Simply connected algebras of polynomial growth

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Abstract. The representation theory of polynomial growth strongly simply connected finite dimensional algebras over an algebraically closed field is established.

Introduction

Throughout by an algebra we mean a finite dimensional K-algebra with an identity over an algebraically closed field K. By a module over an algebra A we mean a right A-module of finite K-dimension.

From Drozd's remarkable Tame and Wild Theorem [21] the class of algebras may be divided into two disjoint classes. One class consists of tame algebras for which the indecomposable modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory is as complicated as the study of finite dimensional vector spaces together with two non-commuting endomorphisms, for which the classification is a well-known unsolved problem. Hence, we can hope to classify the modules only for tame algebras. Frequently, applying covering techniques, we may reduce the study of modules over tame algebras to that for the corresponding simply connected algebras. Here, we are concerned with the representation theory of tame simply connected algebras.

Among the tame algebras we may distinguish the class of representation-finite algebras, having only finitely many isomorphism classes of indecomposable modules. This class of algebras is presently rather well understood (see [7], [13], [15], [16]). In particular, we know that if a representastion-finite algebra A has a faithful indecomposable module X, then A admits a Galois covering $\tilde{A}$ which has an indecomposable module $\tilde{X}$ whose push-down is X and whose support is simply connected (see [14], [15], [16]). The class of sincere representation-finite simply connected algebras has been classified completely by quivers and relations. It consists of 24 infinite regular families of Bongartz [10] (whose Gabriel quiver has at least 14 vertices), being of considerable theoretical interest, and 16.344 exceptional

algebras (see [18], [19], [40]). We note also that, for a representation-finite simply connected algebra A, the Auslander-Reiten quiver Γ_A of A is directed, and hence the indecomposable A-modules are uniquely determined by their dimension-vectors, being the positive roots of the Tits form q_A of A (see [11], [39]).

The representation theory of tame representation-infinite algebras is only emerging. At present the most accessible seem to be the (tame) algebras of polynomial growth, for which there exists an integer m such that the number of one-parameter families is bounded, in each dimension d, by d^m. This class of algebras has been the subject of intensive research over the last 15 years. It contains the class of domestic algebras for which there is a constant bound on the number of one-parameter families in each dimension. Important classes of polynomial growth algebras are representation-infinite tilted algebras of Euclidean type and tubular algebras for which a complete representation theory was established in the Ringel's book [39]. Later O. Kerner proved in [24] that all tame tilted algebras are domestic, and hence of polynomial growth. Moreover, it was shown in [39] that the support of any indecomposable module which does not lie on a cycle of nonzero nonisomorphisms between indecomposable modules (directing module) is a tilted algebra. Hence, all tame algebras having sincere directing indecomposable modules are tilted and domestic. This class of tame algebras has been investigated by J. A. de la Peña in [33], [34]. On the other hand, by a result due to L. Peng–J. Xie [30] and the author [45], the Auslander–Reiten quiver of an algebra A has at most finitely many $D\mathrm{Tr}$-orbits containing directing modules. Hence, in order to develop the representation theory of arbitrary tame algebras, we need informations on cycles of indecomposable modules. This was the main objective of studies in [3], [4], [5], [50] where a theory of algebras with finite cycles of indecomposable modules has been developed.

One of the main objectives of this paper is to describe the structure of the category $\mathrm{mod}\, A$ of modules over an arbitrary strongly simply connected algebra A of polynomial growth. Recall that an algebra A without oriented cycles in its Gabriel quiver is called strongly simply connected [44] if, for any convex subcategory C of A, the first Hochschild cohomology group $H^1(C, C)$ vanishes. For A of finite representation type this coincides with the simply connectedness in the sense of [15] (the geometric realization $|\Gamma_A|$ of the Auslander–Reiten quiver Γ_A of A is simply connected). We characterize the polynomial growth of strongly simply connected algebras (Theorem 4.1) by properties of the infinite radical $\mathrm{rad}^\infty(\mathrm{mod}\, A)$ of $\mathrm{mod}\, A$, the shape and behaviour of the Auslander–Reiten components in $\mathrm{mod}\, A$, and properties of cycles of indecomposable modules in $\mathrm{mod}\, A$. In particular, we get that, for a polynomial growth strongly simply connected A, all but finitely many $D\mathrm{Tr}$-orbits in Γ_A are periodic, the components of Γ_A are standard and partially ordered (in $\mathrm{mod}\, A$), and each of them is either directed or a glueing of finitely many coils [4] by directed parts. Moreover, we obtain that a strongly simply connected algebra is of polynomial growth if and only if A does not contain a convex subcategory which is hypercritical or pg-critical. This gives a handy criterion for

the polynomial growth of strongly simply connected algebras. In the course of our proof of Theorem 4.1 (Section 6) we establish also the following important fact. For an arbitrary strongly simply connected algebra A of polynomial growth, there are tame coil enlargements of critical algebras (coil algebras) [5] $B_1, \ldots, B_n$ which are convex subcategories of A and whose indecomposable modules exhaust all but finitely many (up to isomorphism) indecomposable A-modules. Moreover, the remaining indecomposable modules are directing, and hence their supports are tame tilted algebras. This open a door to a complete classification of indecomposable modules over polynomial growth strongly simply connected algebras. We prove here (Corollary 4.7) that the one-parameter families of indecomposable modules over such algebras are given by the one-parameter families of indecomposable modules over critical and tubular algebras. In Section 4 we give also several characterizations of domestic strongly simply connected algebras (Corollary 4.3). In particular, we get that a strongly simply connected algebra A is domestic if and only if A does not contain a convex subcategory which is hypercritical, pg-critical or tubular.

We shall mention that our Theorem 4.1 is essentially applied in [52] to establish a handy criterion for the polynomial growth (respectively, domestic type) of algebras having strongly simply connected Galois coverings.

The results of this paper were partially announced in [46], [49], [51] and presented during the Conferences in Oberwolfach (1991), Ottawa (1992), Mexico (1994) and the Universities of Bielefeld, Paderborn, Trondheim, Paris VI, Nagoya, Tsukuba, Sherbrooke, Beijing, Singapore, Sao Paulo, Mexico. The author is grateful to these institutions for providing the possibility of discussing our investigations.

1. Terminology and notation

Throughout this paper K will denote a fixed algebraically closed field. By an algebra is meant an associative finite dimensional K-algebra with an identity, which we shall assume (without loss of generality) to be basic. For such an algebra A, there exists an isomorphism $A \simeq KQ/I$, where KQ is the path algebra of the *Gabriel quiver* $Q = Q_A$ of A and I is an admissible ideal in KQ. Equivalently, $A = KQ/I$ may be considered as K-category whose class of objects is the set Q_0 of vertices of Q, and the set of morphisms $A(x, y)$ from x to y is the quotient of the K-space $KQ(x, y)$, formed by the K-linear combinations of paths in Q from x to y, by the subspace $I(x, y) = KQ(x, y) \cap I$. An algebra A with Q_A having no oriented cycle is said to be *triangular*. A full subcategory C of A is said to be *convex* if any path in Q_A with source and target in Q_C lies entirely in Q_C.

For an algebra A, we denote by mod A the category of finitely generated right A-modules and by ind A its full subcategory consisting of indecomposable modules. The term A-module is used for an object of mod A if not specified otherwise.

We shall denote by rad(mod A) the *Jacobson radical* of mod A, that is, the ideal of mod A generated by all noninvertible morphisms in ind A. Then the *infinite radical* $\mathrm{rad}^\infty(\mathrm{mod}\,A)$ of mod A is the intersection of all powers $\mathrm{rad}^i(\mathrm{mod}\,A)$, $i \geqslant 1$, of rad(mod A). We say that $\mathrm{rad}^\infty(\mathrm{mod}\,A)$ is *nilpotent* (respectively, *locally nilpotent*) if there exists $m \geqslant 1$ such that $(\mathrm{rad}^\infty(\mathrm{mod}\,A))^m = 0$ (respectively, $(\mathrm{rad}^\infty(M,M))^m = 0$ for any module M in ind A). A *path of length t* in mod A is a sequence of nonzero nonisomorphisms

$$(*) \qquad M_0 \xrightarrow{f_1} M_1 \to \cdots \to M_{t-1} \xrightarrow{f_t} M_t$$

with $t \geqslant 1$ and the modules M_i in ind A. Such a path is said to be *infinite* if $f_i \in \mathrm{rad}^\infty(\mathrm{mod}\,A)$ for some $1 \leqslant i \leqslant t$. A *cycle* of length t is a path $(*)$ with $M_0 \simeq M_t$. A module M from ind A is said to be *directing* if it does not lie on a cycle in mod A. For a module M in mod A, we denote by $\underline{\dim}\,M$ the *dimension-vector* $(\dim_K M(i))_{i \in Q_0}$ of M. The *support* supp M of a module $M \in \mathrm{mod}\,A$ is the full subcategory of A given by all vertices $i \in Q_A$ such that $M(i) \neq 0$. We denote by $D\colon \mathrm{mod}\,A \to \mathrm{mod}\,A^{\mathrm{op}}$ the standard duality $D = \mathrm{Hom}_K(-, K)$, where A^{op} denotes the oposite algebra of A.

For an algebra A, we shall denote by Γ_A the *Auslander–Reiten quiver* of A, and by τ_A and τ_A^- the Auslander–Reiten translations $D\mathrm{Tr}$ and $\mathrm{Tr}D$, respectively. We shall identify the vertices of Γ_A with the corresponding indecomposable A-modules. A connected component $\mathcal{C}$ of Γ_A is said to be *standard* if the full subcategory of mod A formed by all modules of $\mathcal{C}$ is equivalent to the mesh-category $K(\mathcal{C})$ of $\mathcal{C}$ (see [15], [39]). Moreover, following [47], a connected component $\mathcal{C}$ of Γ_A is said to be *generalized standard* if $\mathrm{rad}^\infty(X,Y) = 0$ for all modules X and Y in $\mathcal{C}$. It is known [28] that every standard component is generalized standard. For the components which do not contain both a projective module or an injective module, the converse implication is also true [48] but in general it is not the case. For a subquiver Γ of Γ_A we denote by supp Γ the *support* of Γ, that is, the full subcategory of A formed by all vertices x of Q_A such that the simple A-modules $S(x)$ at x occur as the composition factors of modules lying in Γ. Finally, the *component quiver* Σ_A of A [46] is defined as follows: the vertices of Σ_A are the connected components of Γ_A, and two components $\mathcal{C}$ and $\mathcal{D}$ are connected in Σ_A by an arrow $\mathcal{C} \to \mathcal{D}$ if and only if $\mathrm{rad}^\infty(X,Y) \neq 0$ for some modules X in $\mathcal{C}$ and Y in $\mathcal{D}$. Observe that a connected component $\mathcal{C}$ of Γ_A is generalized standard if and only if Σ_A has no loop at $\mathcal{C}$.

Let A be an algebra and $K[x]$ the polynomial algebra in one variable. Following [21] A is said to be *tame* if, for any dimension d, there exists a finite number of $K[x]$–A-bimodules M_i, $1 \leqslant i \leqslant n_d$, which are finitely generated and free as left $K[x]$-modules, and all but a finite number of isomorphism classes of indecomposable A-modules of dimension d are of the form $K[x]/(x-\lambda) \otimes_{K[x]} M_i$ for some $\lambda \in K$ and some i. Let $\mu_A(d)$ be the least number of $K[x]$–A-bimodules satisfying the above conditions. Then A is said to be of *polynomial growth* (respectively, of *linear*

growth) if there exists a positive integer m such that $\mu_A(d) \leqslant d^m$ (respectively, $\mu_A(d) \leqslant md$) for any $d \geqslant 1$ [43]. Further, A is *domestic* if $\mu_A(d) \leqslant m$ for a fixed $m \geqslant 1$ and any $d \geqslant 1$. From the validity of the second Brauer–Thrall conjecture we know that A is representation-finite if and only if $\mu_A(d) = 0$ for any $d \geqslant 1$. There is still an open problem whether all polynomial growth algebras are of linear growth.

We recall that, if B is a representation-infinite tilted algebra of Euclidean type Δ, then one of the following holds:

(a) B is a domestic tubular extension of a tame concealed algebra C and

$$\Gamma_B = \mathcal{P}_0 \vee \mathcal{T}_0 \vee \mathcal{I}_0,$$

where $\mathcal{P}_0$ is the preprojective component of Γ_C, $\mathcal{T}_0$ is a $\mathbb{P}_1(K)$-family of pairwise orthogonal and standard ray tubes, obtained from the stable tubes of Γ_C by ray insertions, and $\mathcal{I}_0$ a preinjective component containing a slice of type Δ.

(b) B is a domestic tubular coextension of a tame concealed algebra C and

$$\Gamma_B = \mathcal{P}_\infty \vee \mathcal{T}_\infty \vee \mathcal{I}_\infty,$$

where $\mathcal{P}_\infty$ is a preprojective component containing a slice of type Δ, $\mathcal{T}_\infty$ is a $\mathbb{P}_1(K)$-family of pairwise orthogonal and standard coray tubes, obtained from the stable tubes of Γ_C by coray insertions, and $\mathcal{I}_\infty$ is the preinjective component of Γ_C. The ordering from the left to the right indicates that there are nonzero morphisms only from any of these classes to itself and to the classes on its right (see [39, (4.9)]).

If B is a tubular algebra then B is a tubular extension (respectively, coextension) of a tame concealed algebra C (respectively, C') and

$$\Gamma_B = \mathcal{P}_0 \vee \mathcal{T}_0 \vee \left(\bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q \right) \vee \mathcal{T}_\infty \vee \mathcal{I}_\infty,$$

where $\mathcal{P}_0$ is the preprojective component of Γ_C, $\mathcal{T}_0$ is a $\mathbb{P}_1(K)$-family of pairwise orthogonal ray tubes, obtained from the stable tubes of Γ_C by ray insertions, $\mathcal{I}_\infty$ is the preinjective component of $\Gamma_{C'}$, $\mathcal{T}_\infty$ is a $\mathbb{P}_1(K)$-family of pairwise orthogonal coray tubes, obtained from the stable tubes of $\Gamma_{C'}$ by coray insertions, and, for each $q \in \mathbb{Q}^+$, $\mathcal{T}_q$ is a $\mathbb{P}_1(K)$-family of pairwise orthogonal stable tubes. Moreover, all components of Γ_B are standard (see [49, (5.2)]).

The *Tits form* of a triangular algebra $A = KQ/I$ is an integral quadratic form $q_A \colon \mathbb{Z}^n \to \mathbb{Z}, n = |Q_0|$, defined, for $x = (x_i) \in \mathbb{Z}^n$, by

$$q_A(x) = \sum_{i \in Q_0} x_i^2 - \sum_{(i \to j) \in Q_1} x_i x_j + \sum_{i,j \in Q_0} r(i,j) x_i x_j,$$

where Q_1 denotes the set of arrows in Q and $r(i, j)$ is the cardinality of $L \cap I(i, j)$, for a minimal set of generators $L \subset \bigcup_{i,j} I(i, j)$ of the ideal I. If moreover gl.dim $A \leqslant 2$, then q_A coincides (see [11]) with the *Euler form* χ_A of A which, for an A-module X, is defined by

$$\chi_A(\underline{\dim} X) = \sum_{i=0}^{\infty} (-1)^i \dim_K \operatorname{Ext}^i_A(X, X),$$

(see [39, (2.4)]). It is known (see [31]) that, if A is tame, then q_A is *weakly nonnegative*, that is, $q_A(x) \geqslant 0$ whenever $x \in \mathbb{N}^n$.

Finally, for an algebra B and a B-module R we denote by $B[R]$ the one-point extension algebra

$$\begin{bmatrix} K & R \\ 0 & B \end{bmatrix}$$

For a basic background on the representation theory applied here we refer the reader to [22], [39], [4], [5].

2. Simply connected algebras

Let A be a triangular algebra and Q its Gabriel quiver. For each vertex x of Q, denote by $Q(x)$ the subquiver of Q obtained by deleting all those vertices of Q being a source of a path in Q with target x (including the trivial path from x to x). We shall denote by $A(x)$ the full subcategory of A whose objects are the vertices of $Q(x)$. Moreover, for each vertex x of Q, denote by $P(x)$ the indecomposable projective A-module at x, and by $R(x)$ the radical of $P(x)$. Then $R(x)$ is said to be *separated* if $R(x)$ is a direct sum of pairwise nonisomorphic indecomposable modules and whose supports are contained in pairwise different connected components of $Q(x)$. Following [8] we say that A has the *separation property* if $R(x)$ is separated for any vertex x of Q. It was shown in [44] that if A has the separation property then A is simply connected in the sense of [1], that is, for any presentation $A \simeq KQ/I$ of A as a bound quiver algebra, the fundamental group $\pi_1(Q, I)$ of (Q, I) is trivial. Recall also that A is called *strongly simply connected* [44] if every convex subcategory of A is simply connected. The following characterizations of strongly simply connected algebras have been proved in [44, (4.1)].

(2.1) PROPOSITION. *For a triangular algebra A the following conditions are equivalent:*

  (i) *A is strongly simply connected.*
 (ii) *Every convex subcategory of A has the separation property.*
(iii) *Every convex subcategory of A^{op} has the separation property.*
(iv) *$H^1(C, C) = 0$ for every convex subcategory C of A.*

We note also that a representation-finite algebra A is simply connected if and only if A is strongly simply connected (see [16]). Moreover, the class of strongly simply connected algebras contains the class of completely separating algebras investigated by P. Dräxler [20].

We shall now exhibit some important classes of strongly simply connected algebras playing a crucial role in our investigations of polynomial growth algebras. Observe that a hereditary algebra is simply connected if and only if it is the path algebra of a tree. Let Δ be a finite connected quiver whose underlying graph $\overline{\Delta}$ is a tree, and $H = K\Delta$. Then it is known that H is representation-infinite and tame if and only if $\overline{\Delta}$ is one of the Euclidean graphs

$\tilde{\mathbb{D}}_n$
$n \geq 4$

$\tilde{\mathbb{E}}_6$

$\tilde{\mathbb{E}}_7$

$\tilde{\mathbb{E}}_8$
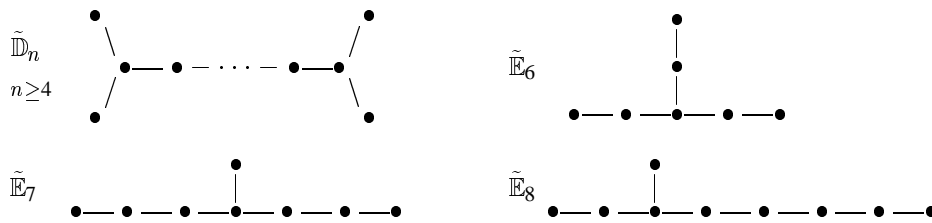
Hence $H = K\Delta$ is wild if and only if $\overline{\Delta}$ contains one of the following graphs

T_5

$\tilde{\tilde{\mathbb{D}}}_n$

$\tilde{\tilde{\mathbb{E}}}_6$

$\tilde{\tilde{\mathbb{E}}}_7$

$\tilde{\tilde{\mathbb{E}}}_8$
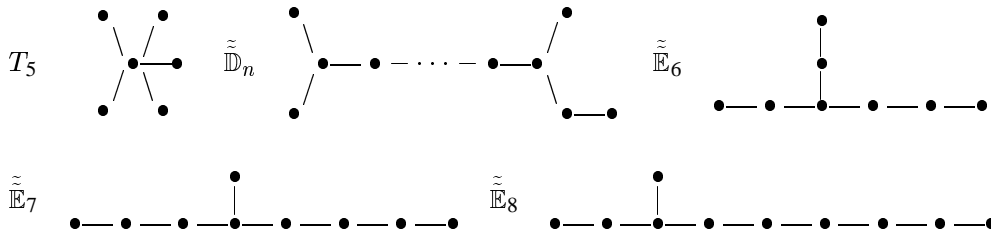
where in the case of $\tilde{\tilde{D}}_n$ the number of vertices is $n + 2, 4 \leqslant n \leqslant 8$.

Assume that $H = K\Delta$ is representation-infinite ($\overline{\Delta}$ is not a Dynkin quiver) and T is a preprojective tilting H-module, that is, $\operatorname{Ext}_H^1(T, T) = 0$ and T is a direct sum of $n = |\Delta_0|$ pairwise nonisomorphic indecomposable H-modules lying in the τ_H-orbits of projective modules. Then $C = \operatorname{End}_A(T)$ is called a *concealed algebra* of type $\overline{\Delta}$. It is known that $\operatorname{gl.dim} C \leqslant 2$ and C has the same representation type as H (see [39]). A concealed algebra of type $\overline{\Delta} = \tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7$ or $\tilde{\mathbb{E}}_8$ (respectively, $\overline{\Delta} = T_5, \tilde{\tilde{\mathbb{D}}}_n, \tilde{\tilde{\mathbb{E}}}_6, \tilde{\tilde{\mathbb{E}}}_7$ or $\tilde{\tilde{\mathbb{E}}}_8$) is said to be *critical* (respectively, *hypercritical*). The critical (respectively, hypercritical) algebras are strongly simply connected and have been classified completely by quivers and relations in [12], [23] (respectively, [26], [55], [56]). It is known (see [11], [12]) that a strongly simply connected algebra A is representation-finite if and only if A does not contain a critical convex subcategory, and if and only if q_A is weakly positive (that is, $q_A(x) > 0$ whenever $x \neq 0$ and has nonnegative coordinates). It is expected that a strongly simply connected algebra A is tame if and only if q_A is weakly nonnegative. We shall show in the paper that this is true for strongly simply connected algebras without

pg-critical convex subcategories defined below. We shall use the following result proved in [32, (3.1)].

(2.2) PROPOSITION. *Let A be a strongly simply connected algebra. Then q_A is weakly nonnegative if and only if A does not contain a hypercritical convex subcategory.*

For tubular extensions (respectively, coextensions) of critical algebras we have the following fact.

(2.3) PROPOSITION. *Let B be a tubular extension (respectively, coextension) of a critical algebra C. Then the following conditions are equivalent:*

 (i) B *is tame.*
 (ii) B *is tubular or representation-infinite tilted of Euclidean type.*
(iii) q_B *is weakly nonnegative.*

Proof. Follows from [39] and [36, (3.3)].

Following [29] by a pg-critical algebra we mean an algebra Λ satisfying the following conditions:

(i) Λ is one of the matrix algebras

$$
B[R] = \begin{bmatrix} K & R \\ 0 & B \end{bmatrix}, \qquad
B[N,t] = \begin{bmatrix}
K & K & \dots & K & K & K & N \\
  & K & \dots & K & K & K & 0 \\
  &   & \ddots & \vdots & \vdots & \vdots & \vdots \\
  &   &   & K & K & K & 0 \\
  &   &   &   & K & 0 & 0 \\
0 &   &   &   & & K & 0 \\
  &   &   &   &   &   & B
\end{bmatrix},
$$

where B is a representation-infinite tilted algebra of Euclidean type $\tilde{\mathbb{D}}_n$, $n \geqslant 4$, with a complete slice in the preinjective component of Γ_B, R (respectively, N) is an indecomposable regular B-module of regular length 2 (respectively, length 1) lying in a tube of Γ_B with $n-2$ rays, $t+1$ $(t \geqslant 2)$ is the number of isoclasses of simple $B[N,t]$-modules which are not B-modules.

(ii) Every proper convex subcategory of Λ is of polynomial growth.

The pg-critical algebras have been classified by quivers and relations in [29]. In particular, it is shown in [29] that all pg-critical algebras are simply connected of global dimension 2. There are 31 frames of such algebras, and among them only 16 frames are strongly simply connected.

The following proposition describes the representation type of hypercritical, pg-critical and tubular algebras.

(2.4) PROPOSITION. (i) *Every hypercritical algebra is wild*.

(ii) *Every pg-critical algebra is tame but not of polynomial growth*.

(iii) *Every tubular algebra is nondomestic of linear growth*.

Proof. (i) It follows from [38] and [39, (4.2)], that, if B is hypercritical, the Γ_B has no homogeneous tube, and so B is wild by [17, Corollary E]. See also [32, (3.1)], for a different proof.

(ii) For a proof we refer to [42].

(iii) See [43, Lemma 3.6].

(2.5) PROPOSITION. *Let B be a tubular algebra and*

$$\Gamma_B = \mathcal{P} \vee \mathcal{T}_0 \vee \left(\bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q \right) \vee \mathcal{T}_\infty \vee \mathcal{I}_\infty,$$

the standard decomposition of Γ_B. Let M be an indecomposable B-module lying in $\mathcal{P}_0 \vee \mathcal{T}_0 \vee (\bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q)$, and $\Lambda = B[M]$. Then q_Λ is not weakly nonnegative.

Proof. Let ω be the extension vertex of $B[M]$. Since M belongs to $\mathcal{P}_0 \vee \mathcal{T}_0 \vee (\bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q)$ we have $\mathrm{pd}_B M \leqslant 1$ and hence $\mathrm{gl.dim} \, \Lambda \leqslant 2$. In particular, $q_\Lambda = \chi_\Lambda$. Moreover, there exists $p \in \mathbb{Q}^+$ such that M belongs to $\mathcal{P}_0 \vee \mathcal{T}_0 \vee (\bigvee_{q < p} \mathcal{T}_q)$. Since $\mathcal{T}_p$ is separating (see [39, (3.1)]), there exist modules X and Y lying in different homogenous tubes of $\mathcal{T}_p$ such that $\mathrm{Hom}_B(M, X) \neq 0$ and $\mathrm{Hom}_B(M, Y) \neq 0$. Take $Z = X \oplus Y$ and put $v = \underline{\dim} Z + e_\omega$. Then

$$\begin{aligned} q_\Lambda(v) = \chi_\Lambda(v) &= \langle \underline{\dim} Z + e_\omega, \underline{\dim} Z + e_\omega \rangle \\ &= \langle \underline{\dim} Z, \underline{\dim} Z \rangle + \langle e_\omega, \underline{\dim} Z \rangle + \langle \underline{\dim} Z, e_\omega \rangle + \langle e_\omega, e_\omega \rangle \\ &= \langle e_\omega, \underline{\dim} Z \rangle + 1 = \langle \underline{\dim} P(\omega) - \underline{\dim} M, \underline{\dim} Z \rangle + 1 \\ &= -\langle \underline{\dim} M, \underline{\dim} Z \rangle + 1 = -\dim_K \mathrm{Hom}_B(M, Z) + 1 < 0, \end{aligned}$$

because $\mathrm{Ext}_B^1(M, Z) \simeq D \, \mathrm{Hom}_B(Z, \tau_B M) = 0$. Therefore, q_Λ is not weakly nonnegative.

(2.6) PROPOSITION. *Let B be a tubular extension of a critical algebra C, $\mathcal{T}$ a ray tube in Γ_B obtained from a stable tube of Γ_C by ray insertions, and R an indecomposable B-module in $\mathcal{T}$ having two direct successors. Assume that $\Lambda = B[R]$ is strongly simply connected and does not contain a pg-critical convex subcategory. Then q_Λ is not weakly nonnegative.*

Proof. By Propositions 2.3 and 2.5, we may assume that B is tilted of Euclidean type $\Delta \, (= \tilde{\mathbb{D}}_n, n \geqslant 4$, or $\tilde{\mathbb{E}}_p, 6 \leqslant p \leqslant 8)$. Let r be the length of maximal sectional path in $\mathcal{T}$ with source R and consisting of arrows pointing to the mouth. We know from [39, (4.9)] that $B = \mathrm{End}_H(T)$, where H is a hereditary algebra of type Δ and T a tilting H-module without preinjective direct summands. Moreover, $R = \mathrm{Hom}_H(T, R')$ for an indecomposable regular H-module R' of regular length r lying in a stable tube $\mathcal{T}'$ of Γ_H. Since by our assumption Λ has no pg-critical

convex subcategory, in the case $\Delta = \tilde{\mathbb{D}}_n$, we have additionally that $r \geqslant 3$ or $\mathcal{T}'$ is not of rank $n - 2$. Let ω be the extension vertex of $\Lambda = B[R]$. Then by [31, (2.1)], there are preinjective H-modules $V_1, \ldots, V_m$ and $a \in \mathbb{N}$ such that $q_\Lambda(\Sigma_{i=1}^m \underline{\dim} \operatorname{Hom}_H(T, V_i) + ae_\omega) < 0$, and hence q_Λ is not weakly nonnegative.

(2.7) COROLLARY. *Let Λ be a strongly simply connected algebra of the form*

$$\Lambda = \begin{bmatrix} F & N^r \\ 0 & C \end{bmatrix},$$

where C is a critical algebra, N a simple regular C-module, F is given by one of the quivers

with the right quiver bound by the commutativity relation, and such that $\Lambda(x, y) = N(y) \otimes_K F(x, \omega)$ for any vertices $x \in Q_F$ and $y \in Q_C$. Then either Λ is pg-critical or q_Λ is not weakly nonnegative.

Proof. Assume that Λ is not pg-critical. In order to show that q_Λ is not weakly nonnegative, we may assume that $a - b$ is oriented as $a \leftarrow b$. Let B be the convex subcategory of Λ given by all objects except b. Then B is a tubular extension of C and $\Lambda = B[R]$, where R is the radical of $P(b)$. It is easy to see that R is an indecomposable B-module lying in the ray tube $\mathcal{T}$ of Γ_B, obtained from the stable of Γ_C containing N by ray insertions, and has two direct successors in $\mathcal{T}$. Applying now the above proposition we infer that q_Λ is not weakly nonnegative.

We shall need also the following fact.

(2.8) PROPOSITION. *Let A be a connected hereditary algebra of wild type, T a tilting H-module without preprojective direct summands, and $B = \operatorname{End}_H(T)$. Then q_B is not weakly nonnegative.*

Proof. See [24, (6.2)].

3. Infinite radical of module categories

In this section we shall establish properties of the infinite radical of module categories over pg-critical, hypercritical and tubular algebras.

For a one-point extension $B[R]$ of an algebra B by a B-module R, we identify mod $B[R]$ with the category of triples (V, X, φ) where V is a K-vector space, X is a B-module and $\varphi: V \to \operatorname{Hom}_B(R, X)$ a K-linear map. Then a morphism $(V, X, \varphi) \to (W, Y, \psi)$ is given by a pair (f, g) where $f: V \to W$ is a K-linear map, $g: X \to Y$ is a B-homomorphism and $\psi f = \operatorname{Hom}_B(R, g)\varphi$. For

a background on the one-point extensions of algebras and vector space category methods we refer to [39] and [41].

We shall need the following lemma.

(3.1) LEMMA. *Let B be an algebra, R a B-module, and $\varphi : Y \to Z$ a nonzero map in* ind B. *Assume that there are nonzero nonisomorphisms $f_i : W_i \to W_{i+1}, i \geqslant 1$, in* ind B *and nonzero maps $g : R \to Y, f_0 : R \to W_1$ such that $\varphi g = 0, f_i \dots f_0 \neq 0$ for $i \geqslant 1$, and* $\operatorname{Hom}_B(R, h) = 0$ *for any $h : Y \to W_i$ or $h : W_i \to Y, i \geq 1$. Then φ belongs to* $\operatorname{rad}^\infty(\operatorname{mod} B[R])$.

Proof. Since $\operatorname{Hom}_B(R, h) = 0$ for all maps $h : Y \to W_i$ or $h : W_i \to Y, i \geqslant 1$, we may consider the indecomposable $B[R]$-modules $Z_i = (K, Y \oplus W_i, \Delta_i), i \geqslant 1$, where $\Delta_i : K \to \operatorname{Hom}_B(R, Y \oplus W_i) = \operatorname{Hom}_B(R, Y) \oplus \operatorname{Hom}_B(R, W_i)$ is given by $\Delta_i(1) = (g, f_{i-1} \dots f_0)$. Further, let L be the indecomposable $B[R]$-module (K, Y, γ) where $\gamma : K \to \operatorname{Hom}_B(R, Y)$ is given by $\gamma(1) = g$. For each $i \geqslant 1$, denote by $\alpha_i : Z_i \to Z_{i+1}$ the map $(1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & f_i \end{smallmatrix}\right))$, and by $\beta_i : Z_i \to L$ the canonical map $(1, (1, 0))$ induced by the projection $Y \oplus W_i \to Y$. Further, let $\alpha_0 : Y = (0, Y, 0) \to (K, Y \oplus W_1, \Delta_1) = Z_1$ be the monomorphism given by the canonical map $Y \to Y \oplus W_1$. Finally, since $\varphi g = 0$, we get also a nonzero map $\rho = (0, \varphi) : L = (K, Y, \gamma) \to (0, Z, 0) = Z$. Observe that $\varphi = \rho \beta_{i+1} \alpha_i \dots \alpha_0$ for any $i \geqslant 1$. Clearly, the maps $\alpha_i : Z_i \to Z_{i+1}$ are nonisomorphisms. Therefore, $\varphi \in \operatorname{rad}^\infty(\operatorname{mod} B[R])$.

(3.2) PROPOSITION. *Let A be a pg-critical algebra. Then $\bigcap_{m \geqslant 1}(\operatorname{rad}^\infty(\operatorname{mod} A))^m \neq 0$ and* $\operatorname{rad}^\infty(\operatorname{mod} A)$ *is not locally nilpotent.*

Proof. Consider first the case when A is of the form $B[N, t]$. Then A is obtained from the one-point extension $B[N]$ by identifying its extension vertex with the vertex ω of the following quiver
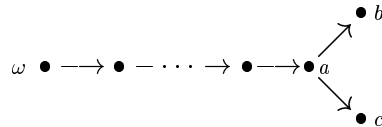
Denote by B' the convex subcategory of A given by all objects of A except b. Observe that B' is a representation-infinite tilted algebra of Euclidean type $\tilde{\mathbb{D}}_{n+t}$ (if B is of type $\tilde{\mathbb{D}}_n$) having a complete slice in the preinjective component of $\Gamma_{B'}$ and $R' = P_{B'}(a)$ is an indecomposable regular B'-module of regular length 2 lying in a ray tube of $\Gamma_{B'}$ having $n + t - 2$ rays. Hence $A' = B'[R']$ is a pg-critical algebra obtained from B by reversing the arrow $a \to b$ on $a \leftarrow b$. Consider the APR-tilting module $T = \tau_A^-(S_A(b)) \oplus P$, where P is given by $A = S_A(b) \oplus P$, associated to the simple A-module $S_A(b)$ at b. Then $A' \simeq \operatorname{End}_A(T)$ and, by [6, Theorem 1.11], the functor $\operatorname{Hom}_A(T, -)$ induces an equivalence between the full subcategory of mod A formed by all modules having no $S_A(b)$ as a direct summand and the full subcategory of mod A' formed by all modules having no $S_{A'}(b)$ as a direct summand. Clearly, $\bigcap_{m \geqslant 1}(\operatorname{rad}^\infty(\operatorname{mod} A))^m \neq 0$ (respectively, $\operatorname{rad}^\infty(\operatorname{mod} A)$

is not locally nilpotent) if and only if $\bigcap_{m \geqslant 1}(\operatorname{rad}^\infty(\operatorname{mod} A'))^m \neq 0$ (respectively, $\operatorname{rad}^\infty(\operatorname{mod} A')$ is not locally nilpotent). Therefore, we may assume that A is a one-point extension $A = B[R]$ where B is a tilted algebra of type $\tilde{\mathbb{D}}_s$, being the corresponding tubular extension of the critical algebra C of type $\tilde{\mathbb{D}}_n$, R is an indecomposable B-module lying in a ray tube $\mathcal{C}$ of Γ_B obtained from a stable tube $\mathcal{T}$ of rank $n - 2$ of Γ_C by a sequence of ray insertions, and such that we have in $\mathcal{C}$ a full translation subquiver of the form

$$
\begin{array}{ccccccccc}
Y_1 & \longrightarrow & Y_2 & \to \cdots \to & Y_i & \longrightarrow & Y_{i+1} & \to \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & \\
R = X_1 & \longrightarrow & X_2 & \to \cdots \to & X_i & \longrightarrow & X_{i+1} & \to \cdots
\end{array}
$$

with Y_1 lying on the mouth of $\mathcal{C}$. Since $\mathcal{C}$ does not contain injective modules, the modules $Z_i = \tau_B^- Y_{i-1}, i \geqslant 2$, are nonzero. Fix $r \geqslant 2$ and take an arbitrary irreducible map $\varphi_r : Y_r \to Z_r$ in $\operatorname{mod} B$. We claim that φ_r, considered as a map $(0, \varphi_r) : (0, Y_r, 0) \to (0, Z_r, 0)$ in $\operatorname{mod} B[R] = \operatorname{mod} A$, belongs to $\operatorname{rad}^\infty(\operatorname{mod} A)$. Choose arbitrary irreducible maps $u_i : X_i \to X_{i+1}, i \geqslant 1$, and $v : X_r \to Y_r$. Consider the modules $W_i = X_{r+i}, i \geqslant 1, Y = Y_r, Z = Z_r$ and the maps $\varphi = \varphi_r : Y \to Z, g = vu_{r-1} \ldots u_1 : R \to Y, f_0 = u_r \ldots u_1 : R \to W_1$, and $f_i = u_{r+i} : W_i \to W_{i+1}, i \geqslant 1$. Since $\mathcal{C}$ is a standard ray tube in Γ_B and Y_1 lies on the mouth of $\mathcal{C}$ we get that $g \neq 0, \varphi g = 0$ and $f_i \ldots f_0 \neq 0$ for any $i \geq 1$. Moreover, $\operatorname{Hom}_B(R, h) = 0$ for any $h : Y \to W_i$ or $h : W_i \to Y, i \geq 1$, in $\operatorname{mod} B$. Hence, applying Lemma 3.1 we infer that $\varphi = \varphi_r$ belongs to $\operatorname{rad}^\infty(\operatorname{mod} B[R])$.

We shall show now that $\bigcap_{m \geqslant 1}(\operatorname{rad}^\infty(\operatorname{mod} A))^m \neq 0$. Take an indecomposable C-module U lying on the mouth of $\mathcal{T}$ and consider the sectional path

$$
\Sigma : \cdots \to U_i \to U_{i-1} \to \cdots \to U_1 \to U_0 = U
$$

in $\mathcal{T}$ consisting of arrows pointing to the mouth of $\mathcal{T}$. Let P' be an indecomposable projective C-module such that $\operatorname{Hom}_C(P', U) \neq 0$. Let $f : P' \to U$ be a nonzero map. Clearly, $f \in \operatorname{rad}^\infty(P', U)$ because P' is not in $\mathcal{T}$. Since $\mathcal{T}$ is a standard stable tube in Γ_C, there are irreducible morphisms $g_i : U_i \to U_{i-1}$ and morphisms $h_i : P' \to U_i, i \geqslant 1$, in $\operatorname{mod} C$ such that $f = g_1 \ldots g_i h_i$ for any $i \geqslant 1$. The tube $\mathcal{C}$ of Γ_B is obtained from $\mathcal{T}$ by a sequence of ray insertions, and hence each arrow $U_i \to U_{i-1}$ of $\mathcal{T}$ either remains an arrow of $\mathcal{C}$ or is replaced by a finite sectional path $U_i \to \cdots \to U_{i-1}$ in $\mathcal{C}$, consisting of modules pointing to the mouth of $\mathcal{C}$. Hence the modules $U_i, i \geqslant 0$, lie on a sectional path Σ' of $\mathcal{C}$ consisting of modules pointing to the mouth. Observe that the intersection of Σ' with the sectional path

$$
\Omega : Y_1 \to Y_2 \to \cdots \to Y_j \to Y_{j+1} \to \cdots,
$$

is infinite. Therefore, there exists a sequence $1 \leqslant i_1 < i_2 < i_3 < \ldots$ of positive integers such that, for each $s \geqslant 1$, the subpath of Σ' with source U_{i_s} and target $U_{i_{s-1}}$ contains a module, say Y_{j_s}, of Ω, different from $U_{i_{s-1}}$. Then $g_{i_s} = \eta_s \psi_s \xi_s$ for an irreducible map $\psi_s : Y_{j_s} \to Z_{j_s}$ and some maps $\xi_s : U_{i_s} \to Y_{j_s}$ and $\eta_s : Z_{j_s} \to U_{i_{s-1}}$ in $\operatorname{mod} B$. We know that the maps ψ_s belong to $\operatorname{rad}^\infty(\operatorname{mod} B[R]) = \operatorname{rad}^\infty(\operatorname{mod} A)$, and so g_{i_s} also belong to $\operatorname{rad}^\infty(\operatorname{mod} A)$. Consequently, $f \in \bigcap_{m \geqslant 1}(\operatorname{rad}^\infty(\operatorname{mod} A))^m$. This proves that $\bigcap_{m \geqslant 1}(\operatorname{rad}^\infty(\operatorname{mod} A))^m \neq 0$. Finally, we prove that $\operatorname{rad}^\infty(\operatorname{mod} A)$ is not locally nilpotent. Fix $m \geqslant 1$. Take an indecomposable C-module X in $\mathcal{T}$ of quasi-length $(m+1)(n-1)$, where $n-2$ is the rank of $\mathcal{T}$. Since the full subcategory of $\operatorname{mod} C$ given by the modules in the standard stable tube $\mathcal{T}$ is serial, we infer that $\operatorname{End}_C(X) \simeq K[T]/(T^{m+1})$ (see [39, (3.1)]). Consider a path of irreducible epimorphisms

$$X = X_0 \xrightarrow{u_1} X_1 \longrightarrow \cdots \xrightarrow{u_{n-2}} X_{n-2} = W$$

and a path of irreducible monomorphisms

$$W = Y_{n-2} \xrightarrow{v_{n-2}} \cdots \longrightarrow Y_1 \xrightarrow{v_1} Y_0 = X,$$

in $\operatorname{mod} C$, and put $v = v_1 \ldots v_{n-2}$, $u = u_{n-2} \ldots u_1$, $h = vu$. Then $h \in \operatorname{rad}(X, X)$ and $h^m \neq 0$. As above we conclude that in $\operatorname{mod} B$ the morphism u has a decomposition $u = \eta \psi \xi$ for an irreducible map $\psi : Y_i \to Z_i$ and some maps $\xi : X \to Y_i$ and $\eta : Z_i \to X_{n-2}$. Since ψ belongs to $\operatorname{rad}^\infty(\operatorname{mod} A)$, we get that $h = vu$ also belongs to $\operatorname{rad}^\infty(\operatorname{mod} A)$. Therefore, $(\operatorname{rad}^\infty(X, X))^m \neq 0$. This proves that $\operatorname{rad}^\infty(\operatorname{mod} A)$ is not locally nilpotent.

(3.3) PROPOSITION. *Let A be an algebra such that $\bigcap_{m \geqslant 1}(\operatorname{rad}^\infty(\operatorname{mod} A))^m = 0$ or $\operatorname{rad}^\infty(\operatorname{mod} A)$ is locally nilpotent. Then A is tame.*

Proof. Suppose that A is not tame. Then By [21] (see also [17]), A is wild, that is, denoting by $K\langle x, y \rangle$ the free K-algebra in two noncommuting variables x and y, there exists a $K\langle x, y \rangle$–A-bimodule M, free and finitely generated as a left $K\langle x, y \rangle$-module, such that the functor $F : -\otimes_{K\langle x,y \rangle} M : \operatorname{mod} K\langle x, y \rangle \to \operatorname{mod} A$ preserves indecomposability and isomorphism classes; in particular F is faithful. Let now Λ be a *pg*-critical algebra. It is well-known that there exists a full exact embedding $G : \operatorname{mod} \Lambda \to \operatorname{mod} K\langle x, y \rangle$. Hence, we get a faithful functor $FG : \operatorname{mod} \Lambda \to \operatorname{mod} A$. From Proposition 3.2 we infer that there is a nonzero morphism f in $\bigcap_{m \geqslant 1} \operatorname{rad}^\infty(\operatorname{mod} \Lambda)^m$. Then $FG(f) \neq 0$ and belongs to $\bigcap_{m \geqslant 1}(\operatorname{rad}^\infty(\operatorname{mod} A))^m$. Similarly, for each $m \geq 1$, there exists a module $X \in \operatorname{ind} \Lambda$ and a nonzero morphism h_m in $(\operatorname{rad}^\infty(X, X))^m$. Then $FG(h_m) \neq 0$ and belongs $(\operatorname{rad}^\infty(FG(X), FG(X)))^m$. Therefore, we get that $\bigcap_{m \geq 1}(\operatorname{rad}^\infty(\operatorname{mod} A))^m \neq 0$ and $\operatorname{rad}^\infty(\operatorname{mod} A)$ is not locally nilpotent, a contradiction. This proves that A is tame.

As a direct consequence of Propositions 2.4 and 3.3 we get the following fact.

(3.4) COROLLARY. *Let A be a hypercritical algebra. Then $\bigcap_{m \geqslant 1}(\operatorname{rad}^\infty(\operatorname{mod} A))^m \neq 0$ and $\operatorname{rad}^\infty(\operatorname{mod} A)$ is not locally nilpotent.*

We shall need also the following fact.

(3.5) PROPOSITION. *Let A be a tubular algebra. Then*

$$\operatorname{rad}^{\infty}(A, D(A)) = (\operatorname{rad}^{\infty}(A, D(A)))^2 = (\operatorname{rad}^{\infty}(A, D(A)))^3 = \dots$$

In particular, $\bigcap_{m \geqslant 1}(\operatorname{rad}^{\infty}(\operatorname{mod} A))^m \neq 0$.

   Proof. It is a direct consequence of the separation property of tubular families in Γ_A [39, (5.2)]. For a complete proof we refer to [25, (1.5)].

4. Strongly simply connected algebras of polynomial growth

The following theorem gives different characterizations of polynomial growth strongly simply connected algebras and their module categories.

(4.1) THEOREM. *Let A be a strongly simply connected algebra. The following conditions are equivalent:*

   (i) *A is of polynomial growth.*
  (ii) *A is of linear growth.*
 (iii) *A does not contain a convex subcategory which is pg-critical or hypercritical.*
 (iv) *$\operatorname{rad}^{\infty}(\operatorname{mod} A)$ is locally nilpotent.*
  (v) *Σ_A is directed.*
 (vi) *Every connected component of Γ_A is standard.*
(vii) *A is a multicoil algebra.*

   The proof will be presented in Section 6. Here, we shall derive several important consequences of the above theorem. First observe that the condition (iii) gives a handy criterion for a strongly simply connected algebra to be of polynomial (respectively, linear) growth. The condition (v) gives a partial order of all connected component of Γ_A in mod A. In particular, it implies that every cycle in mod A is finite (A is cycle-finite in the sense of [2]), and so $\operatorname{rad}^{\infty}(M, M) = 0$ for any module M from ind A. Moreover, the condition (vii) gives a more precise description of cycles in mod A. Namely, there exists a natural generalization of the notion of a tube called a coil (see [3], [4], [5]). Then a multicoil consists, roughly speaking, of a finite number of coils glued together by some directed parts, and a multicoil algebra is an algebra A having the property that every cycle in mod A consists of modules of a (standard) coil of a multicoil of Γ_A.

   The following corollary is a direct consequence of Theorem 4.1, Proposition 2.2 and the fact that the hypercritical algebras are wild.

(4.2) COROLLARY. *Let A be a strongly simply connected algebra having no pg-critical convex subcategory. The following conditions are equivalent:*
   (i) *A is tame.*
  (ii) *A is of polynomial growth.*
 (iii) *q_A is weakly nonnegative.*

The following corollary gives characterizations of domestic strongly simply connected algebras and their module categories.

(4.3) COROLLARY. *Let A be a strongly simply connected algebra. The following conditions are equivalent:*

 (i) *A is domestic.*
 (ii) *A does not contain a convex subcategory which is hypercritical, pg-critical or tubular.*
(iii) $\bigcap_{m \geqslant 1}(\mathrm{rad}^{\infty}(\mathrm{mod}\, A))^{m} = 0$.
(iv) $\mathrm{rad}^{\infty}(\mathrm{mod}\, A)$ *is nilpotent.*
 (v) Σ_A *is directed and there is a common bound on the length of paths in Σ_A.*

Proof. It follows from Theorem 4.1 that, if A is of polynomial growth, then every cycle in $\mathrm{mod}\, A$ is finite. Then the equivalence of (i) and (ii) is a direct consequence of Theorem 5.1 in [50]. Clearly, (v) implies (iv), and (iv) implies (iii). Moreover, by Propositions 3.2, 3.5 and Corollary 3.4, (iii) implies (ii). Therefore, it remains to show that (ii) implies (v). Assume that (ii) holds. Then, by Theorem 4.1, Σ_A is directed and every connected component of Γ_A is (generalized) standard. Applying now [47, Corollary 3.11], we infer that all but finitely many connected components of Γ_A are stable tubes. Further, since A is a cycle-finite algebra, we infer by Theorem 4.1 in [50] that the support $\mathrm{supp}\,\mathcal{T}$ of any stable tube $\mathcal{T}$ in Γ_A is a convex critical or tubular subcategory of A. Moreover, the stable tubes in the Auslander-Reiten quiver of any critical algebra are pairwise orthogonal and standard. Therefore, the fact that A has no tubular convex subcategory, implies that the paths in Σ_A are of bounded length. This finishes the proof.

Recall that an Auslander-Reiten quiver Γ_A is called *quasi-periodic* if all but finitely many τ_A-orbits in Γ_A are periodic.

(4.4) COROLLARY. *Let A be a strongly simply connected algebra of polynomial growth. Then Γ_A is quasi-periodic and all but finitely many components in Γ_A are stable tubes. If A is domestic then all but finitelly many components in Γ_A are stable tubes of rank* 1.

Proof. From [47, Theorem 2.3], every generalized standard component of Γ_A admits at most finitely many nonperiodic τ_A-orbits. Further, since A is cycle-finite, we know by [50, Proposition 3.3], that every regular generalized standard component of Γ_A is a stable tube. Hence our claim follows from Theorem 4.1(vi) and [50, Theorem 4.4].

For a component $\mathcal{C}$ of Γ_A we denote by $|\mathcal{C}|$ the *geometric realization* of $\mathcal{C}$ as defined in [15, (4.1)]. Moreover, following [4, (3.3)], a coil Γ in Γ_A is said to be *proper* if each of its vertices belongs to an oriented cycle of Γ. As a direct consequence of our proof of Theorem 4.1 given in Section 6 we get the following fact.

(4.5) COROLLARY. *Let A be a strongly simply connected algebra of polynomial growth, C a (connected) component of Γ_A, X a vertex of Γ_A, and r the number of pairwise disjoint proper coils in C. Then the fundamental group $\pi_1(|C|, X)$ is a free (noncommutative) group in r generators.*

We shall now derive some informations on the supports of indecomposable modules over strongly simply connected algebras of polynomial growth.

(4.6) COROLLARY. *Let A be a strongly simply connected algebra of polynomial growth and $\mathcal{T}$ a stable tube in Γ_A. Then the support of $\mathcal{T}$ is a critical or tubular convex subcategory of A.*

Proof. See [3, (4.1)], or [50, (4.2)].

The following corollary describes the one-parameter families of indecomposable modules over strongly simply connected algebras of polynomial growth.

(4.7) COROLLARY. *Let A be a strongly simply connected algebra of polynomial growth. Assume that M is a module in $\mathrm{ind}\, A$ such that there are infinitely many pairwise nonisomorphic modules N in $\mathrm{ind}\, A$ with $\underline{\dim}\, N = \underline{\dim}\, M$. Then $\mathrm{supp}\, M$ is a convex critical or tubular subcategory of A.*

Proof. Since A is tame, we know from [17] that all but finitely many isoclasses of indecomposable A-modules of dimension-vector $\underline{\dim}\, M$ lie in the stable tubes of rank 1. Hence, by our assumption, $\underline{\dim}\, M = \underline{\dim}\, N$ for a module N lying in a stable tube $\mathcal{T}$ of rank 1. Consequently, by the above corollary, $\mathrm{supp}\, M = \mathrm{supp}\, N = \mathrm{supp}\, \mathcal{T}$ is a convex critical or tubular subcategory of A.

By a *coil algebra* we mean a tame coil enlargement of a tame concealed algebra (in the sense of [5]). As a direct consequence of the proof of Theorem 4.1 presented in Section 6 we get also the following fact.

(4.8) COROLLARY. *Let A be a strongly simply connected algebra of polynomial growth. Then there exist convex coil subcategories $B_1, \ldots B_m$ of A whose indecomposable modules exhaust all but finitely many isoclasses of indecomposable A-modules. Moreover, if the support $D = \mathrm{supp}\, X$ of an indecomposable A-module X is not contained in one of the categories $B_1, \ldots, B_m$, then X is directing, and so D is a tame tilted algebra.*

We shall note that the module categories over coil algebras are well understood (see [5]). Moreover, a complete classification of coil algebras with sincere nondirecting indecomposable modules lying in nonstable coils will be presented in a joint paper with I. Assem and B. Tomé. Hence, in view of Corollary 4.5, in order to classify the indecomposable modules over strongly simply connected algebras of polynomial growth, it remains to describe the supports of directing modules. If X is a directing module over such an algebra A then $\mathrm{supp}\, X$ is a convex subcategory of A [11, Proposition 3.2], and is tilted [39, p. 376]. Hence, we need a classification of tame tilted strongly simply connected algebras with sincere directing modules. Those of finite representation type are classified in [10] and [18], [40]. It was shown

in [33] that the representation-infinite tame tilted algebras with sincere directing modules have, in each dimension d, at most 2 one-parameter families of modules. The families of two-parametric tame algebras with sincere directing modules and having at least 20 vertices in the Gabriel quiver have been classified in [34]. The classification of the remaining ones is still an open problem.

The support of a nondirecting indecomposable module over a polynomial growth strongly simply connected algebra A is not necessarily a convex subcategory of A. The following simple example is due to P. Dräxler. Namely, let C be the critical algebra of type $\tilde{\mathbb{D}}_5$ given by the fully commutative quiver
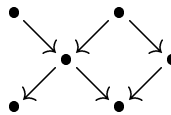
and M be the indecomposable C-module

$$
\begin{array}{ccc}
K & & K \\
\searrow{\scriptstyle\begin{bmatrix}0\\1\end{bmatrix}} & {\scriptstyle\begin{bmatrix}1\\0\end{bmatrix}}\swarrow & \searrow \\
K^2 & & 0 \\
{\scriptstyle\begin{bmatrix}1\\1\end{bmatrix}}\swarrow & {\scriptstyle[0,\,1]}\searrow & \swarrow \\
K & & K
\end{array}
$$

Then clearly supp M is not a convex subcategory of C. But from the above discussion and our proof of Theorem 4.1 we can deduce the following fact.

(4.9) COROLLARY. *Let A be a strongly simply connected algebra of polynomial growth, M a module in* ind A *and $\Lambda = $ supp M.*

 (i) *If Λ is representation-finite then Λ is a convex subcategory of A.*
(ii) *If Λ is representation-infinite, then it contains a full subcategory C which is critical.*

For $M \in$ ind A, denote by $c(M)$ the number of pairwise different critical convex subcategories of the convex hull of supp M in A. Then we have the following fact.

(4.10) COROLLARY. *Let A be a strongly simply connected algebra of polynomial growth and M a module in* ind A. *Then $c(M) \leqslant 3$. Moreover, if A is domestic, then $c(M) \leqslant 2$.*

Proof. It is a direct consequence of Corollaries 4.7 and 4.8, [4], [5, (4.2)], and [33].

In the joint work with J. A. de la Peña [35] we proved that, if A is a sincere tame strongly simply connected algebra and admits a convex subcategory which is either tubular or representation-infinite tilted of type $\tilde{E}_p$, $6 \leqslant p \leqslant 8$, then A is of polynomial growth and its quiver Q_A has at most 19 vertices. Then it follows (see [35, Corollary 6.2]) that every sincere polynomial growth strongly simply connected algebra with at least 14 vertices is domestic.

We end this section with an example illustrating our criteria for the polynomial growth and domestic type of strongly simply connected algebras.

Let $A = KQ/I$ where Q is the quiver
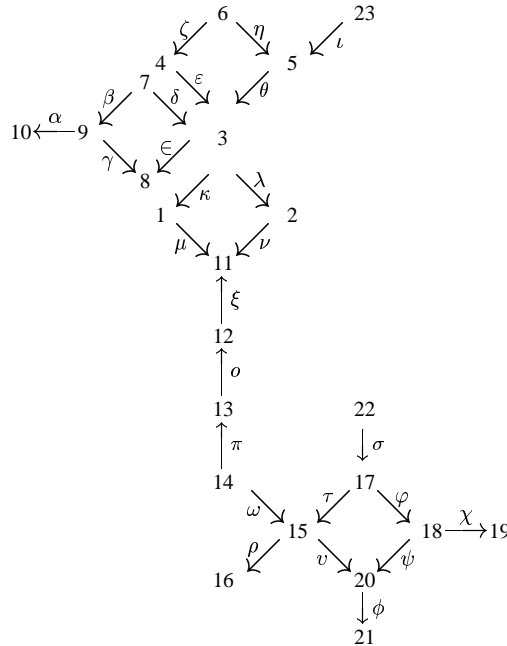
and I is the ideal of KQ generated by $\beta\gamma - \delta \in, \zeta\varepsilon - \eta\theta, \kappa\mu - \lambda\nu, \delta\kappa, \delta\lambda, \varepsilon \in, \theta \in, \iota\theta\lambda, \pi o\xi, \omega\rho, \omega\upsilon, \tau\upsilon - \varphi\psi$. Clearly, A is strongly simply connected. A simple inspection of the frames of critical algebras presented in [12] and [23] shows that the following algebras form a complete set of critical convex subcategories of A:

C_1 (of type $\tilde{\mathbb{D}}_4$) given by the vertices $1, 2, 3, 4, 5$;
C_2 (of type $\tilde{\mathbb{E}}_8$) given by the vertices $1, 3, 4, 5, 6, 7, 9, 10, 23$;
C_3 (of type $\tilde{\mathbb{E}}_6$) given by the vertices $15, 16, 17, 18, 19, 20, 21$;
C_4 (of type $\tilde{\mathbb{E}}_6$) given by the vertices $15, 16, 17, 18, 19, 20, 22$;
C_5 (of type $\tilde{\mathbb{E}}_8$) given by the vertices $12, 13, 14, 15, 16, 17, 18, 19, 22$.

Similarly, a simple inspection of the frames of hypercritical algebras presented in [55] shows that A does not contain a convex subcategory which is hypercritical. Further, there is no extension of C_1 to a pg-critical convex subcategory of A, and hence A does not contain a convex subcategory which is pg-critical. Hence, applying Theorem 4.1, we conclude that A is of polynomial growth. Further, it is easy to see that the following algebras form a complete set of tubular convex subcategories of A:

Λ_1 (of type $(2, 3, 6)$) given by the vertices $1, 2, 3, 4, 5, 6, 7, 9, 10, 23$;
Λ_2 (of type $(2, 3, 6)$) given by the vertices $12, 13, 14, 15, 16, 17, 18, 19, 20, 22$;
Λ_3 (of type $(3, 3, 3)$) given by the vertices $15, 16, 17, 18, 19, 20, 21, 22$.

In particular, by Corollary 4.3, A is not domestic. On the other hand, the full convex subcategory A' of A formed by all objects of A except 22 and 23 is a representation-infinite domestic strongly simply connected algebra, containing only two critical convex subcategories, namely C_1 and C_3. Finally, we note that the algebras:

B_1 given by the vertices $1, 2, 3, \ldots, 12, 13, 23$;

and

B_2 given by the vertices $12, 13, 14, \ldots, 20, 21, 22$,

are convex coil subcategories of A whose indecomposable modules exhaust (up to isomorphism) all indecomposable A-modules.

5. Partial order of sources

Let $A = KQ/I$ be an algebra with the separation property and $\mathcal{S}$ the set of all sources in Q. The aim of this section is to establish a partial order $\leqslant$ in $\mathcal{S}$ in the case when all component quivers $\Sigma_{A(x)}, x \in \mathcal{S}$, are directed. This will play a crucial role in our proof of Theorem 4.1. We start with the following technical lemma.

(5.1) LEMMA. *Let $z \in \mathcal{S}$ and $\mathcal{C}$ be a connected component of $\Gamma_{A(z)}$. Assume that $\Sigma_{A(z)}$ is directed and there exists an infinite path*

$$Z = Z_0 \to Z_1 \to \cdots \to Z_s = N,$$

in $\mathrm{mod}\, A(z)$ *such that Z belongs to $\mathcal{C}$ and N is an indecomposable direct summand of $R(z)$. Moreover, let*

$$\mathcal{D} = \mathcal{C}_0 \to \cdots \to \mathcal{C}_t = \mathcal{C},$$

$t \geqslant 0$, *be a path in $\Sigma_{A(z)}$ and Y a module in $\mathcal{D}$. Then every indecomposable A-module M lying on a path in* $\mathrm{mod}\, A$ *with target Y is an $A(z)$-module. In particular, $\mathcal{D}$ is a full component of Γ_A.*

Proof. Let $M = M_0 \to M_1 \to \cdots \to M_r = Y$ be a path in $\mathrm{mod}\, A$ with target Y. Suppose that $\mathrm{Hom}_A(P(z), M) \neq 0$. We may assume that $M_1, \ldots, M_r$ are $A(z)$-modules. We shall show that $\Sigma_{A(z)}$ admits an oriented cycle $\mathcal{C} \to \cdots \to \mathcal{D} \to \cdots \to \mathcal{C}$, which will contradict our assumption on $\Sigma_{A(z)}$. Since $A = A(z)[R(z)]$ we may consider M as a triple (V, X, φ) where V is a K-vector space, X an $A(z)$-module, and $\varphi : V \to \mathrm{Hom}_{A(z)}(R(z), X)$ a K-linear map. Observe that M is not isomorphic to the simple injective A-module $S(z) = (K, 0, 0)$, because there is a nonzero nonisomorphism $M \to M_1$ with M_1 indecomposable. Hence $X \neq 0$ and let $X = X_1 \oplus \ldots \oplus X_r$ be a decomposition of X into a direct sum of indecomposable $A(z)$-modules. Since $\mathrm{Hom}_A(P(z), M) \neq 0$ and M is indecomposable, we infer that $\varphi \neq 0$ and $\mathrm{Hom}_{A(z)}(R(z), X_i) \neq 0$ for any $1 \leq i \leq r$. Moreover, since $\mathrm{Hom}_A(M, M_1) \neq 0$ and $M_1 = (0, M_1, 0)$, there exists $1 \leqslant j \leqslant r$ such that $\mathrm{Hom}_{A(z)}(X_j, M_1) \neq 0$. This shows that there is in $\mathrm{mod}\, A(z)$ a path from X_j to Y.

On the other hand, there is in mod $A(z)$ a path from Z to N, and $\Sigma_{A(z)}$ admits a path $\mathcal{D} = \mathcal{C}_0 \to \cdots \to \mathcal{C}_t = \mathcal{C}$. Hence, the supports of X_j and N belong to the same connected component of $Q(z)$, because the radical $R(z)$ of $P(z)$ is separated. But then $\mathrm{Hom}_{A(z)}(R(z), X_j) \neq 0$ implies that $\mathrm{Hom}_{A(z)}(N, X_j) \neq 0$, and so mod $A(z)$ admits an infinite path $Z \to Z_1 \to \cdots \to N \to \cdots \to Y$. Thus, we get in $\Sigma_{A(z)}$ an oriented cycle $\mathcal{C} \to \cdots \to \mathcal{C}_0 \to \cdots \to \mathcal{C}_t = \mathcal{C}$, a contradiction. Therefore, $\mathrm{Hom}_A(P(z), M) = 0$, and M is an $A(z)$-module.

For $x, y \in \mathcal{S}$, we set $x <^* y$ if and only if there is an infinite path

$$M = Z_0 \to Z_1 \to \cdots \to Z_s = N,$$

in mod $A(y)$ with M an indecomposable direct summand of $R(x)$ and N an indecomposable direct summand of $R(y)$. Then we have the following proposition.

(5.2) PROPOSITION. *Assume that for each $x \in \mathcal{S}$ the component quiver $\Sigma_{A(z)}$ is directed. Then the transitive closure $\leqslant$ of $<^*$ is a partial order in $\mathcal{S}$.*

Proof. Assume that $x_0 <^* x_1 <^* \cdots <^* x_r$ for some $x_0, x_1, \ldots, x_r$ in $\mathcal{S}$ and $r \geqslant 1$. We shall show that $x_0 \neq x_r$. We may assume by induction that $x_0, x_1, \ldots x_{r-1}$ (respectively, $x_1, \ldots, x_r$) are pairwise different. By our assumption, for each $1 \leqslant i \leqslant r$, there exists an infinite path

$$p_i \colon M_{i-1} = Z_0^i \to \cdots \to Z_{s_i}^i = N_i,$$

in mod $A(x_i)$ with M_{i-1} an indecomposable direct summand of $R(x_{i-1})$ and N_i an indecomposable summand of $R(x_i)$. Denote by $\mathcal{D}_0, \ldots, \mathcal{D}_{r-1}$ the components of $\Gamma_{A(x_r)}$ containing the modules $M_0, \ldots, M_{r-1}$, respectively. Observe that, in order to prove our claim, it is enough to show that $\mathcal{D}_0$ is a full component of Γ_A. Indeed, then $\mathcal{D}_0$ consists of $A(x_r)$-modules and contains $P(x_0)$, because Γ_A admits an arrow $M_0 \to P(x_0)$. Hence $P(x_0) \neq P(x_r)$, and so $x_0 \neq x_r$. Since p_{r-1} is an infinite path from M_{r-1} to N_r in mod $A(x_r)$, applying Lemma 5.1 for $t = 0, \mathcal{D} = \mathcal{D}_{r-1}, Z = M_{r-1}$ and $N = N_r$, we get that $\mathcal{D}_{r-1}$ is a full component of Γ_A. Hence, we may assume that $r \geq 2$. Now, since $x_1, \ldots, x_{r-1}$ are sources in Q different from x_r, we conclude that $P(x_1), \ldots, P(x_{r-1})$ are $A(x_r)$-modules. Then, for each $1 \leqslant i \leqslant r-1$, the component $\mathcal{D}_i$ admits arrows $M_i \to P(x_i)$ and $N_i \to P(x_i)$. Applying Lemma 5.1 again, we prove by descending induction on i that, for each $r - 1 \geqslant i \geqslant 1, p_{i-1}$ consists of $A(x_r)$-modules, $\mathcal{D}_{i-1}$ is a full component of Γ_A, and $\Sigma_{A(x_r)}$ admits a path $\mathcal{D}_{i-1} \to \cdots \to \mathcal{D}_{r-1}$. For $i = 1$, we get that $\mathcal{D}_0$ is a full component of Γ_A. This finishes the proof.

6. Proof of Theorem 4.1

Let A be a strongly simply connected algebra. We may assume that A is connected. Clearly, (ii) implies (i). By Proposition 2.4, (i) implies (iii). Further each of the conditions (v) and (vi) implies that $\mathrm{rad}^\infty(M, M) = 0$ for any $M \in \mathrm{ind}A$, and then

(iv) holds. From Proposition 3.2 and Corollary 3.4, (iv) implies (iii). Moreover, the implication (vii) $\Rightarrow$ (ii) is a direct consequence of [3, Theorem 4.6], and Proposition 2.4(iii) (see also [50, Theorem 4.3]). Therefore, it remains to prove that (iii) implies (v), (vi) and (vii).

Assume that $A = KQ/I$ does not contain a convex subcategory which is pg-critical or hypercritical. Then, by Proposition 2.2, the Tits form q_A of A is weakly nonnegative. We shall prove (v), (vi) and (vii) by induction on the number of vertices in Q. We use the notations of Sections 4 and 5. Observe that, for any source x of Q, the algebra $A(x)$ is strongly simply connected and does not contain a convex subcategory which is pg-critical or hypercritical. Hence, we may assume that, for any source x in Q, $A(x)$ is a multicoil algebra, $\Sigma_{A(x)}$ is directed and every component of $\Gamma_{A(x)}$ is standard. Therefore, applying Proposition 5.2, we endow the set $\mathcal{S}$ of all sources in Q by the partial order $\leqslant$ induced by $<^*$. Denote by $\mathcal{M}$ the set of all maximal elements in $\mathcal{S}$ with respect to $\leqslant$. We divide our proof into two main steps.

(1) Assume first that, for each $x \in \mathcal{M}$, $R(x)$ is a direct sum of indecomposable modules whose successors in $\Gamma_{A(x)}$ are all directing $A(x)$-modules. Fix $x \in \mathcal{M}$ and consider a decomposition $R(x) = M_1 \oplus \cdots \oplus M_r$ of $R(x)$ into a direct sum of indecomposable $A(x)$-modules. For each $1 \leqslant i \leqslant r$, denote by $\mathcal{C}_i$ the component of $\Gamma_{A(x)}$ containing the module M_i, by $\mathcal{D}_i$ the full translation subquiver of $\mathcal{C}_i$ formed by all successors of M_i in $\mathcal{C}_i$, and by $\mathcal{E}_i$ the full translation subquiver of $\mathcal{C}_i$ given by the remaining indecomposable modules. Clearly, $\mathcal{D}_i$ is closed under successors and $\mathcal{E}_i$ is closed under predecessors in $\mathcal{C}_i$. Moreover, by our assumption, each $\mathcal{D}_i$ consists entirely of directing $A(x)$-modules, and hence the number of $\tau_{A(x)}$-orbits in $\mathcal{D}_i$ is finite (see [53]). Further, let $\mathcal{C}$ be the component of Γ_A containing $P(x)$, $\mathcal{D}$ the full translation subquiver of $\mathcal{C}$ formed by all successors of modules $M_1, \ldots, M_r$ in $\mathcal{C}$, and $\mathcal{E}$ the full translation subquiver of $\mathcal{C}$ given by the remaining indecomposable A-modules. We shall show that $M_1, \ldots, M_r, P(x)$ are directing modules in $\bmod A$ and $\mathcal{D}$ contains all indecomposable projective A-modules being successors of modules $M_1, \ldots, M_r$ in $\bmod A$. Since the component quiver $\Sigma_{A(x)}$ is directed and $R(x)$ is separated, we have $\operatorname{Hom}_{A(x)}(R(x), Z) = 0$ for any indecomposable $A(x)$-module Z being predecessor in $\bmod A(x)$ of an indecomposable module lying in one of the subquivers $\mathcal{E}_1, \ldots, \mathcal{E}_r$. Applying now [39, (2.5)], to the one-point extension $A = A(x)[R(x)]$ we infer that $\mathcal{E}$ is the disjoint union of $\mathcal{E}_1, \ldots, \mathcal{E}_r$, and, if $\Gamma \to \cdots \to \mathcal{C}$ is a path in Σ_A with $\Gamma \neq \mathcal{C}$, then Γ is a full component from $\Gamma_{A(x)}$. Clearly, this implies that the modules $M_1, \ldots, M_r, P(x)$ are directing in $\bmod A$.

Let now $M_i = Y_0 \to Y_1 \to \cdots \to Y_t = P(b)$ be a path in $\bmod A$, for some $1 \leqslant i \leqslant r$. We claim that this path is finite, and hence $P(b)$ belongs to $\mathcal{D}$. Suppose that this is not the case. Then there is an infinite path in $\bmod A$ of the form $M_i = Y_0 \to Y_1 \to \cdots \to Y_t \to \cdots \to Y_m = P(c)$ with $c \in \mathcal{S}$. Clearly, $\operatorname{Hom}_A(P(c), M_i) = 0$ because c is source. Observe that then $m \geqslant 2$ and $\operatorname{Hom}_A(P(c), Y_j) \neq 0$ for some $1 \leqslant j \leqslant m - 1$. Indeed, otherwise we have $x <^* c$ which contradicts $x \in \mathcal{M}$. In particular, $P(c)$ is nondirecting in $\bmod A$, and by

the above considerations $c \notin \mathcal{M}$. Hence $c <^* d$ for some $d \in \mathcal{S}$. Consequently, there is an infinite path $Z = Z_0 \to Z_1 \to \cdots \to Z_s = N$ in $\mathrm{mod}\, A(d)$ with Z an indecomposable direct summand of $R(c)$ and N an indecomposable direct summand of $R(d)$. Since Z and $P(c)$ belong to the same component, say $\mathcal{C}'$, of $\Gamma_{A(d)}$ we infer by Lemma 5.1 that $Y_0, Y_1, \ldots, Y_m$ are $A(d)$-modules. In particular, $P(c)$ is nondirecting in $\mathrm{mod}\, A(d)$. Since $A(d)$ is a multicoil algebra, we then conclude that $P(c)$ lies on a cycle in a coil Γ' of $\mathcal{C}'$. Observe also that the path $Z = Z_0 \to Z_1 \to \cdots \to Z_s = N$ is not formed by directing modules lying on a sectional path of $\mathcal{C}'$, because it is infinite (see [9]) and $\mathcal{C}'$ is a standard multicoil. Then we infer that $\mathrm{mod}\, A(d)$ admits an infinite path of the form

$$M_i = Y_0 \to Y_1 \to \cdots \to Y_m = P(c) \to \cdots \to Z_s = N.$$

But this implies $x <^* d$, again a contradiction with $x \in \mathcal{M}$. Therefore, $\mathcal{D}$ contains any indecomposable projective A-module which is successor of a module M_i, $1 \leqslant i \leqslant r$, in $\mathrm{mod}\, A$. The above arguments show also that, if $P(c)$ with $c \in \mathcal{S}$ is a successor of $P(x)$ in $\mathcal{D}$, then $c \in \mathcal{M}$. From the first part of our proof we know that, for any $c \in \mathcal{M}$, $P(c)$ is a directing A-module. Hence we may choose $x \in \mathcal{M}$ in such a way that any indecomposable proper successor of $P(x)$ in $\mathrm{mod}\, A$ is not projective. For each $1 \leqslant i \leqslant r$, denote by Ω_i the set of all modules X in $\mathcal{D}_i$ such that any path in $\mathcal{C}_i$ from M_i to X is sectional. Observe that each Ω_i is finite because the modules from Ω_i belong to pairwise different $\tau_{A(x)}$-orbits in $\mathcal{D}_i$. Further, put $\Omega = \left(\bigcup_{1 \leqslant i \leqslant r} \Omega_i \right) \cup \{P(x)\}$. Then, by the above discussion, we get that Ω is finite and consists of directing A-modules. Moreover, it follows from our new choice of x that any module U in $\mathcal{D}$ is of the form $\tau_A^{-l} V$ for some $V \in \Omega$, $l \geqslant 0$. Observe also that $\mathcal{D}$ has no oriented cycles. Indeed, since any proper successor of $P(x)$ in $\mathcal{D}$ is not projective, any cycle $U_0 \to U_1 \to \cdots \to U_t \to U_0$ in $\mathcal{D}$ gives a cycle $\tau_A^l U_0 \to \tau_A^l U_1 \to \cdots \to \tau_A^l U_t \to \tau_A^l U_0$, for some $l \geqslant 0$ passing through a module from Ω, a contradiction because Ω consists of directing modules. Finally, suppose that there is in $\mathrm{mod}\, A$ a cycle

$$W = W_0 \to W_1 \to W_2 \to \cdots \to W_n \to W_{n+1} = W,$$

with W from $\mathcal{D}$. Since $\mathcal{D}$ has no oriented cycle there exists $1 \leqslant q \leqslant n$ such that $W_l \in \mathcal{D}$ for any $l > q$ but $W_q \notin \mathcal{D}$. But then from the shape of $\mathcal{C}$ described above, we infer that $\mathrm{mod}\, A$ admits a cycle of the form

$$W = W_0 \to W_1 \to \cdots \to W_q \to L \to W_{q+1} \to \cdots \to W_{n+1} = W,$$

with L from Ω. But it is not possible because Ω consists of directing modules. Therefore, $\mathcal{D}$ consists entirely of directing modules. We know that any module in $\mathcal{C}$ belongs either to $\mathcal{E}$ or to $\mathcal{D}$. Hence $\mathcal{C}$ either consists of directing modules or $\mathcal{C}$ is a multicoil (if one of $\mathcal{E}_i$ contains a coil). We shall show now that $\mathcal{C}$ is also standard. Observe that, since the modules M_i, $1 \leqslant i \leqslant r$, are not predecessors of modules

from $\mathcal{E}$ in mod A, we have $\operatorname{Hom}_A(X, Y) = 0$ for all $X \in \mathcal{D}$ and $Y \in \mathcal{E}$. This implies that $\mathcal{C}$ is generalized standard. But $\mathcal{E}$ is standard, $\mathcal{D}$ is directed with finitely many τ_A-orbits, so applying arguments as in [48], we conclude that $\mathcal{C}$ is in fact a standard component of Γ_A. We shall now describe Γ_A. Observe that any component $\mathcal{X}$ of $\Gamma_{A(x)}$ which has no module M with $\operatorname{Hom}_{A(x)}(R(x), M) \neq 0$ remains a full component in Γ_A, by [39, (2.5)]. Hence, we must describe the components of Γ_A containing successors of $P(x)$ in mod A. Consider the right stable part $\mathcal{D}_r$ of $\mathcal{D}$ obtained from $\mathcal{D}$ by removing the τ_A-orbits of injective modules. If $\mathcal{D}_r$ is empty, $\mathcal{C}$ is the unique component of Γ_A which is not a component of $\Gamma_{A(x)}$. Clearly, then $A = A(x)[B(x)]$ is a multicoil algebra, Σ_A is directed and every component of Γ_A is standard. Assume that $\mathcal{D}_r$ is not empty. The $\mathcal{D}$ admits a full translation subquiver $\mathcal{F}$ of $\mathcal{D}_r$ which is closed under successors in $\mathcal{D}$ (hence in $\mathcal{C}$), contains all but finitely many modules of $\mathcal{D}$, and is a disjoint union of translation quivers of the form $\mathcal{F}_j = (-\mathbb{N})\Delta_j$, $1 \leqslant j \leqslant t$, for some finite quivers $\Delta_1, \ldots, \Delta_t$ without oriented cycles (see [27] for the shape of right stable quivers without oriented cycles). Fix $1 \leqslant j \leqslant t$. Since $\mathcal{F}_j$ consists of directing modules, applying the dual of Proposition 2.4 in [53], we conclude that there exists a hereditary algebra H_j of type Δ_j and a tilting H_j-module T_j without preprojective direct summands such that the tilted algebra $B_j = \operatorname{End}_{H_j}(T_j)$ is a convex subcategory of A and $\mathcal{F}_j$ is a full translation subquiver of a connecting component $\mathcal{P}_j$ of Γ_{B_j} which is closed under successors. Since q_A is weakly nonnegative, we conclude that q_{B_j} is weakly nonnegative, and hence, by Proposition 2.8, Δ_j is an Euclidean quiver. In particular, $\mathcal{P}_j$ is the preprojective component of Γ_{B_j}. Recall that (see Section 1) Γ_{B_j} consists of $\mathcal{P}_j$, a $\mathbb{P}_1(K)$-family $\mathcal{T}_j$ of tubes without projective modules and a preinjective component $\mathcal{I}_j$. Moreover, all components of Γ_{B_j} are standard, Σ_{B_j} is directed, and B_j is a multicoil algebra. Now, since $P(x)$ has no proper projective successors in mod A, it follows that any component $\mathcal{X}$ in Γ_A different from $\mathcal{C}$ and containing a successor of $P(x)$ in mod A is a component from the family $\bigvee_{1 \leqslant j \leqslant t}(\mathcal{T}_j \vee \mathcal{I}_j)$. We know also that $\operatorname{Hom}(\mathcal{T}_p \vee \mathcal{I}_p, \mathcal{T}_q \vee \mathcal{I}_q) = 0$ for $p \neq q$. Therefore, we get that Σ_A is directed, all components of Γ_A are standard, and A is a multicoil algebra.

(2) Assume now that, for some $y \in \mathcal{M}$ and an indecomposable direct summand U of $R(y)$, there exists a path $U = U_0 \to U_1 \cdots \to U_r = Z$ in $\Gamma_{A(y)}$ with Z nondirecting. We shall show that then there exists $x \in \mathcal{M}$ such that $R(x)$ admits a nondirecting indecomposable direct summand M. If U is nondirecting, we set $x = y$ and $M = U$. Assume U is directing. Since $A(y)$ is a multicoil algebra, Z lies in a coil Ω of a standard multicoil $\mathcal{X}$ of $\Gamma_{A(y)}$. By [4, Lemma 3.3], we may assume that the coil Ω is proper, that is, any module from Ω lies on a cycle in Ω. Observe that Ω contains at least one projective module, because there is in $\mathcal{X}$ a path from the directing module U to the module Z lying in Ω. Further, it follows from the structure of standard coils that, if $\operatorname{Hom}_{A(y)}(P(a), P(b)) \neq 0$ for $P(a)$ from Ω and $P(b)$ from $\mathcal{X}$, then $P(b)$ lies in Ω. Hence, we infer that there exists in $\mathcal{X}$ a path $U = U_0 \to U_1 \to \cdots \to U_s \to P(x)$ such that $P(x)$ lies in Ω and $\operatorname{Hom}_{A(y)}(P(x), P(c)) = 0$ for any projective module $P(c)$ from $\mathcal{X}$ nonisomorphic

to $P(x)$. We claim that $x \in \mathcal{M}$. First we show that $x \in \mathcal{S}$. Suppose $x \notin \mathcal{S}$. Then we have in $\operatorname{mod} A(y)$ a path $P(x) \to P(e_1) \to \cdots \to P(e_r) = P(z)$ with $z \in \mathcal{S}$. By our choice of x we infer that $P(e_1)$ does not belong to $\mathcal{X}$. Then, since $\Sigma_{A(y)}$ is directed, we have $\operatorname{Hom}_{A(y)}(P(z), U_l) = 0$ for any $0 \leqslant l \leqslant s$. Therefore we get an infinite path in $\operatorname{mod} A(z)$ of the form

$$U = U_0 \to U_1 \to \cdots \to U_s \to P(x) \to \cdots \to P(e_{r-1}) \to V,$$

where V is an indecomposable direct summand of $R(z)$. But then $y <^* z$, a contradiction because $y \in \mathcal{M}$. Hence, $x \in \mathcal{S}$. Suppose now that $x <^* w$ for some $w \in \mathcal{S}$. Then $\operatorname{mod} A(w)$ admits an infinite path

$$N = Z_0 \to Z_1 \to \cdots \to Z_q = T,$$

where N is an indecomposable direct summand of $R(x)$ and T an indecomposable direct summand of $R(w)$. Denote by $\mathcal{Y}$ and $\mathcal{Z}$ the components of $\Gamma_{A(w)}$ containing the modules N and T, respectively. Clearly, $\mathcal{Y} \neq \mathcal{Z}$, because $\Sigma_{A(w)}$ is directed. Applying now Lemma 5.1 we conclude that $\mathcal{Y}$ is in fact a component of Γ_A and for any path in $\operatorname{mod} A$ with target lying in $\mathcal{Y}$ its source is an $A(w)$-module. Let p be minimal such that Z_p does not belong to $\mathcal{Y}$. Then $\operatorname{rad}^\infty(Z_{p-1}, Z_p) \neq 0$, and we have in $\operatorname{mod} A(w)$ paths

$$N = Z_0 \to \ldots \to Z_{p-1} \to W_1 \to \cdots \to W_m \to Z_p \to \cdots \to Z_q = T,$$

for all $m \geqslant 1$. On the other hand, since N is a direct predecessor of $P(x)$ and $P(x)$ lies in a proper coil Δ of the standard multicoil $\mathcal{Y}$ of $\Gamma_{A(w)}$, we infer that $P(x)$ is a predecessor of W_m in Δ for a sufficiently large m. We then conclude that $\operatorname{mod} A(w)$ admits an infinite path $U = U_0 \to \cdots \to P(x) \to \cdots \to W_m \to Z_p \to \cdots \to Z_q = T$, and so $y <^* w$. This contradiction proves that $x \in \mathcal{M}$. Let M be a nondirecting indecomposable direct summand of $R(x)$ and B the connected part of $A(x)$ such that M is a B-module. Since B is a convex subcategory of $A(x)$, the conditions (v), (vii) and (viii) hold for B. Further, M is nondirecting in $\operatorname{mod} B$, and so it belongs to a proper coil Γ of a standard multicoil $\mathcal{C}$ of Γ_B. We shall show that the vector space category $\operatorname{Hom}_B(M, \mathcal{C})$, given by the objects $\operatorname{Hom}_B(M, X)$ with X from $\mathcal{C}$, is the K-linear category of one of the following partially ordered sets:

(a) $\operatorname{Hom}_B(M, X_0) \to \operatorname{Hom}_B(M, X_1) \to \operatorname{Hom}_B(M, X_2) \to \cdots$

$$\operatorname{Hom}_B(M, Y_t) \leftarrow \cdots \leftarrow \operatorname{Hom}_B(M, Y_1)$$

(b)

$$\operatorname{Hom}_N(M, X_0) \to \operatorname{Hom}_B(M, X_1) \to \cdots$$

with $t \geqslant 1$ and X_0 injective,

(c)

$$\begin{array}{ccccc}
\mathrm{Hom}_B(M,Y_1) & \to \mathrm{Hom}_B(M,Y_2) & \to & \cdots & \to \mathrm{Hom}_B(M,Y_t) \\
\uparrow & \uparrow & & & \uparrow \\
\mathrm{Hom}_B(M,X_0) & \to \mathrm{Hom}_B(M,X_1) & \to \cdots \to \mathrm{Hom}_B(M,X_{t-1}) & \to & \mathrm{Hom}_B(M,X_t) \to \cdots
\end{array}$$

with $t \geqslant 2$ and X_{t-1} injective,

where $M = X_0, X_1, X_2, \ldots$ lie on a unique infinite sectional path $X_0 \to X_1 \to X_2 \to \cdots$ in $\mathcal{C}$ with source $M = X_0$, and $Y_1, \ldots Y_t$ are directing modules lying on a sectional path $Y_1 \to Y_2 \cdots \to Y_t$ in $\mathcal{C}$. Moreover, we claim that, if $R(x)$ is decomposable, then $A(x) = B \times D$, $R(x) = M \oplus N$, where D is connected, N is an indecomposable D-module, $\mathrm{Hom}_B(M, \mathcal{C})$ is of the form (a), and the vector space category $\mathrm{Hom}_D(N, \mathrm{ind}\, D)$ is the K-linear category of a partially ordered set

$$\mathrm{Hom}_D(N, Y_1) \to \mathrm{Hom}_D(N, Y_2) \to \cdots \to \mathrm{Hom}_D(N, Y_t)$$

with $t \geqslant 1$ and $N = Y_1, \ldots, Y_t$ directing modules lying on a unique sectional path $Y_1 \to \cdots \to Y_t$ in Γ_D.

We divide our considerations into several steps.

(α) Suppose that $\mathcal{C} = \Gamma$ is a stable tube. Then $C = \mathrm{supp}\,\Gamma$ is a convex subcategory of B (see [2, (3.1)]) whose Auslander-Reiten quiver admits a sincere stable tube. Since, by our assumption, C is also a multicoil algebra (hence cycle-finite), we get from [3, (4.1)] (see also [50, (4.1)]), that C is either critical or tubular. Clearly, $E = C[M]$ is also a convex subcategory of A, and hence q_E is weakly nonnegative and E does not contain a pg-critical convex subcategory. Then, by Proposition 2.5, C is critical. Applying now Proposition 2.6, we infer that M is a simple regular module, and so E is a tubular extension of C. Therefore, E is either tubular or representation-infinite tilted algebra of Eulidean type with a complete slice in the preinjective component, because q_E is weakly nonnegative (Proposition 2.3). Clearly, in this case, $\mathcal{C}$ admits a unique sectional path $M = X_0 \to X_1 \to X_2 \to \cdots$ consisting of arrows pointing to infinity, and $\mathrm{Hom}_B(M, \mathcal{C})$ is of type (a).

(β) Assume that $\mathcal{C}$ is not a stable tube. Then by [4, (5.9)], there exists a critical convex subcategory C of B and a stable tube $\mathcal{T}$ in Γ_C such that $\Lambda = \mathrm{supp}\,\Gamma$ is obtained from C by a sequence of admissable operations of types (ad 1), (ad 2), (ad 3), (ad 1*), (ad 2*) or (ad 3*), Γ_Λ admits a coil Γ' obtained from the stable tube $\mathcal{T}$ be the corresponding sequence of admissible operations, and such that Γ is the full translation subquiver Γ'_γ of Γ' consisting of all modules lying on cycles in Γ' (see [4, Section 3] for details). Then $\Lambda = \mathrm{supp}\,\Gamma$ is a coil enlargement of C in the sense of [5]. Moreover, Λ is a convex subcategory of B. Applying now [5, (3.5)], we infer that there is a unique tubular extension $E = \Lambda^+$ of C which is a convex subcategory of Λ and such that Λ can be obtained from E by a sequence of one-point coextensions of types (ad 1*), (ad 2*), or (ad 3*). Furthermore, there is a ray tube $\mathcal{T}^+$ in Γ_E such that Γ' can be obtained from $\mathcal{T}^+$ by the corresponding

sequence of admissible operations of types (ad 1*), (ad 2*), or (ad 3*), being coray insertions. Since M lies in $\Gamma = \Gamma'_\gamma$, there exists an infinite sectional path

$$\Sigma\colon M = X_0 \to X_1 \to X_2 \to X_3 \to \cdots$$

in Γ consisting of arrows pointing to infinity. Further, since Γ' is obtained from the ray tube $\mathcal{T}^+$ by a sequence of coray insertions (operations of types (ad 1*), (ad 2*), (ad 3*)), we get that, for infinitely many $i \geqslant 0$, X_i is an E-module. We claim now that $\operatorname{Hom}_B(M, \mathcal{C})$ does not contain a full subcategory which is the K-linear category of the partially ordered set

$$\begin{array}{ccccccc}
\operatorname{Hom}_B(M, U_s) & \longrightarrow & \operatorname{Hom}_M(M, U_{s+1}) & \longrightarrow & \operatorname{Hom}_B(M, U_{s+2}) & \to \cdots \\
\uparrow & & \uparrow & & \uparrow & \\
\operatorname{Hom}_B(M, X_s) & \longrightarrow & \operatorname{Hom}_M(M, X_{s+1}) & \longrightarrow & \operatorname{Hom}_B(M, X_{s+2}) & \to \cdots
\end{array}$$

$s \geqslant r$, with $U_s, U_{s+1}, \ldots$, lying on an infinite sectional path $U_s \to U_{s+1} \to U_{s+2} \to \cdots$ in Γ' which is parallel to $X_s \to X_{s+1} \to X_{s+2} \to \cdots$. Suppose that this is not the case. Then by the above remark $\mathcal{T}^+$ admits a subquiver of the form

$$\begin{array}{ccccccc}
U_{i_1} & \longrightarrow & U_{i_2} & \longrightarrow & U_{i_3} & \to \cdots \\
\uparrow & & \uparrow & & \uparrow & \\
X_{i_1} & \longrightarrow & X_{i_2} & \longrightarrow & X_{i_3} & \to \cdots
\end{array}$$

where $M' = X_{i_1}$ is the restriction of $M = X_0$ to $E = \Lambda^+$. Moreover, $E' = E[M']$ is a convex subcategory of A, and hence is strongly simply connected. Applying now Proposition 2.6 to the one-point extension $E' = E[M']$ we get that either E' contains a convex pg-critical subcategory or $q_{E'}$ is not weakly nonnegative, which contradicts our assumptions on A. The proved fact shows that Σ is a unique infinite sectional path in $\mathcal{C}$ with source M. We claim now that if $X_i \to P$ is an arrow in $\mathcal{C}$ with P projective then P lies on Σ. Suppose that $X_i \to P$, for some $i \geqslant 0$, is an arrow in $\mathcal{C}$ with P projective not lying on Σ. From the above remarks we know that then $\mathcal{C}$ has no infinite sectional path with source P and parallel to Σ. Hence, there is $j > i$ such that $X_i, \ldots, X_{j-1}$ are noninjective but X_j is injective. Applying now Lemma $(3 + 3 + 2)$ in [2, (2.1)], we obtain an exact sequence

$$0 \to X_i \to P \oplus X_j \oplus V \to \tau_B^- X_{j-1} \to 0.$$

On the other hand we have $\dim_K X_i < \dim_K P$ and $\dim_K \tau_B^- X_{j-1} < \dim_K X_j$. This contradiction proves our claim. We recall also that by [5, (3.5)], there is a unique tubular coextension $F = \Lambda^-$ of C and a coray tube $\mathcal{T}^-$ in Γ_F such that Λ is obtained from F by a sequence of admissible operations of types (ad 1), (ad 2) or (ad 3), and Γ' is obtained from $\mathcal{T}^-$ by the corresponding sequence of admissible

operations (ray insertions). Hence, if $X_i \to Z$ is an arrow in $\mathcal{C}$ with $Z \neq X_{i+1}$ then Z is from $\mathcal{T}^-$. In particular, any module $X_i, i \geqslant 0$, has at most two direct successors in $\mathcal{C}$. Therefore, if M has exactly one direct successor, then the vector space category $\mathrm{Hom}_B(M, \mathcal{C})$ is of type (a).

(γ) Assume now that $\mathcal{C}$ admits a full translation subquiver

$$
\begin{array}{ccccccc}
Y_1 & \longrightarrow & Y_2 & \to \cdots \to & Y_t & & \\
\uparrow & & \uparrow & & \uparrow & & \\
X_0 & \longrightarrow & X_1 & \to \cdots \to & X_{t-1} & \longrightarrow & X_t & \longrightarrow & X_{t+1} & \to \cdots
\end{array}
$$

with $t \geqslant 2$ and X_{t-1} injective. We claim that then $R(x) = M$, $\mathrm{Hom}_B(M, \mathcal{C})$ is of the form (c), induced by the above subquiver of $\mathcal{C}$, and no module $Y_j, 1 \leqslant j \leqslant t$, is predecessor of a projective module in $\mathcal{C}$. Suppose that $R(x) \neq M$ or $\mathcal{C}$ admits an arrow $Y_j \to W$ with $1 \leqslant j \leqslant t$ and $W \neq Y_{j+1}$ (possibly W is projective). We know that the modules $Y_1, \ldots, Y_t$ are from the coray tube $\mathcal{T}^-$, and so their supports are in one coextension branch of the tubular coextension F of C. We look now on the supports of the modules $Y_1, \ldots, Y_t$ and apply the rules for the Auslander-Reiten sequences in the module categories over special biserial simply connected algebras described in [54, p. 180] (see also [37]). Then a simple analysis (see the proof of Lemma 3.5 in [3]) shows that $A = A(x)[R(x)]$ admits a convex subcategory of a form described in Corollary 2.7. This contradicts our assumption on A. Hence $R(x) = M$ and $\mathrm{Hom}_B(M, \mathcal{C})$ is of the required form (c). Suppose now that one of the modules $Y_1, \ldots, Y_t$ is a predecessor of a projective module in $\mathcal{C}$. Then $\mathcal{C}$ admits a path of the form

$$
Y_j = V_0 \to V_1 \to \cdots \to V_{p-1} \to V_p = P(y),
$$

with $j \geqslant 2, p \geqslant 2, V_1 = \tau_B^- Y_{j-1}$ and $V_0, V_1, \ldots, V_p$ directing in $\mathrm{mod}\, B$. Let

$$
P(y) = P(y_0) \to P(y_1) \to \cdots \to P(y_l) = P(z),
$$

be a path in $\mathrm{mod}\, B$ with $z \in \mathcal{S}$. We then have a path

$$
M = X_0 \to \cdots \to X_{j-1} \to V_0 \xrightarrow{\varphi} V_1 \to \cdots \to V_{p-1} \to \cdots \to L
$$

with L and indecomposable direct summand of $R(z)$. Since $V_0, \ldots, V_{p-1}$ are directing modules in $\mathrm{mod}\, B$, we infer that in fact it is a path in $\mathrm{mod}\, A(z)$. We claim that φ belongs to $\mathrm{rad}^\infty(\mathrm{mod}\, A(z))$. Indeed, this follows from Lemma 3.1 if we take $R = X_0, Y = Y_j, Z = \tau^- Y_{j-1}, W_i = X_{t+i}, i \geqslant 1, f_i : W_i \to W_{i+1}, i \geqslant 1$, arbitrary irreducible maps and $g : R \to Y, f_0 : R \to W_1$ the compositions of the corresponding irreducible maps. Hence we get an infinite path in $\mathrm{mod}\, A(z)$ from M to L, and so $x < z$. But this is a contradiction with the maximality of x in $\mathcal{S}$. Therefore, the modules $Y_1, \ldots, Y_t$ have no projective successors in $\mathcal{C}$.

(δ) Assume now that $\mathrm{Hom}_B(M, \mathcal{C})$ is of the form (a) but $R(x) = M \oplus N$ with $N \neq 0$. We know that the restriction M' of M to the tubular extension $E = \Lambda^+$ of C is nonzero. Then applying Corollary 2.7 we infer that N is indecomposable, and moreover uniserial. Hence, $A(x) = B \times D$ where D is connected and N is in $\mathrm{mod}\, D$. Let $\mathcal{D}$ be the component of Γ_D containing N. We claim that there is a sectional path

$$N = Y_1 \to Y_2 \to \cdots \to Y_t$$

in $\mathcal{D}$ formed by directing modules such that the vector space category $\mathrm{Hom}_D\,(N, \mathrm{ind}\, D)$ is the K-linear category of the partially ordered set

$$\mathrm{Hom}_D(N, Y_1) \to \mathrm{Hom}_D(N, Y_2) \to \cdots \to \mathrm{Hom}_D(N, Y_t).$$

Since N is uniserial and D is strongly simply connected, as a convex subcategory of A, the support of N is given by a convex line in Q_D of the form $a_1 \to a_2 \to \cdots \to a_r$. Let G be the maximal convex subcategory of D which is the bound quiver algebra of a finite connected bound subquiver of the following infinite tree
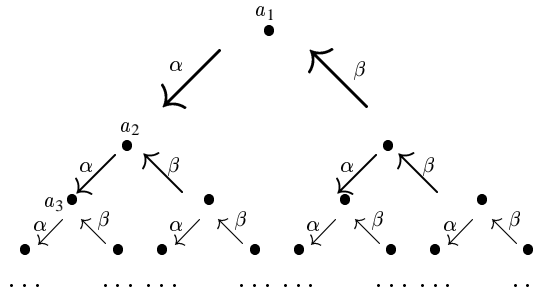
bound by all possible relations $\beta\alpha = 0$, and N is the projective G-module $P_G(a_1)$. It is known (see [49, (4.4)]) that G is a tilted algebra of Dynkin A_t and Γ_G admits a complete section (slice) Θ of the form

$$N = Y_1 \to Y_2 \to \cdots \to Y_t.$$

Using the maximality of G, our assumptions on A, and applying Corollary 2.7 again, we infer that, if $G[V]$ is a one-point extension of G inside D with V indecomposable, then V is a proper successor of a module Y_i and does not lie on Θ. In particular, $\mathrm{Hom}_D(N, \mathrm{ind}\, D)$ is the required K-linear category of a finite linear partially ordered set. The maximality of x in $\mathcal{S}$ implies also that any successor of N in Γ_D is a G-module. Indeed, if this is not the case, then there is in $\mathrm{mod}\, D$ a path of the form

$$Y_j = V_0 \xrightarrow{\varphi} V_1 \to \cdots \to V_{p-1} \to P(y_1) \to \cdots \to P(y_l) \to L$$

where $j \geqslant 2$, $V_1 = \tau_G^- Y_{j-1}, V_0, \ldots, V_{p-1}$ are directing D-modules (in fact G-modules) and L is an indecomposable direct summand of $R(z)$ for some $z \in \mathcal{S}$. Clearly, then it is a path in $\mod A(z)$. We claim that $\varphi \in \mathrm{rad}^\infty (\mod A(z))$. This follows from Lemma 3.1 if we put $R = R(x), Y = V_0, Z = V_1, W_i = X_i$ for $i \geqslant 1$, and take for $f_i : W_i \to W_{i+1}$ arbitrary irreducible maps, for $f_0 : R \to W_1 = X_1$ the composition of the projection $R = M \oplus N \to X_0 = M$ with an irreducible map $X_0 \to X_1 = W_1$, and for $g : R \to Y$ the composition of the projection $R = M \oplus N \to N$ with irreducible maps $Y_1 \to \cdots \to Y_j$. Since $\mathrm{Hom}_{A(x)}(P(z), Y_j) = 0$ and $\mathrm{Hom}_{A(x)}(P(z), X_i) = 0$ for all $i \geqslant 1$ we get from the proof of Lemma 3.1 that in fact $\varphi \in \mathrm{rad}^\infty (\mod A(z))$. Therefore, we have in $\mod A(z)$ an infinite path from N to L, a contradiction with the maximality of x in $\mathcal{S}$.

(σ) Assume now that M is injective and has two direct successors in $\mathcal{C}$. Applying Corollary 2.7 we infer that then $R(x) = M$. Let Y_1 be the direct successor of M in $\mathcal{C}$ different from X_1. Then Y_1 is a module from $\mathcal{T}^-$. Observe that M is not necessarily an F-module but its restriction to F is an injective F-module having two direct successors in $\mathcal{T}^-$. Let

$$Y_1 = U_1 \to U_2 \to \cdots \to U_r,$$

be the maximal sectional path in $\mathcal{T}^-$ with source Y_1. Clearly, it consists of injective F-modules. Applying Corollary 2.7 again we infer that none of modules $U_1, \ldots, U_{r-1}$ is a direct summand of the radical of a projective module inside B. Suppose now that there is in $\mathcal{T}^-$ a sectional path

$$V = V_0 \to V_1 \to \cdots \to V_s = U_k,$$

with $1 \leqslant k \leqslant r - 1$ such that V is a summand of the radical of a projective module $P(d)$ in B. We may assume that k is minimal with this property. Since $F = \Lambda^-$ is a tubular coextension of C, the supports of all successors of V in $\mathcal{T}^-$ are contained in one (coextension) branch of F. Looking on the support of V, and applying Corollary 2.7 and the formulas for the Auslander-Reiten sequences in module categories over special biserial simply connected algebras (see [54], [37]), we conclude that the vector space category $\mathrm{Hom}_F(V, \mathrm{ind} F)$ is the K-linear category of one of the following finite partially ordered sets:

$$\mathrm{Hom}_F(V, V_0) \to \mathrm{Hom}_F(V, V_1) \to \cdots \to \mathrm{Hom}_F(V, V_s) \tag{a$'$}$$

$$\mathrm{Hom}_F(V, W_l) \leftarrow \cdots \to \mathrm{Hom}_F(V, W_1) \leftarrow \mathrm{Hom}_F(V, V_0) \to \cdots \to$$

$$\mathrm{Hom}_F(V, V_s) \tag{b$'$}$$

with $l \geqslant 1$ and $V = V_0$ injective, (c$'$)

$$
\begin{array}{ccccccc}
\operatorname{Hom}_F(V,W_1) & \to & \operatorname{Hom}_F(V,W_2) & \to & \cdots & \to & \operatorname{Hom}_F(V,W_l) \\
\uparrow & & \uparrow & & & & \uparrow \\
\end{array}
$$

$$
\operatorname{Hom}_F(V,V_0) \to \operatorname{Hom}_F(V,V_1) \to \cdots \to \operatorname{Hom}_F(V,V_{l-1}) \to \operatorname{Hom}_F(V,V_l) \to \cdots \to \operatorname{Hom}_F(V,V_s)
$$

with $2 \leqslant l \leqslant s$ and V_{l-1} injective.

Moreover, we conclude as above that, if the radical $R(d)$ of $P_A(d)$ is decomposable, then $\operatorname{Hom}_F(V,\operatorname{ind} F)$ is of the form (a'), $R(d) = V \oplus W$, where W is a uniserial module whose support is the K-linear category H a linear quiver $l \to l-1 \to \cdots \to 2 \to 1, l \geqslant 1$. Then, applying [4, (2.1)], and [37], we infer that, for $F' = F[V]$ or $F' = (F \times H)[V \oplus W]$, the component of $\Gamma_{F'}$ containing $P(d)$ is obtained from the coray tube $\mathcal{T}^-$ by inserting a rectangle and glueing with $\mathcal{T}^-$ as follows:
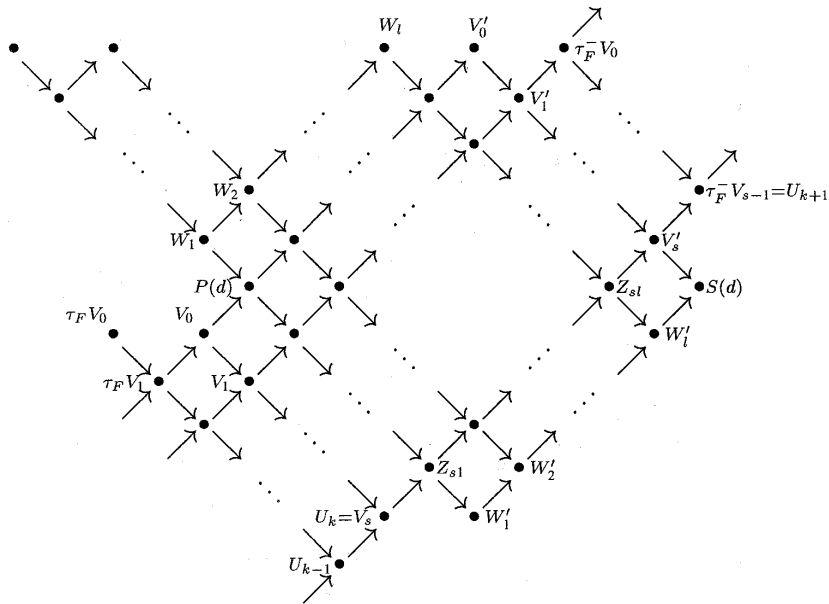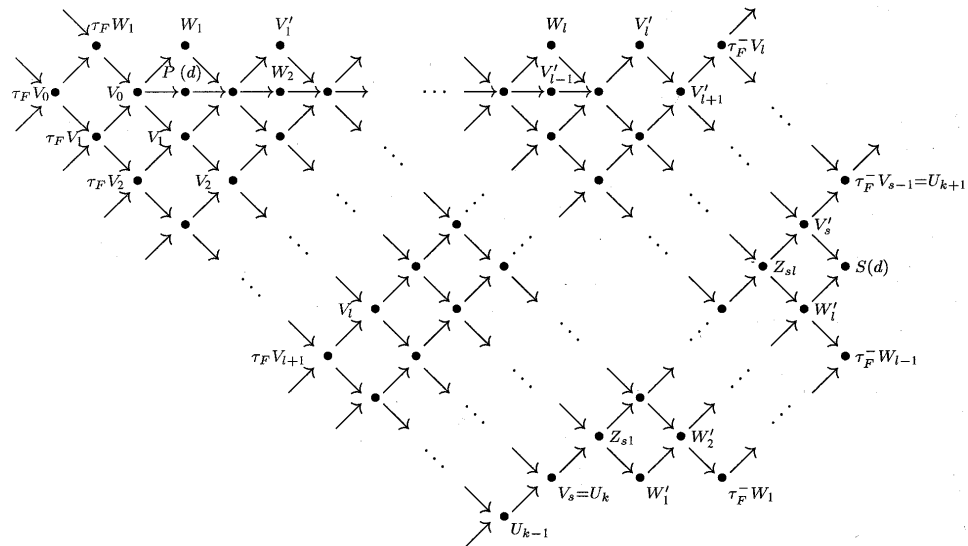

if $\operatorname{Hom}_F(V,\operatorname{ind} F)$ is of type (a') and $R(d)$ decomposable;
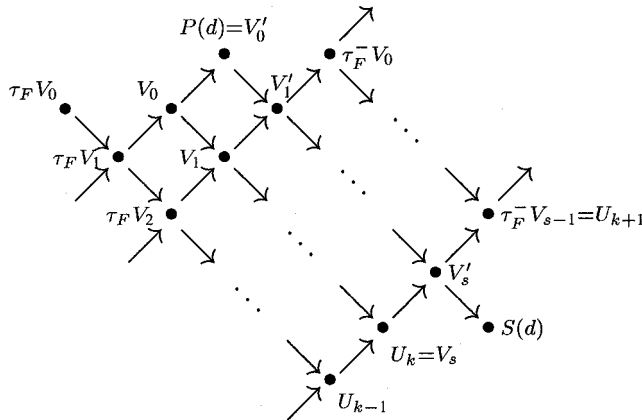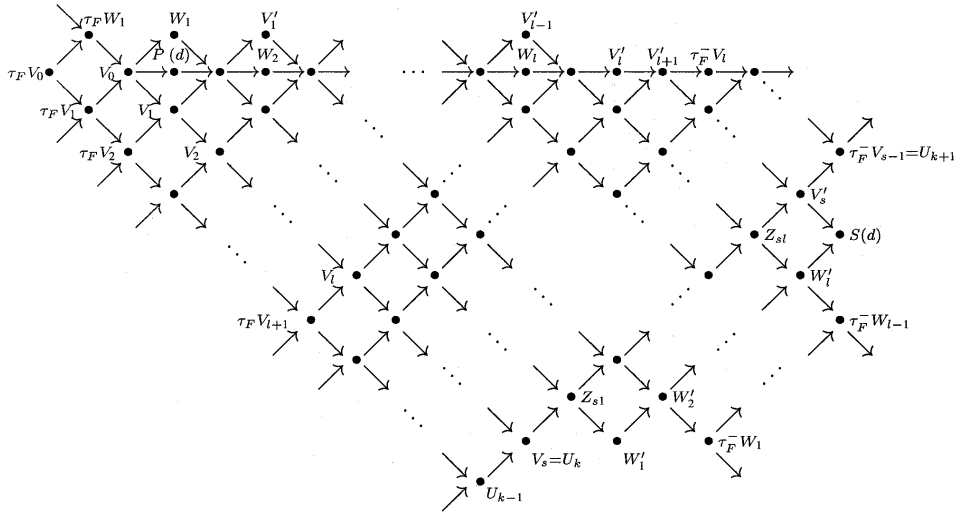
if $\mathrm{Hom}_F(V, \mathrm{ind}F)$ is of type (a′) and $R(d)$ indecomposable;
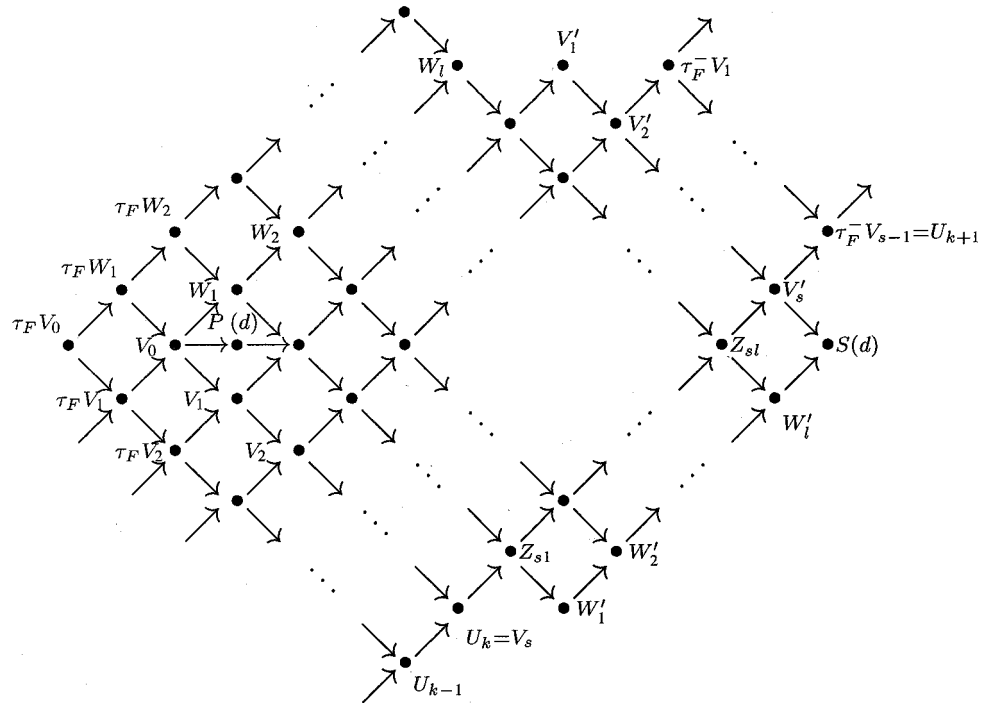
if $\mathrm{Hom}_F(V, \mathrm{ind}\, F)$ is of type (b′);

if $\operatorname{Hom}_F(V, \operatorname{ind} F)$ is of type (c$'$) and l is odd; or

if $\operatorname{Hom}_F(V, \operatorname{ind} F)$ is of type (c$'$) and l is even.

Observe that the sectional path $Y_1 = U_1 \to \cdots \to U_r$ is in this process extended to one of the sectional paths

$$Y_1 \to U_1 \to \cdots \to U_k \to V_s' \to U_{k+1} \to \cdots \to U_r$$

or

$$Y_1 = U_1 \to \cdots \to U_{k-1} \to U_k \to Z_{s1} \to \cdots \to Z_{sl}$$
$$\to V_s' \to U_{k+1} \to \cdots \to U_r.$$

Moreover, from the choice of k and the shape of the inserted rectangle, we infer that none of $U_1, \ldots, U_k$ is the target of a sectional path with source being a direct summand of the radical of indecomposable projective module which is nonisomorphic to $P(d)$. If one of the modules $Z_{s1}, \ldots, Z_{sl}, V_s, U_{k+1}, \ldots, U_r$ lies on a sectional path in $\mathcal{C}$ with source being direct predecessor of a projective module we repeat the above analysis. Since $\mathcal{C}$ admits only finitely many projective modules, we conclude that in fact $\mathcal{C}$ admits a finite sectional path

$$\Theta : Y_1 \to Y_2 \to \cdots \to Y_t$$

and $\operatorname{Hom}_B(M, \mathcal{C})$ is of the form (b). Observe also that after the extension of $A(x)$ to $A = A(x)[R(x)]$, any irreducible map $Y_i \to Z$ with $Z \not\simeq Y_{i+1}$ will belong, by Lemma 3.1, to $\operatorname{rad}^\infty (\operatorname{mod} A)$. In particular, any projective module which is successor of a module $Y_i, 1 \leqslant i \leqslant t$, lies on the path Θ.

Applying now [4] and [5] we conclude that, in all cases considered above, M is the pivot of an admissible operation of type (ad 1), (ad 2), or (ad 3), and the component $\mathcal{C}'$ of Γ_A containing $P(x)$, is a standard multicoil. Moreover, it follows from Lemma 5.1 and the maximality of x in $\mathcal{S}$, that any projective module which is successor of $P(x)$ in mod A belongs to $\mathcal{C}'$.

We shall describe the remaing components of Γ_A. First observe that, if $\mathcal{E}$ is a component in $\Gamma_{A(x)}$, such that $\operatorname{Hom}_{A(x)}(R(x), E) = 0$ for all modules E from $\mathcal{E}$, then $\mathcal{E}$ remains a full component in Γ_A, by [39, (2.5)]. Let $\mathcal{F}$ be a component of Γ_A different from $\mathcal{C}'$ but containing a module Z with $\operatorname{Hom}_A(P(x), Z) \neq 0$. By [5, (4.1)], there exists a maximal tubular extension Ω of C which is a convex subcategory of A and $\mathcal{F}$ is a component of Γ_Ω. It follows from Proposition 2.3 and our assumptions on A that Ω is either (representation-infinite) tilted of Euclidean type with a complete slice in the preinjective component or tubular. In the first case, $\mathcal{F}$ is the unique preinjective component of Γ_Ω. In the second case, $\mathcal{F}$ is, in the notations of Section 1, a component in

$$\left(\bigvee_{\gamma \in \mathbb{Q}^+} \mathcal{T}_\gamma \right) \vee \mathcal{T}_\infty \vee \mathcal{I}_\infty.$$

We know that all components in Γ_Ω are standard and the component quiver Σ_Ω is directed. Hence, $\mathcal{F}$ is either a standard preinjective component or a standard tube without projective modules. Applying now our assumptions on $A(x)$, we conclude that A is a multicoil algebra, all components in Γ_A are standard, and Σ_A is directed. This finishes the proof of Theorem 4.1.

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