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GRAPHS AND k-SOCIETIES(1)

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A graph G is a couple (X, R) where X is a set, $R \subset X \times X$. If G is an undirected graph without loops (R a symmetric irreflexive relation), we can interpret G as a couple (X, R), where R is a set of two-element subsets of X, i.e. $R \subset \mathscr{P}(X)$. This interpretation is generalized in the notion of society.

A *society* $\mathscr{G}$ is a couple (X, R), where $R \subset \mathscr{P}(X)$; a *k-society* is a society (X, R) with $|A| = k$ for each $A \in R$.

Let $\mathscr{R}$ be the category of all graphs and all compatible mappings. Compatible mappings between two societies are defined similarly to those between two graphs (mapping $f: X \to Y$ is a *compatible mapping* of the society $\mathscr{G} = (X, R)$ into the society $\mathscr{H} = (Y, S)$ if $A \in R \Rightarrow f(A) \in S$). Let the category of all k-societies and all their compatible mappings be $\mathscr{S}_k$. Obviously $\mathscr{S}_2$ is the category of all undirected graphs without loops, hence $\mathscr{S}_2 \neq \mathscr{R}$. Let $\mathscr{S}$ be the category of all societies and all compatible mappings.

In [2] a full embedding of $\mathscr{R}$ into $\mathscr{S}_2$ is given, which is of course also a full embedding of $\mathscr{R}$ into $\mathscr{S}$. In this paper we give a full embedding $\mathscr{R} \to \mathscr{S}_k$ for every $k \geq 2$, thus we prove that each category $\mathscr{S}_k$ $(k \geq 2)$ is binding (cf. [3]). The problem was suggested by Z. Hedrlín.

For the notions and definitions concerning graphs, see [1].

Our method is based on the idea that, relative to compatible mappings, certain graphs behave like k-societies.

Let $k \geq 2$ be fixed from now on.

Let $\mathscr{G} = (X, R)$ be a k-society. A graph $\mathscr{G}^* = (X, R^*)$ is naturally associated with $\mathscr{G}$, where

$$R^* = \{(a, b) \mid a \neq b \text{ and there exists an } A \in R \text{ such that } a \in A, b \in A\}.$$

Obviously the graph $\mathscr{G}^*$ satisfies the following conditions:

(i) $\mathscr{G}^*$ is undirected and has no loops,

(*) (ii) each edge of $\mathscr{G}^*$ belongs to some complete k-subgraph of $\mathscr{G}^*$ (complete k-subgraph is a complete subgraph of cardinality k).

Let $C(\mathscr{G}, \mathscr{H})$ be the set of all compatible mappings $\mathscr{G} \to \mathscr{H}$; let us write $C(\mathscr{G})$ instead of $C(\mathscr{G}, \mathscr{G})$.

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(1) Sometimes, instead of society (k-society), the words set-system (uniform set-system) or hypergraph are used.

(2) Part of this paper was written while the authors were supported by the National Research Council of Canada.

LEMMA 1. *Let $\mathscr{G}=(X, R), \mathscr{H}=(Y, S)$ be k-societies. Then $C(\mathscr{G}, \mathscr{H}) \subset C(\mathscr{G}^*, \mathscr{H}^*)$.*

Proof. Take $f \in C(\mathscr{G}, \mathscr{H})$. If $(a, b) \in R^*$ then there exists an $A \in R$ such that $a, b \in A$. We have $f(A)=B \in S$, hence $f(a) \neq f(b)$, $(f(a), f(b)) \in S^*$.

In general, having a graph G with the property (*) we can find several societies $\mathscr{G}$ such that $\mathscr{G}^*=G$. Among those there is always a society $\mathscr{G}$ with $C(\mathscr{G})=C(G)$.

A set $A \subset X$ in a graph $G=(X, R)$ is called a carrier of a complete subgraph of G if $(A, R \cap A \times A)$ is a complete graph.

Let $G=(X, R)$ be a graph with (*). We define the society $G^\circ=(X, R^\circ)$ as follows: $A \in R^\circ$ if and only if A is the carrier of a complete k-subgraph of G. Indeed we have $(G^\circ)^*=G$, but not necessarily $(\mathscr{G}^*)^\circ=\mathscr{G}$.

LEMMA 2. *Let $G=(X, R)$, $H=(Y, S)$ be graphs satisfying (*). Then $C(G, H) = C(G^\circ, H^\circ)$.*

Proof. $C(G, H) \supset C(G^\circ, H^\circ)$ by Lemma 1. Let $f \in C(G, H)$, $A \in R^\circ$. Since A is the carrier of a complete k-subgraph of G and f is compatible, $f(A)$ is the carrier of a complete k-subgraph of H, $f(A) \in S^\circ$.

In other words: if we denote the category of all graphs satisfying (*) and all their compatible mappings by $\mathscr{S}_2^k$, then we can define a functor $\Phi_1: \mathscr{S}_2^k \to \mathscr{S}_k$ by $\Phi_1(G)=G^\circ$, $\Phi_1(f)=f$ and by Lemma 2, Φ_1 is a full embedding of $\mathscr{S}_2^k$ into $\mathscr{S}_k$. Thus for the construction of a full embedding of $\mathscr{R}$ into $\mathscr{S}_k$, it suffices to find a full embedding Φ_2 of $\mathscr{R}$ into $\mathscr{S}_2^k$. Then $\Phi_1 \circ \Phi_2$ will be the desired embedding.

E. Mendelsohn in [3] gives a general construction, which in slight modification will provide us with the full embedding Φ_2. He defines the *šíp- součin* (*šíp-product*) $(X, R, A, B) * (Y, S)$ of a '*šíp*' (X, R, A, B) (i.e. graph (X, R) with distinguished two isolated subsets A, B and an isomorphism i of $(A, R \cap A \times A)$ onto $(B, R \cap B \times B)$) and an arbitrary graph (Y, S). Intuitively the šíp-product is obtained by replacing every arrow of the graph (Y, S) with the starting point a and endpoint b by a copy of the graph (X, R), where the set A 'replaces' the point a and B 'replaces' b. Isolated points of (Y, S) are replaced by copies of $(A, R \cap A \times A)$ by this definition, and loops by graphs (X, R, A, A), where (X, R, A, A) is the quotient graph of (X, R, A, B) under the equivalence generated by $x \sim y \Leftrightarrow x \in A$, $y \in B$ and $i(x)=y$. Let η denote the natural compatible mapping (X, R, A, B) onto (X, R, A, A).

If $f \in C((Y, S), (Y', S'))$ then one can define a mapping $f^*: (X, R, A, B) * (Y, S) \to (X, R, A, B) * (Y', S')$ by

$$f^*([(a, y)]) = [(a, f(y))] \quad \text{for } a \in A, y \in Y$$
$$f^*([(x, s)]) = [(x, {}^2f(s))] \quad \text{for } x \in X, s \in S,$$

where ${}^2f((c, d))=(f(c), f(d))$ Obviously

$$f^* \in C((X, R, A, B) * (Y, S), (X, R, A, B) * (Y', S')),$$

furthermore $1^*_{(Y,S)}=1_{(X,R,A,B)*(Y,S)}$ and $(f \circ g)^*=f^* \circ g^*$.

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A šíp (X, R, A, B) is *strongly rigid* (cf. [3]) if and only if for every graph (Y, S)

(1) $f \in C((A, R \cap A \times A), (X, R, A, B) * (Y, S)) \Rightarrow f(a) = [(a, y)]$ for some fixed $y \in Y$;

(2) $f \in C((X, R), (X, R, A, B) * (Y, S)) \Rightarrow$ either $f(x) = [(x, s)]$ for a fixed $s \in S_1$ or $f(x) = [\eta(x), s]$ for a fixed $s \in S_2$ (here S_1 is the irreflexive part of S, $S_2 = S - S_1$);

(3) $f \in C((X, R, A, A), (X, R, A, B) * (Y, S)) \Rightarrow f([x]_\eta) = [(x, s)]$ for a fixed $s \in S_2$.

E. Mendelsohn shows that if (X, R, A, B) is strongly rigid, then the correspondence $\Phi(f) = f^*$ is an isomorphism between $C((Y, S))$ and $C((X, R, A, B) * (Y, S))$ (Theorem 1 in [3]). By the same argument one can prove the following lemma.

LEMMA 3. *If (X, R, A, B) is strongly rigid, then for any $g \in C((X, R, A, B) * (Y, S), (X, R, A, B) * (Y', S'))$ there exists an $f \in C((Y, S), (Y', S'))$ such that $f^* = g$.*

If we define the functor Φ_2 by $\Phi_2(G) = (X, R, A, B) * G$ and $\Phi_2(f) = f^*$ then we clearly have a full embedding $\mathscr{R} \to \mathscr{S}_2^k$, provided *the šíp (X, R, A, B) is strongly rigid and satisfies property* (*) (if the šíp satisfies (*), then so do all its šíp-products).

Let us first introduce some remarks about undirected graphs without loops.

If G is such a graph we denote its chromatic number by $\chi(G)$. For every compatible mapping f, $\chi(f(G)) \geq \chi(G)$. Thus every compatible mapping maps a complete n-graph onto a complete n-graph.

We define $|G|$ to be $|X|$ for $G = (X, R)$, and write 1_G for 1_X. $W = \{K_1, K_2, \ldots, K_r\}$ is an *n-complete path of length r* if $n \geq 3$ and $K_1, \ldots, K_r$ are complete n-graphs such that $|K_i \cap K_{i+1}| \geq n-1$ for $i = 1, 2, \ldots, r-1$.

Two points x, y are *joined by an n-complete path in the graph G* if there exists an n-complete path $W = \{K_1, \ldots, K_r\}$ such that each K_i is a subgraph of G and $x \in K_1$, $y \in K_r$.

Let $x, y \in G$, $x \neq y$. Let $d_n(x, y)$ denote the length of a shortest n-complete path in G joining x and y if such a path exists, $d_n(x, y) = 0$ otherwise.

Remember that now we are considering undirected graphs without loops.

LEMMA 4. *Let $f \in C(G, H)$. If $d_n(x, y) > 0$ and $f(x) \neq f(y)$, then $d_n(f(x), f(y)) > 0$ and $d_n(f(x), f(y)) \leq d_n(x, y)$.*

The proof can be done by induction.

In particular: if $d_n(x, y) > 0$ for any two points in G, then there is no compatible mapping of G onto a graph with cut point.

A graph G is called rigid if $C(G) = \{1_G\}$.

Now we start to construct a strongly rigid šíp satisfying (*). Let m, n, l be natural numbers. Denote by $I^m_{n,l}$ the following graph (X, R): $X = \{1, 2, \ldots, m+1\}$, $(i, j) \in R \Leftrightarrow$ either $0 < |i-j| \leq n-1$ or $i = 1, j \geq m+1-l$ or $j = 1, i \geq m+1-l$. Call the triple m, n, l admissible if $n > l+2$ and m is a nontrivial multiple of n.

LEMMA 5. *If m, n, l is admissible then $I^m_{n,l}$ is rigid.*

Proof. Note that every edge of $I^m_{n,l}$ belongs to a complete $(l+2)$-graph, furthermore $d_n(x, y) > 0$ for any $x, y \in I^m_{n,l}$ and even each edge except $\{(1, m+1), (1, m), \ldots,$

$(1, m+1-l)\}$ belongs to a complete n-graph. It is easy to check that $\chi(I^m_{n,l})=n+1$, while chromatic number of each proper subgraph of $I^m_{n,l}$ is $<n+1$. Thus every compatible mapping $I^m_{n,l} \to I^m_{n,l}$ is an automorphism (since it is onto $I^m_{n,l}$). Let h be an automorphism of $I^m_{n,l}$. The set of mentioned exceptional edges is mapped onto itself, thus $h(1)=1$, and an easy argument concerning degrees ($\deg h(x) = \deg x$ for any automorphism h) yields $h(m+1)=m+1$, $h(m)=m, \ldots, h(m+1-l) = m+1-l$. Each point i, $n<i<m+2-n$ is then fixed (i.e. $h(i)=i$) by Lemma 4 since $f(i)<i \Rightarrow d_n(m+1, f(i)) > d_n(m+1, i)$ and $f(i)>i \Rightarrow d_n(1, f(i)) > d_n(1, i)$. The remaining points are fixed again—it suffices to consider their degrees. Thus h is the identity.

Define the šíp $S(m, m', l, l', n) = (X, R, A, B)$ as follows:

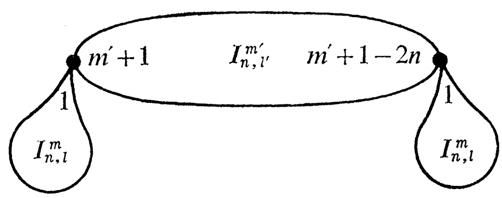
$$\bar{X} = \{a_1, a_2, \ldots, a_{m+1}\} \dot{\cup} \{b_1, b_2, \ldots, b_{m+1}\} \dot{\cup} \{h_1, h_2, \ldots, h_{m'+1}\}$$

$$\bar{R} = \{(a_i, a_j) \mid (i,j) \text{ is an edge in } I^m_{n,l}\} \dot{\cup}$$
$$\{(b_i, b_j) \mid (i,j) \text{ is an edge in } I^m_{n,l}\} \dot{\cup}$$
$$\{(h_i, h_j) \mid (i,j) \text{ is an edge in } I^{m'}_{n,l'}\}$$

and $(X, R) = (\bar{X}, \bar{R})/\sim$

where the equivalence $\sim$ is defined by $a_1 \sim h_{m'+1}$, $h_{m'+1-2n} \sim b_1$.

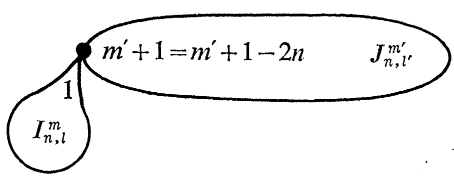
We put $A = \{[a_1], \ldots, [a_{m+1}]\}$, $B = \{[b_1], \ldots, [b_{m+1}]\}$ and the isomorphism $i: a_j \mapsto b_j$. Thus our šíp is formed from two copies of $I^m_{n,l}$ and one copy of $I^{m'}_{n,l'}$ so that the point $m'+1$ from the copy of $I^{m'}_{n,l'}$ is identified with the point 1 from the first copy of $I^m_{n,l}$ and the point $m'+1-2n$ from $I^{m'}_{n,l'}$ is identified with the point 1 from the second copy of $I^m_{n,l}$.



(X,R,A,B)

Note that $(A, R \cap A \times A) \cong I^m_{n,l} \cong (B, R \cap B \times B)$.

Obviously (X, R, A, A) has only one copy of $I^m_{n,l}$ and in the copy of $I^{m'}_{n,l'}$ the points $m'+1$ and $m'+1-2n$ are identified. Let us denote by $J^{m'}_{n,l'} = I^{m'}_{n,l'}/\sim$ where $m'+1 \sim m'+1-2n$.



(X,R,A,A)

LEMMA 6. *Let m', n, l' be admissible, let $m'/n \geq 4$. Then*

(1) $J^{m'}_{n,l'}$ *is rigid*

(2) *The natural mapping* $\xi: I^{m'}_{n,l'} \to I^{m'}_{n,l'}/\sim$ *is the only* $f \in C(I^{m'}_{n,l'}, J^{m'}_{n,l'})$ *with* $f(m'+1) = f(m'+1-2n)$

(3) $C(J^{m'}_{n,l'}, I^{m'}_{n,l'}) = \phi$

(4) *If* $m < m'-2n$ *and* m, n, l *is admissible, then* $C(I^{m}_{n,l}, J^{m'}_{n,l'}) = \phi$.

Proof. (1) The graph $J^{m'}_{n,l'}$ has vertices $[1], [2], \ldots, [m'+1-2n] = [m'+1]$, $[m'+2-2n], \ldots, [m'-1], [m']$. Let J be the full subgraph of $J^{m'}_{n,l'}$ on the vertices $[1], \ldots, [m'+1-2n]$.

One can easily note that J is $n+1$ chromatic, that all its subgraphs have chromatic numbers $< n+1$, and moreover that each $n+1$ chromatic subgraph of $J^{m'}_{n,l'}$ contains J.

This allows us to see that every $f \in C(J^{m'}_{n,l'})$ maps J onto J and (considering degrees in J) that f/J is either the identity, or the mapping which interchanges the vertices $[i]$ and $[m'+2-2n-i]$. In both cases the set $\{[m'+1-l'], [m'+2-l'], \ldots, [m']\}$ is mapped into itself (as the set of common neighbours of $[1]$ and $[m'+1-2n]$) and thus the complete n-graph K on the vertices $[m'+2-n], [m'+3-n], \ldots, [m'+1]$ is mapped onto a complete n-graph containing the vertices $[m'+1-l'], [m'+2-l'], \ldots, [m']$ and $[m'+1] = [m'+1-2n]$ or $[1]$ (remember $n > l'+2$). Since there is no complete n-graph containing both $[m']$ and $[1]$ in $J^{m'}_{n,l'}$ the interchanging $[i] \leftrightarrow [m'+2-i]$ is impossible (thus $f/J = 1_J$) and $f(K) = K$. The vertex $[m'+2-n] = f([m'+2-n])$ because it is the vertex with d_n-distance from $[1]$ smaller than any other point of K. The rest of the argument is similar to Lemma 5.

(2) Obviously $\xi = \eta / I^{m'}_{n,l'}$ for η from the definition of the šíp. Let $f \in C(I^{m'}_{n,l'}, J^{m'}_{n,l'})$ such that $f(m'+1) = f(m'+1-2n)$. Then we can define a mapping $j: J^{m'}_{n,l'} \to J^{m'}_{n,l'}$ by $i([x]) = f(x)$; j is compatible and $f = \xi \circ j$. Thus by (1) $f = \xi$.

(3) Since $\chi(J^{m'}_{n,l'}) = n+1$ and chromatic numbers of all proper subgraphs of $I^{m'}_{n,l'}$ are $< n+1$, any compatible f maps $J^{m'}_{n,l'}$ onto $I^{m'}_{n,l'}$, which contradicts the fact that $|J^{m'}_{n,l'}| < |I^{m'}_{n,l'}|$.

(4) If $m < m'-2n$ then $|I^{m}_{n,l}| = m+1 < m'+1-2n$, while all the subgraphs of $J^{m'}_{n,l'}$ with the cardinality less than $m'+1-2n$ have their chromatic numbers $< n+1$.

LEMMA 7. *If m, n, l and m', n, l' are admissible and $l < l'$ then* $C(I^{m'}_{n,l'}, I^{m}_{n,l}) = \phi$.

Proof. Again the chromatic numbers reasoning implies that each $f \in C(I^{m'}_{n,l'}, I^{m}_{n,l})$ is a mapping onto; on the other hand the edge $(1, m+1)$ does not belong to any complete $l'+2$ graph, while all edges of $I^{m'}_{n,l'}$ do so.

LEMMA 8. *Let the triples m, n, l and m', n, l' be admissible. If $m'/n \geq 4$, $m < m'-2n$, and $l < l'$, then the šíp $S(m, m', l, l', n)$ is strongly rigid.*

Instead of a very formal proof we rather give one somewhat more intuitive. The

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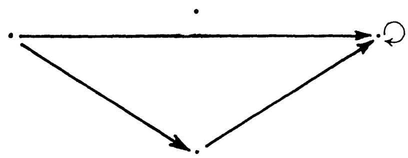
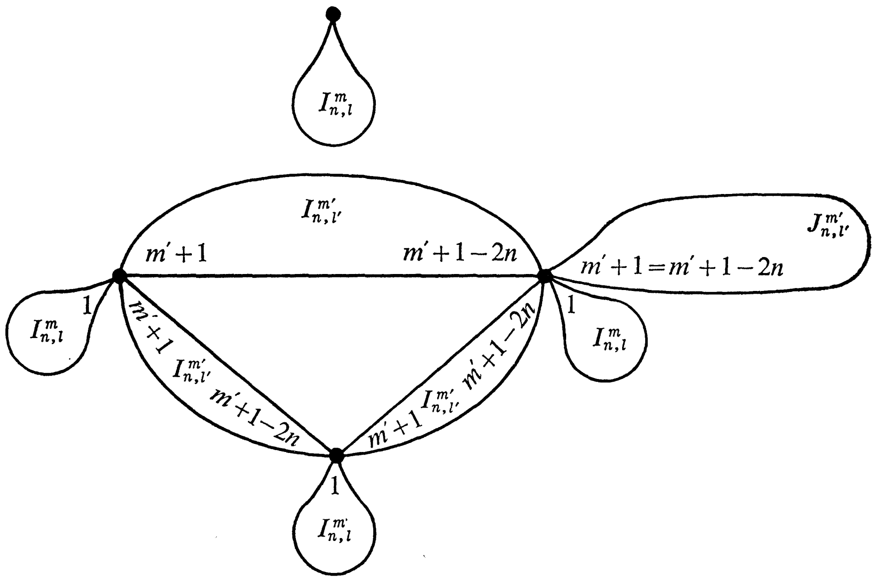
šíp-product $S(m, m', l, l', n) * (Y, S)$ consists of copies of $I^m_{n,l}$, $I^{m'}_{n,l'}$, and $J^{m'}_{n,l'}$ connected only by 'cut points' as in the following example:



(Y,S)



$S(m,m',l,l',n)*(Y,S)$

Let us verify the conditions in definition of the strongly rigid šíp.

(1) Any compatible mapping of $(A, R \cap A \times A) = I^m_{n,l}$ into $S(m, m', l, l', n) * (Y, S)$ maps $I^m_{n,l}$ into a graph without cutpoints, thus a subgraph of $I^{m'}_{n,l'}$, or $J^{m'}_{n,l'}$, or $I^m_{n,l}$. The second case is impossible [by Lemma 6 (4)], and therefore, also the first case is impossible (if $f: I^m_{n,l} \to I^{m'}_{n,l'}$ is compatible, then $\xi \circ f$ is compatible [by Lemma 6 (2)]) and by Lemma 5 we are done.

(2) The two copies of $I^m_{n,l}$ in (X, R, A, B) can either be mapped onto two different copies of $I^m_{n,l}$ in the šíp-product and then $I^{m'}_{n,l'}$ is mapped onto the copy of $I^{m'}_{n,l'}$ spanning them and we are again done by Lemma 5, or they can be mapped onto one copy of $I^m_{n,l}$ and, by Lemma 5, the points $m'+1$ and $m'+1-2n$ from $I^{m'}_{n,l'}$ are mapped onto one point, thus $I^{m'}_{n,l'}$ cannot be mapped into $I^m_{n,l}$ by Lemma 7 and by Lemma 6 (2), we are finished.

(3) Is again obvious from Lemmas 6 (1) and (3), and 7 with 6 (2).

THEOREM. *Let $k \geq 2$. There exists a full embedding of the category of all graphs into the category of all k-societies* (*i.e.* $\mathscr{R} \to \mathscr{S}_k$).

Proof. The šíp $S(2(k+2), 5(k+2), k-1, k, k+2)$ is strongly rigid by Lemma 8, and obviously satisfies (*).

REMARKS and COROLLARIES. (1) For every graph G we have constructed a k-society $\mathcal{G}$ with $C(\mathcal{G}) \cong C(G)$ (and $|\mathcal{G}| \geq |G|$).

Using results from references [2] and [5]:

(2) Each $\mathcal{S}_k$ is binding, in particular for every monoid S^1 there is a k-society $\mathcal{G}$ such that $C(\mathcal{G})$ is isomorphic to S^1.

(3) For every cardinal α there is a rigid k-society $\mathcal{G}$ such that $|\mathcal{G}| \geq \alpha$. Rigid k-society is defined similarly as rigid graph, i.e. $C(\mathcal{G}) = \{1_{\mathcal{G}}\}$.

Finally let us note that

(4) The full embedding Φ_1 is a realization (for definition see [4]) of $\mathcal{S}_2^k$ in $\mathcal{S}_2$.

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