) = \delta_P^{1/2}(a)\rho(a)f(g) = \rho(a)f(g),$$

for all $a \in A_G$ and $g \in G$. Suppose that $\chi \in \mathcal{E}xp_{A_G}(\pi)$ and $f \in V_{\chi,\infty}$ is nonzero. Fix $g_0 \in G$ such that $w = f(g_0)$ is nonzero. There exists $n \in \mathbb{N}$, $n \geq 1$ such that $f \in V_{\chi,n}$. More precisely, $(\pi(a) - \chi(a))^n f = 0_V$, for all $a \in A_G$, where $0_V: G \to V_\rho$ is the zero function. By induction and using (4.1), we see that

$$0 = [(\pi(a) - \chi(a))^n f](g_0) = (\rho(a) - \chi(a))^n (f(g_0)) = (\rho(a) - \chi(a))^n w,$$

for any $a \in A_G$. That is, $w \in (V_\rho)_{\chi,\infty}$ and $(V_\rho)_{\chi,\infty}$ is nonzero; moreover, $\chi \in \mathcal{E}xp_{A_G}(\rho)$. By Lemma 4.15, the map $\mathcal{E}xp_{A_M}(\rho) \to \mathcal{E}xp_{A_G}(\rho)$ defined by restriction is surjective. In particular, there exists $\mu \in \mathcal{E}xp_{A_M}(\rho)$ such that χ is equal to the restriction of μ to A_G. ∎

Let (π, V) be a finitely generated admissible representation of G. Let $P = MN$ be a parabolic subgroup of G with Levi factor M and unipotent radical N. It is a theorem of Jacquet that (π_N, V_N) is also finitely generated and admissible [7, Theorem 3.3.1]. Applying [7, Proposition 2.1.9], we obtain a direct sum decomposition

$$V_N = \bigoplus_{\chi \in \mathcal{E}xp_{A_M}(\pi_N)} (V_N)_{\chi,\infty},$$

where the set $\mathcal{E}xp_{A_M}(\pi_N)$ of quasi-characters of A_M, such that $(V_N)_{\chi,\infty} \neq 0$, is finite. The quasi-characters of A_M appearing in $\mathcal{E}xp_{A_M}(\pi_N)$ are called the exponents of π along P.

Suppose, in addition, that (π, V) is H-distinguished. Fix a nonzero H-invariant form λ on V. For any closed subgroup Z of the centre of G, Kato and Takano [26]

defined

$$(4.2) \qquad \mathcal{E}xp_Z(\pi, \lambda) = \{\chi \in \mathcal{E}xp_Z(\pi) : \lambda|_{V_{\chi,\infty}} \neq 0\},$$

and refer to the set $\mathcal{E}xp_Z(\pi, \lambda)$ as exponents of π relative to λ. The next result, by Kato and Takano [26, Theorem 4.7], is a key ingredient used in the proof of our main theorem.

Theorem 4.17 (The Relative Casselman Criterion) *Let ω be a unitary character of Z_G. Let (π, V) be a finitely generated admissible H-distinguished ω-representation of G. Fix a nonzero H-invariant linear form λ on V. The representation (π, V) is (H, λ)-relatively square integrable if and only if the condition*

$$(4.3) \qquad |\chi(s)| < 1 \quad \text{for all } \chi \in \mathcal{E}xp_{S_M}(\pi_N, r_P\lambda) \text{ and all } s \in S_M^- \smallsetminus S_G S_M^1$$

is satisfied for every proper θ-split parabolic subgroup $P = MN$ of G.

Remark 4.18 Note that the Relative Casselman Criterion reduces to Casselman's Criterion in the group case: $G = G' \times G'$ and $H = \Delta G \cong G'$ is the diagonal subgroup.

Corollary 4.19 (Kato–Takano) *If (π, V) is an H-distinguished discrete series representation of G, then π is H-relatively square integrable.*

The next two lemmas let us prove Proposition 4.22, which allows us to reduce to checking the Relative Casselman Criterion along maximal Δ_0-standard parabolic subgroups (under an additional assumption).

Lemma 4.20 *Let $P = MN$ be a proper θ-split parabolic subgroup of G. Assume that P is H-conjugate to a Δ_0-standard θ-split parabolic subgroup P_Θ. If $P = hP_\Theta h^{-1}$, where $h \in H$, then there is a bijection*

$$\mathcal{E}xp_{S_\Theta}(\pi_{N_\Theta}, r_{P_\Theta}\lambda) \longrightarrow \mathcal{E}xp_S(\pi_N, r_P\lambda) \quad \chi' \longmapsto {}^h\chi',$$

with inverse given by $\chi \longmapsto {}^{h^{-1}}\chi$.

Proof The bijection between $\mathcal{E}xp_{S_\Theta}(\pi_{N_\Theta})$ and $\mathcal{E}xp_S(\pi_N)$ is automatic from the equality $S = hS_\Theta h^{-1}$ (and holds for $h \in (\mathbf{H}\mathbf{M}_0)(F)$). Using that $h \in H$ and H-invariance of λ, one can show that if $r_{P_\Theta}\lambda$ is nonzero on $(V_{N_\Theta})_{\chi',\infty}$, then $r_P\lambda$ is nonzero on $(V_N)_{\chi,\infty}$, where $\chi = {}^h\chi'$. ∎

Lemma 4.21 *Assume that any θ-split parabolic subgroup P of G is H-conjugate to a Δ_0-standard θ-split parabolic. If the condition (4.3) holds for all Δ_0-standard θ-split parabolic subgroups of G, then (4.3) holds for all θ-split parabolic subgroups of G.*

Proof Let $P = MN$ be a proper θ-split parabolic subgroup of G. By assumption, there exists $h \in H$ and a θ-split subset $\Theta \subset \Delta_0$ such that $P = hP_\Theta h^{-1}$. In particular, the (θ, F)-split component S of P is equal to $hS_\Theta h^{-1}$; moreover, $S^- = hS_\Theta^- h^{-1}$. Let $\chi \in \mathcal{E}xp_S(\pi_N, r_P\lambda)$. By Lemma 4.20, $\chi' = {}^{h^{-1}}\chi \in \mathcal{E}xp_{S_\Theta}(\pi_{N_\Theta}, r_{P_\Theta}\lambda)$. Let $s \in S^- \smallsetminus S^1 S_{\Delta_0}$.

Then $s' = h^{-1}sh \in S_\Theta^- \smallsetminus S_\Theta^1 S_{\Delta_0}$. It follows that $|\chi(s)| = |\chi(hs'h^{-1})| = |\chi'(s')| < 1$, where the final inequality holds by the assumption that (4.3) holds for P_Θ. ∎

Proposition 4.22   *Let π be an H-distinguished representation of G and let λ be a nonzero H-invariant linear form on the space of π. Assume that any θ-split parabolic subgroup P of G is H-conjugate to a Δ_0-standard θ-split parabolic. Then π is (H, λ)-relatively square integrable if and only if the condition (4.3) holds for all Δ_0-standard maximal θ-split parabolic subgroups of G.*

Proof   Apply Lemma 4.21, [26, Lemma 4.6], and Theorem 4.17. ∎

The next result is key in our application of Theorem 4.17. It allows us to ignore "bad" exponents relative to λ, as long as the appropriate subquotients are not distinguished.

Proposition 4.23   *Let (π, V) be a finitely generated admissible representation of G. Let $\chi \in \mathcal{E}xp_{Z_G}(\pi)$ and assume that none of the irreducible subquotients of (π, V) with central character χ are H-distinguished. Then for any $\lambda \in \mathrm{Hom}_H(\pi, 1)$, the restriction of λ to $V_{\chi, \infty}$ is equal to zero, i.e., $\lambda|_{V_{\chi, \infty}} \equiv 0$.*

Proof   Suppose, by way of contradiction, that $\lambda|_{V_{\chi, \infty}} \neq 0$. Then $(\pi|_{V_{\chi, \infty}}, V_{\chi, \infty})$ is an admissible finitely generated H-distinguished representation of G. By Lemma 4.2, some irreducible subquotient (ρ, V_ρ) of $V_{\chi, \infty}$ must be H-distinguished. However, the representation (ρ, V_ρ) is also an irreducible subquotient of (π, V) and has central character χ. By assumption, no such (ρ, V_ρ) can be H-distinguished; therefore, we must have that $\lambda|_{V_{\chi, \infty}}$ is identically zero. ∎

5 Tori and Parabolic Subgroups: The Linear and Galois Cases

For the remainder of the paper we work in the linear and Galois cases. Refer to Sections 2.1 and 2.2 for notation.

5.1 Tori and Root Systems Relative to θ

In the linear case, let A_0 be the diagonal maximal F-split torus of G. Note that A_0 is θ-stable. Let S_0 be the (θ, F)-split component of A_0. It is straightforward to check that

$$S_0 = \left\{ \mathrm{diag}(a_1, \ldots, a_{\frac{n}{2}}, a_{\frac{n}{2}}^{-1}, \ldots, a_1^{-1}) : a_i \in F^\times, 1 \leq i \leq \tfrac{n}{2} \right\}.$$

Moreover, S_0 is a maximal (θ, F)-split torus of G. Indeed, it is readily verified that the upper-triangular Borel subgroup of G is a minimal θ-split parabolic subgroup with Levi subgroup A_0. It follows from [18, Proposition 4.7 (iv)] that S_0 is a maximal (θ, F)-split torus of G contained in A_0. In the Galois case, the torus $\mathbf{T}$, obtained as the restriction of scalars of the diagonal torus of $\mathbf{GL}_n$, is a maximal non-split F-torus of $\mathbf{G}$. We identify $T = \mathbf{T}(F)$ with the diagonal matrices in $\mathbf{GL}_n(E)$. Define $T_0 = {}^\gamma T$, where γ is described in Section 2.3. Then $A_0 = {}^\gamma A_T$ is the F-split component of T_0. The tori T, A_T, T_0, and A_0 are all θ-stable. As above, and using (2.2), it is readily

verified that

$$S_0 = \left\{ {}^\gamma \operatorname{diag}(a_1, \ldots, a_{\lfloor \frac{n}{2} \rfloor}, \widehat{1}, a_{\lfloor \frac{n}{2} \rfloor}^{-1}, \ldots, a_1^{-1}) : a_i \in F^\times, 1 \le i \le \lfloor \tfrac{n}{2} \rfloor \right\},$$

is a maximal (θ, F)-split torus of G contained in A_0.

In both cases, let $\Phi_0 = \Phi(G, A_0)$ be the set of roots of G relative to A_0. Explicitly, in the linear case, we have $\Phi_0 = \{\epsilon_i - \epsilon_j : 1 \le i \ne j \le n\}$, where $\epsilon_i \in X^*(A_0)$ is the i-th coordinate (F-rational) character of A_0. Let $\Delta_0 = \{\epsilon_i - \epsilon_{i+1} : 1 \le i \le n-1\}$ be the standard base of Φ_0. The set Φ_0^+ of positive roots (determined by Δ_0) is

$$\Phi_0^+ = \{\epsilon_i - \epsilon_j : 1 \le i < j \le n\}.$$

In the Galois case, we relate Φ_0 to another collection of roots, those relative to A_T. Let $\Phi = \Phi(G, A_T)$ be the root system of G with respect to A_T with standard base Δ. We observe that $\Phi_0 = {}^\gamma\Phi$, where, given a root $\beta \in \Phi$, we have

$$({}^\gamma\beta)(a) = \beta({}^{\gamma^{-1}}a) = \beta(\gamma^{-1}a\gamma),$$

for $a \in A_0$. Moreover, $\Delta_0 = {}^\gamma\Delta$ is a base for Φ_0 and it is clear that

$$\Phi_0^+ = \{{}^\gamma(\epsilon_i - \epsilon_j) : 1 \le i < j \le n\},$$

where, as above, ϵ_i is the i-th coordinate (F-rational) character of the diagonal F-split torus A_T. It is elementary to verify the following.

Lemma 5.1 *The set of simple roots Δ_0 of Φ_0 is a θ-base for Φ_0. In addition, the subset of θ-fixed roots in Φ_0 is empty.*

Corollary 5.2 *The Borel subgroup $P_\varnothing = P_0 = M_0 N_0$ corresponding to $\varnothing \subset \Delta_0$ is a minimal θ-split parabolic subgroup of G.*

Proof The subset Δ_0^θ is a minimal θ-split subset of $\overline{\Delta}_0$; therefore, the parabolic $P_{\Delta_0^\theta}$ is a minimal standard θ-split parabolic subgroup [26]. Since $\Delta_0^\theta = \varnothing$, we have $P_{\Delta_0^\theta} = P_\varnothing = P_0$. In the linear case $M_0 = A_0$, and in the Galois case $M_0 = C_G(A_0) = T_0$. ∎

Following Section 3.1, since $\Delta_0^\theta = \varnothing$, the restricted root system is just the image of Φ_0 under the restriction map $p\colon X^*(A_0) \to X^*(S_0)$. That is, we have $\overline{\Phi}_0 = p(\Phi_0)$ and $\overline{\Delta}_0 = p(\Delta_0)$. Explicitly, in the linear case, $\overline{\Delta}_0 = \{\bar\epsilon_i - \bar\epsilon_{i+1} : 1 \le i \le \frac{n}{2} - 1\} \cup \{2\bar\epsilon_{\frac{n}{2}}\}$, where $\bar\epsilon_i \in X^*(S_0)$ is the i-th coordinate character of S_0 given by

$$\bar\epsilon_i(\operatorname{diag}(a_1, \ldots, a_{\frac{n}{2}}, a_{\frac{n}{2}}^{-1}, \ldots, a_1^{-1})) = a_i.$$

Similarly in the Galois case, we have that

$$\overline{\Delta}_0 = \left\{ {}^\gamma\bar\epsilon_i - {}^\gamma\bar\epsilon_{i+1} : 1 \le i \le \lfloor \tfrac{n}{2} \rfloor - 1 \right\} \cup \{\bar\alpha\},$$

where $\bar\alpha = {}^\gamma\bar\epsilon_{\lfloor \frac{n}{2} \rfloor}$ when n is odd, and $\bar\alpha = 2{}^\gamma\bar\epsilon_{\frac{n}{2}}$ when n is even. The following result is now an immediate consequence of Lemma 5.1.

Lemma 5.3 *For $1 \le k \le \lfloor \frac{n}{2} \rfloor$, let Θ_k denote the $\lfloor \frac{n}{2} \rfloor$ maximal θ-split subsets of Δ_0. In the linear case, for $1 \le k \le n/2 - 1$*

$$\Theta_k = [\overline{\Delta}_0 \smallsetminus \{\bar{\epsilon}_k - \bar{\epsilon}_{k+1}\}] = \Delta_0 \smallsetminus \{\epsilon_k - \epsilon_{k+1}, \epsilon_{n-k} - \epsilon_{n-k+1}\},$$

$$\Theta_{\frac{n}{2}} = [\overline{\Delta}_0 \smallsetminus \{2\bar{\epsilon}_{\frac{n}{2}}\}] = \Delta_0 \smallsetminus \{\epsilon_{\frac{n}{2}} - \epsilon_{\frac{n}{2}+1}\}.$$

Respectively, in the Galois case, for $1 \le k \le \lfloor \frac{n}{2} \rfloor - 1$

$$\Theta_k = [\overline{\Delta}_0 \smallsetminus \{{}^\gamma\bar{\epsilon}_k - {}^\gamma\bar{\epsilon}_{k+1}\}] = \Delta_0 \smallsetminus \{{}^\gamma(\epsilon_k - \epsilon_{k+1}), {}^\gamma(\epsilon_{n-k} - \epsilon_{n-k+1})\},$$

$$\Theta_{\lfloor \frac{n}{2} \rfloor} = [\overline{\Delta}_0 \smallsetminus \{\tilde{\alpha}\}] = \Delta_0 \smallsetminus p^{-1}\{\tilde{\alpha}\},$$

where $\tilde{\alpha} = {}^\gamma\bar{\epsilon}_{\lfloor \frac{n}{2} \rfloor}$ when n is odd, and $\tilde{\alpha} = 2{}^\gamma\bar{\epsilon}_{\frac{n}{2}}$ when n is even.

Note When n is odd, $\Theta_{\lfloor \frac{n}{2} \rfloor} = \Delta_0 \smallsetminus \{{}^\gamma(\epsilon_{\lfloor \frac{n}{2} \rfloor} - \epsilon_{\lfloor \frac{n}{2} \rfloor+1}), {}^\gamma(\epsilon_{\lfloor \frac{n}{2} \rfloor+1} - \epsilon_{\lfloor \frac{n}{2} \rfloor+2})\}$, and when n is even, $\Theta_{\frac{n}{2}} = \Delta_0 \smallsetminus \{{}^\gamma(\epsilon_{\frac{n}{2}} - \epsilon_{\frac{n}{2}+1})\}$.

The next proposition follows immediately from Lemma 5.3.

Proposition 5.4 *The Δ_0-standard maximal θ-split parabolic subgroups of G are:*

$$P_k := P_{\Theta_k} = \begin{cases} P_{(k,n-2k,k)} & 1 \le k \le \frac{n}{2} - 1 \text{ in the linear case,} \\ {}^\gamma P_{(k,n-2k,k)} & 1 \le k \le \lfloor \frac{n}{2} \rfloor - 1 \text{ in the Galois case.} \end{cases}$$

and

$$P_{\lfloor \frac{n}{2} \rfloor} := P_{\Theta_{\lfloor \frac{n}{2} \rfloor}} = \begin{cases} P_{(\frac{n}{2}, \frac{n}{2})} & \text{in the linear case since } n \text{ is even,} \\ {}^\gamma P_{(\frac{n}{2}, \frac{n}{2})} & \text{in the Galois case when } n \text{ is even,} \\ {}^\gamma P_{(\lfloor \frac{n}{2} \rfloor, 1, \lfloor \frac{n}{2} \rfloor)} & \text{in the Galois case when } n \text{ is odd.} \end{cases}$$

Remark 5.5 In both cases, $P_k = M_k N_k$, where $M_k = M_{\Theta_k}$ is the standard Levi factor and N_k is the unipotent radical of P_k. We write A_k for the F-split component and S_k for the (θ, F)-split component of M_k.

In preparation for our proof of Proposition 8.7, we determine the θ-fixed points of the Levi subgroups M_k, $1 \le k \le \lfloor \frac{n}{2} \rfloor$. Recall from (2.1) that, in the Galois case, ϑ is the involution $\gamma \cdot \theta = \mathrm{Int}\, w_\ell \circ \theta = \theta \circ \mathrm{Int}\, w_\ell$. The following is a special case of Lemma 4.3 and also holds for θ_r and ϑ_r (see (2.3)).

Lemma 5.6 *Assume that we are in the Galois case. An element of G of the form ${}^\gamma x$ is θ-fixed (respectively, θ-split) if and only if x is ϑ-fixed (respectively, ϑ-split).*

Proposition 5.7 *Let $1 \le k \le \lfloor \frac{n}{2} \rfloor$. In the linear case, the group M_k^θ of θ-fixed points in $M_k = M_{(k,n-2k,k)}$ is equal to*

$$H_{(k,n-2k,k)} = \begin{cases} \{\mathrm{diag}(A, B, \theta_k(A)) : A \in G_k, B \in H_{n-2k}\} & \text{if } k \ne n/2, \\ \{\mathrm{diag}(A, \theta_k(A)) : A \in G_k\} & \text{if } k = n/2. \end{cases}$$

In the Galois case, $M_k = {}^\gamma M_{(k,n-2k,k)}$ and M_k^θ is G-conjugate to $H_{(k,n-2k,k)}$. Explicitly, we have that $H_{(k,n-2k,k)} = \gamma_{(k,n-2k,k)} M_{(k,n-2k,k)}^\vartheta \gamma_{(k,n-2k,k)}^{-1}$ and

$$M_k^\theta = \gamma M_{(k,n-2k,k)}^\vartheta \gamma^{-1} = \gamma \gamma_{(k,n-2k,k)}^{-1} H_{(k,n-2k,k)} \gamma_{(k,n-2k,k)} \gamma^{-1},$$

where $\gamma_{(k,n-2k,k)} = \mathrm{diag}(\gamma_k, \gamma_{n-2k}, \gamma_k) \in M_{(k,n-2k,k)}$.

Proof Let $1 \le k \le \lfloor \frac{n}{2} \rfloor$ and let $M_\bullet = M_{(k,n-2k,k)}$, (respectively, $M_{(n/2,n/2)}$) when n is even and $k = n/2$. In the linear case, $M_k = M_\bullet$ while, in the Galois case, $M_k = {}^\gamma M_\bullet$. The statement in the linear case follows from the proof in the Galois case (note the relationship between θ and ϑ, (2.2)). Without loss of generality, we work in the Galois case and assume that $k < n/2$. By Lemma 5.6, we have $M_k^\theta = {}^\gamma(M_\bullet^\vartheta)$. Let ${}^\gamma m \in M_k$, where $m \in M_\bullet$. Explicitly, we have $m = \mathrm{diag}(A, B, C)$, where $A, C \in G_k$ and $B \in G_{n-2k}$. One may verify that

$$\vartheta(m) = \mathrm{Int}\, w_\ell \circ \theta(m) = \mathrm{diag}\big(\vartheta_k(C), \vartheta_{n-2k}(B), \vartheta_k(A)\big).$$

It follows that

$$M_\bullet^\vartheta = \big\{ \mathrm{diag}\big(A, B, \vartheta_k(A)\big) : A \in G_k,\ B \in G_{n-2k},\ B = \vartheta_{n-2k}(B) \big\}.$$

Conjugating $M_\bullet^\vartheta$ by the element $\gamma_\bullet = \mathrm{diag}(\gamma_k, \gamma_{n-2k}, \gamma_k)$ in $M_\bullet$ and applying Lemma 5.6, we obtain that $M_\bullet^\vartheta$ is $M_\bullet$-conjugate (and F-isomorphic to) the subgroup $H_\bullet$. ∎

The next result will allow us to apply Proposition 4.22.

Lemma 5.8 *Any θ-split parabolic subgroup P of G is H-conjugate to a Δ_0-standard θ-split parabolic subgroup P_Θ, for some $\Theta \subset \Delta_0$.*

Proof One can check that the degree-one Galois cohomology of $\mathbf{A}_0 \cap \mathbf{H}$ (respectively, $\mathbf{T}_0 \cap \mathbf{H}$) over F is trivial. By a standard argument, we have that $(\mathbf{H}\mathbf{A}_0)(F) = HA_0$ (respectively, $(\mathbf{H}\mathbf{T}_0)(F) = HT_0$). The proposition follows from Corollary 5.2 and [25, Lemma 2.5 (2)]. ∎

5.2 A Class of θ-elliptic Levi Subgroups and θ-stable Parabolic Subgroups

The next two lemmas may be readily verified by hand.

Lemma 5.9 *The Levi subgroup $L_0 = C_G((A_0^\theta)^\circ)$ of G is θ-elliptic and $A_0^\theta = (A_0^\theta)^\circ = A_{L_0}$. Moreover, L_0 is minimal among θ-elliptic Levi subgroup of G that contain A_0.*

Proof First, we observe that since $(A_0^\theta)^\circ$ is θ-stable, the Levi subgroup L_0 is θ-stable. It is immediate that the maximal F-split torus A_0 is contained in L_0 (since A_0 is abelian).

Now, we show that L_0 is θ-elliptic. First, note that the (θ, F)-split component S_G of G is the trivial group. Indeed, in the linear case, θ is inner and we have that $A_G \cong F^\times$ is pointwise θ-fixed. It follows from (3.1) that $S_G = (\{\pm e\})^\circ = \{e\}$. Again, in the Galois case, θ acts trivially on the F-split component of the centre A_G of G and $S_G = \{e\}$. In both the linear and Galois cases, it is readily verified that the F-split component of the

centre of L_0 is equal to $(A_0^\theta)^\circ$, that is, $A_{L_0} = (A_0^\theta)^\circ$. Moreover, in both cases, we have that $(A_0^\theta)^\circ = A_0^\theta$. In particular, A_{L_0} is contained in H and it follows that $S_{L_0} = \{e\}$ [16, §1.3]. By Lemma 3.8, L_0 is θ-elliptic.

Finally, we prove that L_0 is minimal among θ-elliptic Levi subgroups containing A_0. Suppose that $L \subset L_0$ is a proper Levi subgroup of L_0 that contains A_0. We argue that L cannot be θ-elliptic. Since L is proper in L_0, we have that $A_{L_0} = (A_0^\theta)^\circ$ is a proper sub-torus of A_L. Following [16, §1.3], we have an almost direct product $A_L = (A_L^\theta)^\circ S_L$. Observe that, since $A_L \subset A_0$, we have

$$(A_L^\theta)^\circ = (A_L \cap A_0^\theta)^\circ = (A_L \cap A_{L_0})^\circ = A_{L_0}.$$

Since $S_G = S_{L_0} = \{e\} \subset A_{L_0} = (A_0^\theta)^\circ$ and $A_L = (A_L^\theta)^\circ S_L$ properly contains A_{L_0}, it must be the case that S_G is a proper subtorus of S_L; in particular, S_L is non-trivial. It follows from Lemma 3.8 that L is not θ-elliptic and this completes the proof. ∎

Lemma 5.10 *In the Galois case, conjugation by γ maps $N_G(A_T)$ to $N_G(A_0)$ and induces an explicit isomorphism of the Weyl group $W_T = W(G, A_T)$, with respect to A_T, with the Weyl group $W_0 = W(G, A_0)$, with respect to A_0. Moreover, we identify W_T with the group of permutation matrices in G (isomorphic to the symmetric group $\mathfrak{S}_n$) and W_0 with the γ-conjugates of the permutation matrices.*

In both the linear and Galois cases, define $\Delta^{\mathrm{ell}} = w_0 \Delta_0$, where w_0 is defined in (2.4). Since Δ^{ell} is a Weyl group translate of Δ_0, we have that Δ^{ell} is a base of Φ_0. In both cases, set $\Delta_{\mathrm{odd}} = \{\epsilon_i - \epsilon_{i+1} : i \text{ is odd}\}$, and in the Galois case further denote $\Delta_{0,\mathrm{odd}} = {}^\gamma\Delta_{\mathrm{odd}} \subset \Delta_0 = {}^\gamma\Delta$. In the linear case, define the subset $\Delta_{\mathrm{min}}^{\mathrm{ell}}$ of Δ^{ell} by $\Delta_{\mathrm{min}}^{\mathrm{ell}} = w_0 \Delta_{\mathrm{odd}}$ and in the Galois case, define $\Delta_{\mathrm{min}}^{\mathrm{ell}} = w_0 \Delta_{0,\mathrm{odd}}$. In both cases, the subset $\Delta_{\mathrm{min}}^{\mathrm{ell}}$ is exactly the subset of Δ^{ell} that cuts out the torus A_0^θ from A_0. In particular,

$$(5.1) \qquad A_0^\theta = A_{\Delta_{\mathrm{min}}^{\mathrm{ell}}} = \Big(\bigcap_{\beta \in \Delta_{\mathrm{min}}^{\mathrm{ell}}} \ker(\beta : A_0 \to F^\times) \Big)^\circ,$$

and $L_0 = L_{\Delta_{\mathrm{min}}^{\mathrm{ell}}} = C_G(A_{\Delta_{\mathrm{min}}^{\mathrm{ell}}})$. Write $\Delta_{\mathrm{even}} = \{\epsilon_i - \epsilon_{i+1} : 2 \le i \le n-1, \ i \text{ is even}\}$.

Definition 5.11 A Levi subgroup L of G is a standard-θ-elliptic Levi subgroup if and only if L is Δ^{ell}-standard and contains L_0.

The next proposition characterizes the inducing subgroups in Theorem 6.3.

Proposition 5.12 *Let $\Omega^{\mathrm{ell}} \subset \Delta^{\mathrm{ell}}$ such that Ω^{ell} contains $\Delta_{\mathrm{min}}^{\mathrm{ell}}$.*

(i)   *The Δ^{ell}-standard parabolic subgroup $Q = Q_{\Omega^{\mathrm{ell}}}$ associated with Ω^{ell} is θ-stable.*
(ii)   *In particular, the unipotent radical $U = U_{\Omega^{\mathrm{ell}}}$ is θ-stable.*
(iii) *The Levi subgroup $L = L_{\Omega^{\mathrm{ell}}} = C_G(A_{\Omega^{\mathrm{ell}}})$ is a standard-θ-elliptic Levi of G.*
(iv)   *The modular function δ_Q of Q satisfies*

$$\delta_Q^{1/2}\big|_{L^\theta} = \delta_{Q^\theta}.$$

(v)   *We have that*

$$(5.2) \qquad L \cong \prod_{i=1}^k G_{m_i} \quad and \quad L^\theta \cong \prod_{i=1}^k H_{m_i}$$

where $\sum_{i=1}^{k} m_i = n$, *such that when n is odd, exactly one m_i is odd, and when n is even, all of the m_i are even.*

Proof Since $\Delta_{\min}^{\mathrm{ell}} \subset \Omega^{\mathrm{ell}}$, it follows from (5.1) that $A_L = A_{\Omega^{\mathrm{ell}}}$ is contained in $A_{L_0} = A_0^{\theta}$. It follows that L_0 is contained in L and L is a standard-θ-elliptic Levi subgroup by Lemmas 3.7 and 5.9. By Proposition 3.9, Q is a θ-stable parabolic subgroup with θ-stable unipotent radical $U = U_{\Omega^{\mathrm{ell}}}$.

In the Galois case, the statement about modular functions is [13, Lemma 5.5], a proof is given in [29, Lemma 2.5.1]. In the linear case, one may compute the modular functions by hand to verify the desired equality. We omit the straightforward computation.

Finally, we explicitly describe both L and L^{θ}. Note that, in the Galois case, γ centralizes A_T^{ϑ} and by Lemma 5.6 we see that $A_0^{\theta} = \gamma A_T^{\vartheta} \gamma^{-1} = A_T^{\vartheta}$. In both cases, it follows that A_0^{θ} is equal to the w_+-conjugate of the F-split torus

$$A_{(2,\ldots,2,\widehat{1})} = \left\{ \mathrm{diag}\left(a_1, a_1, a_2, a_2, \ldots, a_{\lfloor \frac{n}{2} \rfloor}, a_{\lfloor \frac{n}{2} \rfloor}, \widehat{a}\right) : a, a_i \in F^{\times} \right\},$$

corresponding to the partition $(2, \ldots, 2, \widehat{1})$ of n. Precisely, $A_0^{\theta} = w_+ A_{(2,\ldots,2,\widehat{1})} w_+^{-1}$; moreover, it follows that $L_0 = w_+ M_{(2,\ldots,2,\widehat{1})} w_+^{-1}$. Furthermore, we can realize $\Omega^{\mathrm{ell}} = w_0 \Omega$, where $\Omega \subset \Delta_0$ contains Δ_{odd}, respectively $\Delta_{0,\mathrm{odd}}$. In the linear case, $\Omega^{\mathrm{ell}} = w_+ \Omega$ since $w_0 = w_+$, while in the Galois case, $\Omega^{\mathrm{ell}} = w_0 \Omega = w_+ \gamma \underline{\Omega}$, where $\underline{\Omega} = {}^{\gamma^{-1}} \Omega \subset \Delta$ contains Δ_{odd}. It follows that $L = L_{\Omega^{\mathrm{ell}}}$ is the w_+-conjugate of a block diagonal Levi subgroup $M_{(m_1,\ldots,m_k)}$ that contains $M_{(2,\ldots,2,\widehat{1})}$. We have now established the first isomorphism in (5.2): $L = w_+ M_{(m_1,\ldots,m_k)} w_+^{-1} \cong \prod_{i=1}^{k} G_{m_i}$, where the partition $(m_1, \ldots, m_k)$ of n is refined by $(2, \ldots, 2, \widehat{1})$. In particular, when n is even, each m_i is even and when n is odd, exactly one m_j is odd.

Let $l = w_+ m w_+^{-1} \in L$, where $m \in M = M_{(m_1,\ldots,m_k)}$. To determine L^{θ}, we treat the linear and Galois cases separately. Starting in the linear case, we see that l is θ-fixed if and only if m is fixed by the involution $\theta_+ = w_+ \cdot \theta = \mathrm{Int}(w_+^{-1} w_\ell w_+)$. The element $w_+^{-1} w_\ell w_+$ is the permutation matrix

$$w_+^{-1} w_\ell w_+ = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & \ddots & & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix},$$

which lies in $M_{(2,\ldots,2)} \subset M$. We observe that, since each m_i is even, θ_+ acts on the i-th block $G_{m_i} = \mathbf{GL}_{m_i}(F)$ of M as conjugation by

$$w_{m_i} = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & \ddots & & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix} \in G_{m_i}.$$

Moreover, w_{m_i} is G_{m_i}-conjugate to J_{m_i} (see §2.3). It follows that θ_+ acting on M is M-equivalent to the product involution $\theta_{m_1} \times \cdots \times \theta_{m_k}$; therefore, by Lemma 4.3 we have

$$L^\theta = w_+ M^{\theta_+} w_+^{-1} \cong M^{\theta_+} \cong \prod_{i=1}^k (G_{m_i})^{\theta_{m_i}} = \prod_{i=1}^k H_{m_i},$$

where the second isomorphism is given by conjugation by an element of M. In the Galois case, note that w_+ is θ-fixed and M is θ-stable. Then $l = w_+ m w_+^{-1}$ is θ-fixed if and only if m is θ-fixed. It follows that $L^\theta = w_+ M^\theta w_+^{-1}$ and we have that

$$L^\theta = w_+ M^\theta w_+^{-1} \cong M^\theta = \prod_{i=1}^k (G_{m_i})^{\theta_{m_i}} = \prod_{i=1}^k H_{m_i},$$

as claimed.   ∎

6 Construction of Relative Discrete Series: The Main Theorem

Definition 6.1 A smooth representation τ of a Levi subgroup L of G is *regular* if for every non-trivial element $w \in N_G(L)/L$ we have that ${}^w\tau \not\cong \tau$, where ${}^w\tau = \tau \circ \operatorname{Int} w^{-1}$.

For general linear groups, we can immediately translate Definition 6.1 into the following result.

Lemma 6.2 *Let $(m_1, \ldots, m_k)$ be a partition of n. Let τ_i be an irreducible admissible representation of G_{m_i}, for $1 \le i \le k$. The representation $\tau_1 \otimes \cdots \otimes \tau_k$ of $M_{(m_1, \ldots, m_k)}$ is regular if and only if $\tau_i \not\cong \tau_j$, for all $1 \le i \ne j \le k$.*

Now we come to the main result of the paper.

Theorem 6.3 *Let $Q = LU$ be a proper Δ^{ell}-standard θ-stable parabolic subgroup of G with standard-θ-elliptic Levi factor L and unipotent radical U. Let τ be a regular L^θ-distinguished discrete series representation of L. The parabolically induced representation $\pi = \iota_Q^G \tau$ is irreducible and H-relatively square integrable.*

Proof By assumption, τ is unitary and regular; therefore, π is irreducible by a result of Bruhat [5] (*cf.* [7, Theorem 6.6.1]). Since τ is L^θ-distinguished, π is H-distinguished by Proposition 5.12(iv) and Corollary 4.11. Let λ denote a fixed nonzero H-invariant linear form on π. By Proposition 7.1, λ is unique up to scalar multiples. To complete the proof, it remains to show that π satisfies the Relative Casselman Criterion.

By Lemma 5.8 and Proposition 4.22, it is sufficient to verify that (4.3) is satisfied for every Δ_0-standard maximal θ-split parabolic subgroup. By assumption, $Q = Q_{\Omega^{\mathrm{ell}}}$ for some proper subset $\Omega^{\mathrm{ell}} = w_0 \Omega$ of Δ^{ell} containing $\Delta_{\mathrm{min}}^{\mathrm{ell}}$, where $\Omega \subset \Delta_0$. Let P_Θ be a maximal Δ_0-standard θ-split parabolic subgroup. It follows from Lemma 4.7 that the set $[W_\Theta \backslash W_0 / W_\Omega] \cdot w_0^{-1}$ provides a "nice" choice of representatives for the double-coset space $P_\Theta \backslash G / Q$.

By the Geometric Lemma 4.9 and Lemma 4.14, the exponents of π along P_Θ are given by the union

$$\mathcal{E}xp_{A_\Theta}(\pi_{N_\Theta}) = \bigcup_{y \in [W_\Theta \backslash W_0 / W_\Omega] \cdot w_0^{-1}} \mathcal{E}xp_{A_\Theta}(\mathcal{F}_\Theta^y(\tau)),$$

where the exponents on the right-hand side are the central characters of the irreducible subquotients of $\mathcal{F}^y_\Theta(\tau)$. By Lemma 4.15, the map from $\mathcal{E}xp_{A_\Theta}(\mathcal{F}^y_\Theta(\tau))$ to $\mathcal{E}xp_{S_\Theta}(\pi_{N_\Theta}, r_{P_\Theta}\lambda)$ defined by restriction of characters is surjective. Set $y = ww_0^{-1}$, where $w \in [W_\Theta \backslash W_0 / W_\Omega]$. If $^wM_\Omega \subseteq M_\Theta$, then the parabolic subgroup $P_\Theta \cap {}^wM_\Omega$ of $^wM_\Omega$ is equal to $^wM_\Omega$. The containment $^wM_\Omega \subseteq M_\Theta$ occurs in two cases.

Case 1: when $^wM_\Omega = M_\Theta$, which occurs if and only if $w \in [W_\Theta \backslash W_0 / W_\Omega] \cap W(\Theta, \Omega)$, where $W(\Theta, \Omega) = \{w \in W_0 : w\Omega = \Theta\}$, and

Case 2: when $w \in [W_\Theta \backslash W_0 / W_\Omega]$ is such that $^wM_\Omega \subsetneq M_\Theta$ is a proper Levi subgroup of M_Θ.

In both cases, $\mathcal{F}^y_\Theta(\tau)$ is not M^θ_Θ-distinguished by Proposition 8.7 and the unitary exponents $\chi_{\Theta,y}$ of $\mathcal{F}^y_\Theta(\tau)$ do not contribute to $\mathcal{E}xp_{S_\Theta}(\pi_{N_\Theta}, r_{P_\Theta}\lambda)$ by Proposition 4.23 (cf. (4.2)).

Otherwise, if $^wM_\Omega \nsubseteq M_\Theta$, we have that $P_\Theta \cap {}^wM_\Omega$ is a proper parabolic subgroup of $^wM_\Omega$. By Proposition 8.5, we have that $|\chi(s)|_F < 1$, for all $\chi \in \mathcal{E}xp_{S_\Theta}(\mathcal{F}^y_\Theta(\tau))$, and all $s \in S^-_\Theta \smallsetminus S^1_\Theta S_{\Delta_0}$. In particular, (4.3) holds for all exponents $\chi \in \mathcal{E}xp_{S_\Theta}(\pi_{N_\Theta}, r_{P_\Theta}\lambda)$ relative to λ along all maximal Δ_0-standard θ-split parabolic subgroups P_Θ. By Theorem 4.17, we conclude that π is (H, λ)-relatively square integrable. ∎

Observation 6.4 A representation π of G is H-distinguished if and only if π is ^gH-distinguished; in particular, the property of distinction only depends on the G-conjugacy class of H (or the G-equivalence class of θ, Lemma 4.3). Thus, taking into account Proposition 5.12 and Lemma 6.2, we may rephrase Theorem 6.3 as follows.

(1) Assume that n is even. Let $(m_1, \ldots, m_k)$ be a partition of n such that each m_i is even. Let $\tau_1, \ldots, \tau_k$ be pairwise inequivalent H_{m_i}-distinguished discrete series representations of G_{m_i}. The parabolically induced representation $\tau_1 \times \cdots \times \tau_k$ is an irreducible H-distinguished relative discrete series representation of G.

(2) If n is odd, then we must be in the Galois case. Let $(m_1, \ldots, m_k)$ be a partition of n such that exactly one m_l is odd, and all other m_i are even. Let $\tau_1, \ldots, \tau_k$ be pairwise inequivalent $\mathbf{GL}_{m_i}(F)$-distinguished discrete series representations of $\mathbf{GL}_{m_i}(E)$. The parabolically induced representation $\tau_1 \times \cdots \times \tau_k$ is an irreducible $\mathbf{GL}_n(F)$-distinguished relative discrete series representation of $\mathbf{GL}_n(E)$.

Corollary 6.5 *Let $\pi = \iota^G_Q \tau$ be as in Theorem 6.3. The representation π is a relative discrete series representation that does not lie in the discrete spectrum of G.*

Proof By Theorem 6.3, π is irreducible and H-relatively square integrable; therefore, π is a relative discrete series. Since $\pi = \iota^G_Q \tau$, where Q is proper in G, it follows from the work of Zelevinsky [42] that π does not occur in the discrete spectrum of G. ∎

Remark 6.6 At present, the author does not know if the construction outlined in Theorem 6.3 exhausts all non-discrete relative discrete series in the linear and Galois cases. In order to show that a representation is not (H, λ)-relatively square integrable, it is necessary to show that $r_P\lambda$ is non-vanishing on the generalized eigenspace corresponding to an exponent $\chi \in \mathcal{E}xp_{S_M}(\pi_N, r_P\lambda)$ that fails the condition (4.3). The non-vanishing of $r_P\lambda$ is obscured by the nature of the construction of the form via

Casselman's Canonical Lifting. Due to this lack of precise information, we cannot exclude the possibility that certain representations are RDS. For instance, it may be possible to relax the regularity condition imposed in Theorem 6.3, which is essential in the proof of Proposition 8.7. At this time, the author does not have a method to remove the assumption of regularity from Proposition 8.7, due to a lack of information regarding the support of the $r_P \lambda$.

By [25, Theorem 6.2] (see Theorem 4.13) and the proof of Theorem 6.3, one may obtain the following.

Corollary 6.7   *Let $Q = LU$ be as in Theorem 6.3. If τ is a regular L^θ-distinguished supercuspidal representation of L, then $\pi = \iota_Q^G \tau$ is H-relatively supercuspidal.*

Remark 6.8   Note that Corollary 6.7 can be obtained by more direct methods. See, for instance, the work of Murnaghan [33] for such results in a more general setting.

7 Distinguished Discrete Series: Known Results and Inducing Data

In this section, we survey the known results on distinguished discrete series representations in the linear and Galois cases. Our ultimate goal is to prove Proposition 7.16 and thus Corollary 7.17. First, we note that in the linear case multiplicity-one is due to Jacquet and Rallis [22]. In the Galois case, multiplicity-one is due to Flicker [10].

Proposition 7.1   *Let π be an irreducible admissible representation of G. If π is H-distinguished, then $\mathrm{Hom}_H(\pi, 1)$ is one-dimensional.*

Jacquet and Rallis [22] also proved the next proposition.

Proposition 7.2   *Let M be a maximal Levi subgroup of $\mathbf{GL}_m(F)$, where $m \geq 2$. Let π be an irreducible admissible representation of $\mathbf{GL}_m(F)$. If π is M-distinguished, then π is equivalent to its contragredient $\widetilde{\pi}$.*

Flicker [10] used the methods of [12] to prove the following result.

Proposition 7.3   *Let π be an irreducible admissible representation of $\mathbf{GL}_m(E)$, where $m \geq 2$. If π is $\mathbf{GL}_m(F)$-distinguished, then $\pi \cong {}^\theta \widetilde{\pi}$.*

7.1 Distinguished Discrete Series in the Linear Case

In this subsection, unless otherwise noted, we let $G = \mathbf{GL}_n(F)$, where $n \geq 2$ is even, and we let $H = \mathbf{GL}_{n/2}(F) \times \mathbf{GL}_{n/2}(F)$.

Remark 7.4   It is known that an irreducible square integrable representation π of G is H-distinguished if and only if π admits a Shalika model. It was shown by Jacquet and Rallis [22] that if π is an irreducible admissible representation of G that admits a Shalika model, then π is H-distinguished. For irreducible supercuspidal representations, the converse appears as [23, Theorem 5.5]. Sakellaridis and Venkatesh, and

independently Matringe, proved the converse result for relatively integrable and relatively square integrable representations by the technique of "unfolding" [39, Example 9.5.2], [31, Theorem 5.1]. In fact, Sakellaridis–Venkatesh proved that there is an equivariant unitary isomorphism between $L^2(H\backslash G)$ and $L^2(S\backslash G)$, where S is the Shalika subgroup. Several analogous global results appear in [11].

Let π be a discrete series representation of $\mathbf{GL}_m(F)$, $m \geq 2$. Denote by $L(s, \pi \times \pi)$ the local Rankin–Selberg convolution L-function. It is well known that $L(s, \pi \times \pi)$ has a simple pole at $s = 0$ if and only if π is self-contragredient [21]. By [40, Lemma 3.6], we have a local identity

(7.1) $$L(s, \pi \times \pi) = L(s, \pi, \wedge^2)L(s, \pi, \mathrm{Sym}^2),$$

where $L(s, \pi, \wedge^2)$, respectively $L(s, \pi, \mathrm{Sym}^2)$, denotes the exterior square, respectively symmetric square, L-function of π defined via the Local Langlands Correspondence (LLC). It is also well known [6, 27] that $L(s, \pi, \wedge^2)$ cannot have a pole when m is odd.

Theorem 7.5 ([31, Proposition 6.1])   *Suppose that π is a square integrable representation of G. Then π is H-distinguished if and only if the exterior square L-function $L(s, \pi, \wedge^2)$ has a pole at $s = 0$.*

Remark 7.6   It is now known that for all discrete series (and when n is even, all irreducible generic representations), the Jacquet–Shalika and Langlands–Shahidi local exterior square L-functions agree with the exterior square L-functions defined via the LLC [19, Theorem 4.3, §4.2], [27, Theorems 1.1, 1.2].

Let ρ be an irreducible unitary supercuspidal representation of $\mathbf{GL}_r(F)$, $r \geq 1$. For an integer $k \geq 2$, write $\mathrm{St}(k, \rho)$ for the unique irreducible (unitary) quotient of the parabolically induced representation $\nu^{\frac{1-k}{2}}\rho \times \nu^{\frac{3-k}{2}}\rho \times \cdots \times \nu^{\frac{k-1}{2}}\rho$ of $\mathbf{GL}_{kr}(F)$ [42, Proposition 2.10, §9.1], where $\nu(g) = |\det(g)|_F$, for any $g \in \mathbf{GL}_r(F)$. The representations $\mathrm{St}(k, \rho)$ are often called *generalized Steinberg representations* and they are exactly the nonsupercuspidal discrete series representations of $\mathbf{GL}_{kr}(F)$ [42, Theorem 9.3]. The usual Steinberg representation St_n of $\mathbf{GL}_n(F)$ is obtained as $\mathrm{St}(n, 1)$. Note that $\mathrm{St}(k_1, \rho_1)$ is equivalent to $\mathrm{St}(k_2, \rho_2)$ if and only if $k_1 = k_2$ and ρ_1 is equivalent to ρ_2 [42, Theorem 9.7 (b)].

Theorem 7.7 ([31, Theorem 6.1])   *Suppose that $n = kr$ is even. Let ρ be an irreducible supercuspidal representation of $\mathbf{GL}_r(F)$. Let $\pi = \mathrm{St}(k, \rho)$ be a generalized Steinberg representation of G.*

(i)   *If k is odd, then r must be even, and π is H-distinguished if and only if ρ is $\mathbf{GL}_{r/2}(F) \times \mathbf{GL}_{r/2}(F)$-distinguished.*

(ii)   *If k is even, then π is H-distinguished if and only if $L(s, \rho, \mathrm{Sym}^2)$ has a pole at $s = 0$.*

The following is a corollary of [14, Theorem 1.3] and [32, Proposition 10.1].

Theorem 7.8   *For any even integer $n \geq 2$,*

(i)   *there exist infinitely many equivalence classes of H-distinguished irreducible (unitary) tame supercuspidal representations of G;*

(ii)   *there exist infinitely many equivalence classes of self-contragredient irreducible (unitary) tame supercuspidal representations of G that are not H-distinguished.*

Finally, we note the following.

Proposition 7.9   *For any even integer $n \geq 2$, there are infinitely many equivalence classes of H-distinguished discrete series representations of G. Moreover,*

(i)   *if $n = 2$, there are exactly four H-distinguished twists of the Steinberg representation St_2 of G;*

(ii)   *if $n = 4$, there are exactly four H-distinguished twists of the Steinberg representation St_4 of G, and there are infinitely many equivalence classes of H-distinguished generalized Steinberg representations of G of the form $\mathrm{St}(2, \rho)$;*

(iii) *if $n \geq 6$, there are infinitely many equivalence classes of H-distinguished nonsupercuspidal discrete series representations of G.*

Proof   The main statement follows from Theorem 7.8.

(i) The Steinberg representation St_2 has trivial central character and so it is H-distinguished [36]. A twist $\chi \otimes \mathrm{St}_2$ of St_2 by a quasi-character χ of $F^\times$ has trivial central character if and only if χ is trivial on $(F^\times)^2$. In particular, χ must be a quadratic (unitary) character and, since F has odd residual characteristic, there are four distinct such characters.

(ii) By [11, Corollary 8.5 (ii)], a twisted Steinberg representation $\chi \otimes \mathrm{St}_4$ admits a Shalika model if and only if χ is trivial on $(F^\times)^2$. In this case, $\chi \otimes \mathrm{St}_4$ is H-distinguished (Remark 7.4).

By Theorem 7.8 (ii), there exist infinitely many classes of self-contragredient supercuspidal representations ρ of G_2 that are not H_2-distinguished. In particular, given such a $\rho \cong \widetilde{\rho}$, the Rankin–Selberg L-function $L(s, \rho \times \rho)$ has a pole at $s = 0$ [21]; however, by Theorem 7.5, $L(s, \rho, \wedge^2)$ does not have a pole at $s = 0$. It follows from (7.1) that $L(s, \rho, \mathrm{Sym}^2)$ has a pole at $s = 0$. The claim follows from Theorem 7.7 (ii)

(iii) The last statement is an immediate consequence of Theorems 7.7 and 7.8.   ∎

7.2   Distinguished Discrete Series in the Galois Case

In this subsection, unless otherwise noted, let $G = R_{E/F}\mathbf{GL}_n(F)$, where $n \geq 2$. We identify G with $\mathbf{GL}_n(E)$. Let $H = \mathbf{GL}_n(F)$ be the subgroup of Galois fixed points in G. Let $\eta : E^\times \to \mathbb{C}^\times$ be an extension to $E^\times$ of the character $\eta_{E/F} : F^\times \to \mathbb{C}$ associated with E/F by local class field theory. The following result is due to Anandavardhanan and Rajan [2, §4.4], and also appears in [1, Theorem 1.3] and [30, Corollary 4.2].

Theorem 7.10   *Let ρ be an irreducible supercuspidal representation of $\mathbf{GL}_r(E)$. Then the generalized Steinberg representation $\pi = \mathrm{St}(k, \rho)$ of $\mathbf{GL}_{kr}(E)$ is $\mathbf{GL}_{kr}(F)$-distinguished if and only if ρ is $(\mathbf{GL}_r(F), \eta_{E/F}^{k-1})$-distinguished.*

The next result is due to Prasad for $n = 2$ [35] and Anandavardhanan and Rajan for $n \geq 3$ [2, Theorem 1.5].

Theorem 7.11   *Let χ be a quasi-character of $F^\times$ and identify χ with the quasi-character $\chi \circ \det$ of H. The Steinberg representation* St_n *of G is (H, χ)-distinguished if and only if n is odd and $\chi = 1$, or n is even and $\chi = \eta_{E/F}$.*

Corollary 7.12   *If $n = 2$, then the twist $\eta \otimes \mathrm{St}_2$ of the Steinberg representation* St_2 *of G is H-distinguished.*

The following is a corollary of the work of Hakim and Murnaghan [14, Theorem 1.1] and Murnaghan [32, Proposition 10.1].

Theorem 7.13

(i)   *There are infinitely many equivalence classes of irreducible (unitary) supercuspidal representations of G that are H-distinguished,*
(ii)   *There are infinitely many equivalence classes of irreducible (unitary) supercuspidal representations of G that are $(H, \eta_{E/F})$-distinguished.*

The next result is an immediate consequence of Theorems 7.10, 7.11, and 7.13.

Corollary 7.14   *Let $G = \mathbf{GL}_n(E)$ and let $H = \mathbf{GL}_n(F)$.*

(i)   *If $n \geq 4$ is not an odd prime, then there are infinitely many equivalence classes of H-distinguished nonsupercuspidal discrete series representations of G.*
(ii)   *If n is an odd prime, then the Steinberg representation* St_n *of G is a nonsupercuspidal H-distinguished discrete series.*

Proof   Assume that $n \geq 4$ is not an odd prime. Then $n = kr$ for two integers $k, r \geq 2$. Note that $\eta_{E/F}$ is a quadratic character; in particular, if k is even, then $\eta_{E/F}^{k-1} = \eta_{E/F}$ and if k is odd, then $\eta_{E/F}^{k-1} = 1$. By Theorem 7.13, there are infinitely many equivalence classes of irreducible supercuspidal representations ρ of G_r that are $(G_r, \eta_{E/F}^{k-1})$-distinguished. By Theorem 7.10 and [42, Theorem 9.7 (b)], there are infinitely many equivalence classes of generalized Steinberg representations of G of the form $\mathrm{St}(k, \rho)$ and that are H-distinguished. Of course, the generalized Steinberg representations $\mathrm{St}(k, \rho)$ are nonsupercuspidal discrete series representations. The second statement follows from Theorem 7.11.   ∎

7.3   The Inducing Representations in Theorem 6.3

For the remainder of the paper, fix a proper Δ^{ell}-standard θ-stable parabolic subgroup $Q = Q_{\Omega^{\mathrm{ell}}}$, for some proper subset $\Omega^{\mathrm{ell}} \subset \Delta^{\mathrm{ell}}$ containing $\Delta_{\mathrm{min}}^{\mathrm{ell}}$. As in Proposition 5.12, the subgroup Q admits a standard-θ-elliptic Levi subgroup $L = L_{\Omega^{\mathrm{ell}}}$ and unipotent radical $U = U_{\Omega^{\mathrm{ell}}}$. The next lemma is straightforward to verify by using the description

of L^θ, given in the proof of Proposition 5.12, and Lemma 4.3. The multiplicity-one statement follows from Proposition 7.1.

Lemma 7.15 *Let $L \cong \prod_{i=1}^k G_{m_i}$ be a standard-θ-elliptic Levi subgroup of G. Let $\tau \cong \bigotimes_{i=1}^k \tau_i$ be an irreducible admissible representation of L where each τ_i is an irreducible admissible representation of G_{m_i}, $1 \leq i \leq k$.*

(i) *Then τ is L^θ-distinguished if and only if τ_i is H_{m_i}-distinguished for all $1 \leq i \leq k$.*
(ii) *If τ is L^θ-distinguished, then $\mathrm{Hom}_{L^\theta}(\tau, 1)$ is one-dimensional.*

Proposition 7.16 *Let L be a standard-θ-elliptic Levi subgroup of G. There exist infinitely many equivalence classes of regular non-supercuspidal L^θ-distinguished discrete series representations of L.*

Proof By assumption $n \geq 4$ and n is always taken to be even in the linear case. We have that L is isomorphic to a product $\prod_{i=1}^k G_{m_i}$ of smaller general linear groups, for some partition $(m_1, \ldots, m_k)$ of n. Let τ_i be an irreducible admissible representation of G_{m_i}, $1 \leq i \leq k$. By Lemma 6.2, the representation $\tau_1 \otimes \cdots \otimes \tau_k$ of L is regular if and only if $\tau_i \not\cong \tau_j$ for all $1 \leq i \neq j \leq k$. Moreover, $\tau_1 \otimes \cdots \otimes \tau_k$ is supercuspidal (square integrable) if and only if every τ_i is supercuspidal (square integrable). It is sufficient to prove that for any relevant partition of n (see Proposition 5.12), there exist pairwise inequivalent H_{m_i}-distinguished discrete series representations τ_i such that at least one τ_i is not supercuspidal.

In the linear case, by Proposition 5.12, each $m_i \geq 2$ is even. By Theorem 7.8, there are infinitely many equivalence classes of H_{m_i}-distinguished irreducible supercuspidal representations of G_{m_i}. By Proposition 7.9, there exists at least one non-supercuspidal H_{m_i}-distinguished discrete series representation of G_{m_i} (infinitely many when $m_i \geq 4$). It follows from Lemma 7.15 that there exist infinitely many equivalence classes of regular non-supercuspidal L^θ-distinguished discrete series representations of L.

In the Galois case, by Proposition 5.12, at most one m_i is odd. Without loss of generality, assume that m_k is odd. By Theorem 7.11 (i), the Steinberg representation St_{m_k} of G_{m_k} is a non-supercuspidal H_{m_k}-distinguished discrete series. By Theorem 7.13, there are infinitely many equivalence classes of H_{m_i}-distinguished irreducible supercuspidal representations of G_{m_i}. By Corollaries 7.12 and 7.14, there exists at least one non-supercuspidal H_{m_i}-distinguished discrete series representation of G_{m_i} (infinitely many when $m_i \geq 4$ is not an odd prime). It follows from Lemma 7.15 that there exist infinitely many equivalence classes of regular non-supercuspidal L^θ-distinguished discrete series representations of L. ∎

Corollary 7.17 *There are infinitely many equivalence classes of H-distinguished relative discrete series representations of G of the form constructed in Theorem 6.3. In particular, there are infinitely many classes of such representations where the discrete series τ is not supercuspidal.*

Proof This is immediate from Proposition 7.16 and from [42, Theorem 9.7 (b)]. ∎

8   Computation of Exponents and Distinction of Jacquet Modules

We work under the hypotheses of Theorem 6.3 and use the notation from its proof. In order to discuss Casselman's Criterion for the inducing data of $\pi = \iota_Q^G \tau$, we use the following notation. If $\Theta_1 \subset \Theta_2 \subset \Delta_0$, then we define

$$A_{\Theta_1}^- = \{ a \in A_{\Theta_1} : |\alpha(a)| \leq 1, \text{ for all } \alpha \in \Delta_0 \smallsetminus \Theta_1 \},$$
$$A_{\Theta_1}^{-\Theta_2} = \{ a \in A_{\Theta_1} : |\beta(a)| \leq 1, \text{ for all } \beta \in \Theta_2 \smallsetminus \Theta_1 \}.$$

The set $A_{\Theta_1}^-$ is the dominant part of A_{Θ_1} in G, while $A_{\Theta_1}^{-\Theta_2}$ is the dominant part of A_{Θ_1} in M_{Θ_2}.

In both the linear and Galois cases, we have that $Q = w_0 P_\Omega w_0^{-1}$ is a Weyl group conjugate of a Δ_0-standard parabolic subgroup P_Ω, where $\Omega^{\text{ell}} = w_0 \Omega$. We also have that $L = w_0 M_\Omega w_0^{-1}$. If τ_0 is a representation of M_Ω, then $\tau = {}^{w_0}\tau_0$ is a representation of L. Let P_Θ be a Δ_0-standard maximal θ-split parabolic subgroup corresponding to a maximal proper θ-split subset $\Theta \subset \Delta_0$. Below $y = w w_0^{-1}$ is a "nice" representative of a double-coset in $P_\Theta \backslash G / Q$, where $w \in [W_\Theta \backslash W_0 / W_\Omega]$ (Lemma 4.7).

Lemma 8.1   *The exponents of $\mathcal{F}_\Theta^y(\tau)$ are the restriction to A_Θ of the exponents of ${}^w\tau_0$ along the parabolic subgroup $P_\Theta \cap {}^w M_\Omega$ of ${}^w M_\Omega$. If $\tau = {}^{w_0}\tau_0$ is a discrete series representation of L, and the parabolic subgroup $P_\Theta \cap {}^w M_\Omega$ of ${}^w M_\Omega$ is proper, then for any exponent $\chi \in \mathcal{E}xp_{A_\Theta}(\mathcal{F}_\Theta^y(\tau))$ the inequality $|\chi(a)|_F < 1$ is satisfied for every $a \in A_{\Theta \cap w\Omega}^{-w\Omega} \smallsetminus A_{\Theta \cap w\Omega}^1 A_{w\Omega}$.*

Proof   This is a special case of Lemma 4.16 and the usual Casselman's Criterion ([7, Theorem 6.5.1]) applied to the discrete series representation ${}^w\tau_0$ of ${}^w M_\Omega$.   ∎

Lemma 8.2   *Assume that τ is a regular unitary irreducible admissible representation of L. If $y = w w_0^{-1}$, where $w \in [W_\Theta \backslash W_0 / W_\Omega]$ is such that ${}^w M_\Omega \subset P_\Theta$, then $\mathcal{F}_\Theta^y(\tau)$ is irreducible and the central character $\chi_{\Theta,y}$ of $\mathcal{F}_\Theta^y(\tau)$ is unitary.*

Proof   If ${}^w M_\Omega \subset P_\Theta$, then $P_\Theta \cap {}^w M_\Omega = {}^w M_\Omega$, $N_\Theta \cap {}^w M_\Omega = \{e\}$, and ${}^w M_\Omega \subset M_\Theta$. It follows that the representation $({}^w\tau_0)_{N_\Theta \cap {}^w M_\Omega}$ is equal to ${}^w\tau_0$ and it is irreducible and unitary. Moreover, since τ is a regular representation of L, it follows that ${}^w\tau_0$ is regular as a representation of ${}^w M_\Omega$ regarded as a Levi subgroup of M_Θ. By [7, Theorem 6.6.1], the representation $\mathcal{F}_\Theta^y(\tau)$ is irreducible and unitary. By the irreducibility of $\mathcal{F}_\Theta^y(\tau)$, the only exponent is its central character $\chi_{\Theta,y}$. Since $\mathcal{F}_\Theta^y(\tau)$ is unitary, the character $\chi_{\Theta,y}$ of A_Θ is unitary.   ∎

Remark 8.3   Recall that $W(\Theta, \Omega) = \{w \in W_0 : w\Omega = \Theta\}$. We find ourselves in the situation of Lemma 8.2 in two cases.

Case 1: $w \in [W_\Theta \backslash W_0 / W_\Omega] \cap W(\Theta, \Omega)$, if and only if ${}^w M_\Omega = M_\Theta$,

Case 2: $w \in [W_\Theta \backslash W_0 / W_\Omega]$ is such that ${}^w M_\Omega \subsetneq M_\Theta$ is a proper Levi subgroup of M_Θ.

In order to apply the Relative Casselman Criterion (Theorem 4.17) using Lemma 8.1, we need the following technical fact.

Lemma 8.4   *Let Ω be a proper subset of Δ_0 such that $\Omega^{\mathrm{ell}} = w_0\Omega$ contains $\Delta_{\min}^{\mathrm{ell}}$. Let Θ be a maximal θ-split subset of Δ_0. Let $w \in [W_\Theta \backslash W_0/W_\Omega]$ such that $M_{\Theta \cap w\Omega} = M_\Theta \cap wM_\Omega w^{-1}$ is a proper Levi subgroup of $M_{w\Omega} = wM_\Omega w^{-1}$. Then we have the containment $S_\Theta^- \smallsetminus S_\Theta^1 S_{\Delta_0} \subset A_{\Theta \cap w\Omega}^{-w\Omega} \smallsetminus A_{\Theta \cap w\Omega}^1 A_{w\Omega}$.*

Proof   Recall that for any F-torus $\mathbf{A}$ we write A^1 for the $\mathcal{O}_F$-points of $\mathbf{A}$. First, S_Θ is contained in A_Θ and since $\Theta \cap w\Omega$ is a subset of Θ, we have that

$$\mathbf{A}_\Theta = \big(\bigcap_{\alpha \in \Theta} \ker \alpha\big)^\circ \subset \big(\bigcap_{\alpha \in \Theta \cap w\Omega} \ker \alpha\big)^\circ = \mathbf{A}_{\Theta \cap w\Omega}.$$

At the level of F-points, we have $A_\Theta \subset A_{\Theta \cap w\Omega}$, and similarly for the integer points $A_\Theta^1 \subset A_{\Theta \cap w\Omega}^1$. It follows that $S_\Theta \subset A_{\Theta \cap w\Omega}$ and $S_\Theta^1 \subset A_{\Theta \cap w\Omega}^1$. Also, we have $S_{\Delta_0} \subset A_{\Delta_0} \subset A_\Omega$, and since $S_{\Delta_0} = S_G$ is central in G, we have $S_{\Delta_0} = wS_{\Delta_0}w^{-1} \subset wA_\Omega w^{-1} = A_{w\Omega}$. We now observe that $A_\Theta^- \subset A_{\Theta \cap w\Omega}^{-w\Omega}$ and in particular that $S_\Theta^- \subset A_{\Theta \cap w\Omega}^{-w\Omega}$; it is clear that $S_\Theta^- \subset A_\Theta^-$. Note that $w\Omega$ is a base for the root system of $M_{w\Omega}$ relative to the maximal F-split torus A_0. Suppose that $a \in A_\Theta^-$. Then $|\alpha(a)| \leq 1$, for all $\alpha \in \Delta_0 \smallsetminus \Theta$. Moreover, since $a \in A_\Theta$, we have that $|\alpha(a)| = 1$, for $\alpha \in \Theta$ as well. Let $\beta \in w\Omega \smallsetminus (\Theta \cap w\Omega)$. Then $\beta = w\alpha$ for some $\alpha \in \Omega$. Since $w \in [W_\Theta \backslash W_0/W_\Omega]$, we have that $\beta = w\alpha \in \Phi_0^+$. Write $\beta = \sum_{\epsilon \in \Delta_0} c_\epsilon \cdot \epsilon$, where $c_\epsilon \geq 0$, $c_\epsilon \in \mathbb{Z}$. Then we have that

$$|\beta(a)| = \Big| \prod_{\epsilon \in \Delta_0} \epsilon(a)^{c_\epsilon} \Big| = \prod_{\epsilon \in \Delta_0} |\epsilon(a)|^{c_\epsilon} \leq 1,$$

since $|\epsilon(a)| \leq 1$, for all $\epsilon \in \Delta_0$, and $c_\epsilon \geq 0$. In particular, $a \in A_{\Theta \cap w\Omega}^{-w\Omega}$. Putting this together, we see that $S_\Theta^1 S_{\Delta_0} \subset S_\Theta^- \cap A_{\Theta \cap w\Omega}^1 A_{w\Omega}$; therefore, to prove the desired result, it suffices to prove the opposite inclusion.

     It is at this point that we specialize to the linear and Galois cases. By assumption $\Theta = \Theta_k$, for some $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, as in Proposition 5.4. Suppose that $s \in S_\Theta^- \cap A_{\Theta \cap w\Omega}^1 A_{w\Omega}$. We want to show that $s \in S_\Theta^1 S_{\Delta_0}$. Notice that $S_{\Delta_0} = \{e\}$; therefore, it is sufficient to prove that $s \in S_\Theta^1$. By assumption, $s = tz$ where $t \in A_{\Theta \cap w\Omega}^1$ and $z \in A_{w\Omega}$. Since $w \in [W_\Theta \backslash W_0/W_\Omega]$, we have that $w\Omega \subset \Phi_0^+$; moreover, by the assumption that $M_{\Theta \cap w\Omega}$ is a proper Levi subgroup of $M_{w\Omega}$, we have $\Theta \cap w\Omega \subsetneq w\Omega$ is a proper subset. It follows that $w\Omega$ cannot be contained in Φ_Θ^+. Moreover, there exists $\alpha \in w\Omega \smallsetminus (\Theta \cap w\Omega)$ such that $\alpha \in \Phi_0^+$ and $\alpha \notin \Phi_\Theta^+$. In the Galois case, there is a unique expression $\alpha = \sum_{j=1}^{n-1} c_j {}^\gamma(\epsilon_j - \epsilon_{j+1})$, where $c_j \in \mathbb{Z}$ and $c_j \geq 0$, such that, since $\Theta = \Theta_k$ and $\alpha \notin \Phi_\Theta^+$, at least one of c_k or c_{n-k} is nonzero ($c_{n/2} \neq 0$, when n even, $k = n/2$). In the linear case, γ does not appear in the expression for α. First observe that $\alpha(s) = \alpha(t)\alpha(z) = \alpha(t) \in \mathcal{O}_F^\times$, since $z \in A_{w\Omega}$ and $t \in A_{\Theta \cap w\Omega}^1 = A_{\Theta \cap w\Omega}(\mathcal{O}_F)$. On the other hand, in the Galois case, writing s explicitly as $s = {}^\gamma s'$, where

$$s' = \begin{cases} \mathrm{diag}(\underbrace{a, \ldots, a}_{k}, \underbrace{1 \ldots, 1}_{\lfloor \frac{n}{2} \rfloor - 2k}, \underbrace{a^{-1}, \ldots, a^{-1}}_{k}) & 1 \leq k \leq \lfloor \frac{n}{2} \rfloor, \\[2ex] \mathrm{diag}(\underbrace{a, \ldots, a}_{n/2}, \underbrace{a^{-1}, \ldots, a^{-1}}_{n/2}) & n \text{ even}, k = n/2, \end{cases}$$

(with $s = s'$ and n even in the linear case). Applying α to s, we have

$$\alpha(s) = \Big(\sum_{j=1}^{n-1} c_j {}^\gamma(\epsilon_j - \epsilon_{j+1}) \Big) ({}^\gamma s') = (\epsilon_k - \epsilon_{k+1})(s')^{c_k} (\epsilon_{n-k} - \epsilon_{n-k+1})(s')^{c_{n-k}}$$

$$= a^{c_k} a^{c_{n-k}}$$

So in the Galois case, $\alpha(s) = a^c \in \mathcal{O}_F^\times$, where $c = c_k + c_{n-k}$ (or $c = 2c_{\frac{n}{2}}$ when n is even and $k = n/2$), and similarly in the linear case. Then we have $|a|_F^c = 1$ for c a positive integer so $|a|_F = 1$. In particular, we have that $a \in \mathcal{O}_F^\times$, and $s \in S_\Theta^1 = S_\Theta(\mathcal{O}_F)$, as desired.   ∎

Proposition 8.5   *If $y = ww_0^{-1} \in \big[W_\Theta \backslash W_0 / W_\Omega \big] \cdot w_0^{-1}$ is such that $P_\Theta \cap {}^w M_\Omega$ is a proper parabolic subgroup of ${}^w M_\Omega$, then the exponents $\chi \in \mathcal{E}xp_{S_\Theta}(\mathcal{F}_\Theta^y(\tau))$ of $\mathcal{F}_\Theta^y(\tau)$ satisfy $|\chi(s)|_F < 1$, for all $s \in S_\Theta^- \smallsetminus S_\Theta^1 S_{\Delta_0}$.*

Proof   First by Lemma 8.4 we have that $S_\Theta^- \smallsetminus S_\Theta^1 S_{\Delta_0} \subset A_{\Theta \cap w\Omega}^{-w\Omega} \smallsetminus A_{\Theta \cap w\Omega}^1 A_{w\Omega}$. By Lemma 4.15, any exponent $\chi \in \mathcal{E}xp_{S_\Theta}(\mathcal{F}_\Theta^y(\tau))$ is the restriction to S_Θ of an exponent $\widehat{\chi} \in \mathcal{E}xp_{A_\Theta}(\mathcal{F}_\Theta^y(\tau))$; therefore, the result follows from Lemma 8.1.   ∎

Finally, we study the M_Θ^θ-distinction of the Jacquet module π_{N_Θ}. In preparation for this, we characterized the θ-fixed points of the standard Levi $M_r = M_{\Theta_r}$ of the maximal θ-split parabolic subgroups $P_r = P_{\Theta_r}$ in Proposition 5.7. The characterization is in terms of the groups $M_{(r,n-2r,r)}$ and $H_{(r,n-2r,r)}$. First, we present Lemma 8.6, which characterizes the $H_{(r,n-2r,r)}$-distinction in the Galois case. We omit the elementary verification (and the obvious modification when n is even and $r = n/2$).

Lemma 8.6   *Assume that we are in the Galois case. Fix an integer $1 \le r \le \lfloor \frac{n}{2} \rfloor$ and assume that $r \ne n/2$. Let $\pi_1 \otimes \pi_2 \otimes \pi_3$ be an irreducible admissible representation of $M_{(r,n-2r,r)}$. Then $\pi_1 \otimes \pi_2 \otimes \pi_3$ is $H_{(r,n-2r,r)}$-distinguished if and only if π_2 is H_{n-2r}-distinguished and $\pi_3 \cong {}^{\theta_k}\widetilde{\pi}_1$.*

Proposition 8.7   *Let $\tau \cong \bigotimes_{i=1}^k \tau_i$ be an irreducible admissible regular representation of L. Assume that τ is L^θ-distinguished. Let P_Θ be a maximal Δ_0-standard parabolic subgroup corresponding to a maximal θ-split subset Θ of Δ_0. Let $y = ww_0^{-1}$, where $w \in \big[W_\Theta \backslash W_0 / W_\Omega \big]$. In both Case 1 and Case 2, $\mathcal{F}_\Theta^y(\tau)$ cannot be M_Θ^θ-distinguished.*

Proof   By assumption, we are in either Case 1 or Case 2 of Remark 8.3. In particular, we have that $y = ww_0^{-1}$, where $w \in \big[W_\Theta \backslash W_0 / W_\Omega \big]$, such that

$$\mathcal{F}_\Theta^y(\tau) = \iota_{M_\Theta \cap {}^y Q}^{M_\Theta}({}^y\tau) = \iota_{M_\Theta \cap {}^w M_\Omega}^{M_\Theta}({}^w\tau_0) \quad \text{and} \quad \tau_0 = {}^{w_0^{-1}}\tau = {}^\gamma\Big(\bigotimes_{i=1}^k \tau_i \Big)$$

is the representation of M_Ω corresponding to the representation τ of $L = L_{\Omega^{\mathrm{ell}}} = w_0 M_\Omega w_0^{-1}$, where γ simply does not appear in the linear case. By Lemma 7.15, τ is L^θ-distinguished if and only if each τ_i is H_{m_i}-distinguished for all $1 \le i \le k$. By Lemma 5.3, Θ is equal to Θ_r for some $1 \le r \le \lfloor \frac{n}{2} \rfloor$ and $M_\Theta = M_{\Theta_r} = M_r$. Without loss of generality, $r < n/2$. By Proposition 5.4, in the linear case $M_r = M_{(r,n-2r,r)}$ and in the Galois case $M_r = {}^\gamma M_{(r,n-2r,r)}$. We again use the shorthand $M_\bullet = M_{(r,n-2r,r)}$ and

$H_\bullet = H_{(r,n-2r,r)}$. The θ-fixed point subgroup of M_r is described in Proposition 5.7. In the linear case, we have that

$$M_r^\theta = H_\bullet = \{\mathrm{diag}(A, B, \theta_r(A)) : A \in G_r, B \in H_{n-2r}\},$$

while in the Galois case, we have $M_r^\theta = \gamma\gamma_\bullet^{-1} H_\bullet \gamma_\bullet \gamma^{-1}$, where

$$\gamma_\bullet = \gamma_{(r,n-2r,r)} = \mathrm{diag}(\gamma_r, \gamma_{n-2r}, \gamma_r) \in M_{(r,n-2r,r)}.$$

By Lemma 4.3, in the Galois case, $\mathcal{F}_\Theta^y(\tau)$ is M_Θ^θ-distinguished if and only if $^{\gamma^{-1}}\mathcal{F}_\Theta^y(\tau)$ is $H_\bullet$-distinguished.

Without loss of generality, we complete the proof in the Galois case. To obtain the proof in the linear case replace the application of Proposition 7.3 with Propostion 7.2.

Case 1. Suppose that Θ and Ω are associate and $w \in [W_\Theta \backslash W_0 / W_\Omega] \cap W(\Theta, \Omega)$. Then $M_\Theta = {}^w M_\Omega$ and $\mathcal{F}_\Theta^y(\tau) = {}^w \tau_0 = {}^{w\gamma}(\tau_1 \otimes \tau_2 \otimes \tau_3)$, where τ_1, τ_3 are representations of G_r and τ_2 is a representation of G_{n-2r}. By convention, $\gamma^{-1} w\gamma$ is a permutation matrix (*cf.* Lemma 5.10). Moreover, $^{\gamma^{-1}}(\mathcal{F}_\Theta^y(\tau)) = {}^{\gamma^{-1}w\gamma}(\tau_1 \otimes \tau_2 \otimes \tau_3)$ is equal to $\tau_{x(1)} \otimes \tau_{x(2)} \otimes \tau_{x(3)}$, for some compatible permutation x of $\{1, 2, 3\}$. By Lemma 8.6, $^{\gamma^{-1}}\mathcal{F}_\Theta^y(\tau)$ is $H_\bullet$-distinguished if and only if $\tau_{x(3)} \cong {}^{\theta_r}\widetilde{\tau_{x(1)}}$ and $\tau_{x(2)}$ is H_{n-2r}-distinguished. By Proposition 7.3, each τ_i satisfies $\tau_i \cong {}^{\theta_{m_i}}\widetilde{\tau_i}$. However, since τ is regular, Lemma 6.2 implies that the τ_i are pairwise inequivalent. In particular, we have that $^{\theta_r}\widetilde{\tau_{x(1)}} \cong \tau_{x(1)} \not\cong \tau_{x(3)}$. Therefore, $^{\gamma^{-1}}(\mathcal{F}_\Theta^y(\tau))$ is not $H_\bullet$-distinguished and $\mathcal{F}_\Theta^y(\tau)$ is not M_Θ^θ-distinguished.

Case 2. Suppose that $w \in [W_\Theta \backslash W_0 / W_\Omega]$ is such that $wM_\Omega w^{-1} \subset M_\Theta$ is a proper Levi subgroup. In this case, $\mathcal{F}_\Theta^y(\tau) = \iota_{M_\Theta \cap {}^w P_\Omega}^{M_\Theta} {}^w \tau_0$ and by [7, Theorem 6.6.1], the representation $^{\gamma^{-1}}(\mathcal{F}_\Theta^y(\tau))$ is an irreducible unitary representation of $M_\bullet$. Indeed, $M_\Theta = {}^\gamma M_\bullet$ and $M_\Omega = {}^\gamma M$, with $M_\bullet = M_{(r,n-2r,r)}$ and $M = M_{(m_1,\ldots,m_k)}$; moreover, $w' = \gamma^{-1} w\gamma$ is an element of $[W_{M_\bullet} \backslash W / W_M]$ and conjugates M into $M_\bullet$. Writing P for $P_{(m_1,\ldots,m_k)}$, we have

$$^{\gamma^{-1}}(\mathcal{F}_\Theta^y(\tau)) = {}^{\gamma^{-1}}\Big(\iota_{M_\Theta \cap {}^w P_\Omega}^{M_\Theta} {}^w \tau_0\Big) = {}^{\gamma^{-1}}\Big(\iota_{{}^\gamma M_\bullet \cap {}^{w\gamma} P}^{{}^\gamma M_\bullet} {}^{w\gamma}(\tau_1 \otimes \cdots \otimes \tau_k)\Big)$$

$$\cong \iota_{M_\bullet \cap {}^{w'} P}^{M_\bullet} {}^{w'}(\tau_1 \otimes \cdots \otimes \tau_k),$$

and this representation is isomorphic to $\pi_1 \otimes \pi_2 \otimes \pi_3$, where π_1, π_2 are irreducible admissible representations of G_r and π_2 is an irreducible admissible representation of G_{n-2r}. Since $w' \in [W_{M_\bullet} \backslash W / W_M]$, by Proposition 4.8, the group $M_\bullet \cap {}^{w'} P$ is a parabolic subgroup of $M_\bullet$ (a product of parabolic subgroups on each block of $M_\bullet$). It follows that each of the π_j, $j = 1, 2, 3$, are irreducibly induced representations of the form $\tau_{a_1} \times \cdots \times \tau_{a_r}$, for some subset of the representations $\{\tau_1, \ldots, \tau_k\}$. Again, by Lemma 8.6 $^{\gamma^{-1}}(\mathcal{F}_\Theta^y(\tau))$ is $H_\bullet$-distinguished if and only if π_2 is G_{n-2r}-distinguished and $\pi_3 \cong {}^{\theta_r}\widetilde{\pi_1}$. Suppose that $\pi_1 = \tau_{a_1} \times \cdots \times \tau_{a_l}$, and $\pi_3 = \tau_{b_1} \times \cdots \times \tau_{b_s}$. Then we have that $\widetilde{\pi_1} \cong \widetilde{\tau_{a_1}} \times \cdots \times \widetilde{\tau_{a_l}} \cong {}^{\theta_{m_{a_1}}}\tau_{a_1} \times \cdots \times {}^{\theta_{m_{a_l}}}\tau_{a_l} \cong {}^{\theta_r}\pi_1$. Moreover, we have that $\pi_1 \cong {}^{\theta_r}\widetilde{\pi_1}$. Since τ is regular, by Lemma 6.2, the discrete series τ_i are pairwise inequivalent; therefore, by [42, Theorem 9.7(b)], we have that $\pi_1 \not\cong \pi_3$. That is, we have $^\theta \widetilde{\pi_1} \cong \pi_1 \not\cong \pi_3$. In particular, $^{\gamma^{-1}}(\mathcal{F}_\Theta^y(\tau))$ is not $H_\bullet$-distinguished, and $\mathcal{F}_\Theta^y(\tau)$ is not M_Θ^θ-distinguished.

■

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