COMPOSITIO MATHEMATICA

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Compositio Math. **159** (2023), 380–418.

doi:10.1112/S0010437X22007904

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY EST. 1865

Compositio Math. 159 (2023) 380–418
doi:10.1112/S0010437X22007904

The explicit Zelevinsky–Aubert duality

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Abstract

In this paper, we give an explicit computable algorithm for the Zelevinsky–Aubert duals of irreducible representations of p-adic symplectic and odd special orthogonal groups. To do this, we establish explicit formulas for certain derivatives and socles. We also give a combinatorial criterion for the irreducibility of certain parabolically induced representations.

Contents

1. Introduction

Let F be a local non-Archimedean field. In 1980, Zelevinsky [Zel80] defined an involution $\tau \mapsto \hat{\tau}$ on the Grothendieck group of finite-length smooth representations of $\mathrm{GL}_n(F)$ and conjectured that this involution should preserve irreducibility. Assuming this conjecture, in 1986, Mœglin and Waldspurger [MW86] studied the involution and gave an algorithm for computing the Langlands (or Zelevinsky) data of $\hat{\tau}$ for every irreducible representation τ of $\mathrm{GL}_n(F)$. Later, another explicit formula was given by Knight and Zelevinsky [KZ96].

Motivated by the Alvis–Curtis duality for finite groups [Alv79, Alv82, Cur80], Kato [Kat93] defined an involution on the Grothendieck group of smooth finite-length Iwahori-fixed representations of a split reductive group over F. In 1996, Aubert showed that Kato's involution could be extended to the category of finite-length smooth representations of any reductive group G and

Received 6 October 2021, accepted in final form 29 August 2022, published online 20 February 2023.
2000 Mathematics Subject Classification 22E50 (primary), 11S37 (secondary).
Keywords: Zelevinsky–Aubert duality, derivatives, socles.

proved that it indeed preserves irreducibility. Furthermore, using different approaches, Schneider and Stuhler [SS97], as well as Bernstein, Bezrukavnikov and Kazhdan [Ber92, BBK18, Bez04], defined involutions on the category of smooth representations of G. For irreducible representations of $\mathrm{GL}_n(F)$, all these involutions coincide (up to the contragredient and up to a sign) with the involution defined by Zelevinsky.

For simplicity, when restricted to the set of irreducible smooth representations of a reductive group G, this involution is commonly known as the *Zelevinsky–Aubert duality*, and it is the main topic of this article. This duality has many interesting applications to Koszul duality (see [MR15]) and to the Langlands program (see for example [Tad18] or [Wal18]). One important property of the Zelevinsky–Aubert duality is that it does not preserve the fact of being tempered. For this reason, in the proof of Arthur's local classification, the first step beyond tempered representations is to consider the Zelevinsky–Aubert dual of tempered representations [Art13, § 7]. However, one expects that the duality will preserve unitarity, so it should be an important tool for determining the unitary dual of classical groups [Tad22].

Our goal is to extend the result of Mœglin and Waldspurger to the Zelevinsky–Aubert duality, that is, we give an algorithm for computing the Langlands data of $\hat{\pi}$ in terms of those of π, for every irreducible representation π of G. As we will use the endoscopic classification of Arthur [Art13] and Mœglin's construction of the local packets [Mœg11], we focus on the case where F is a local non-Archimedean field of characteristic 0 and G is either a symplectic or an odd special orthogonal group.

There have been several attempts to explicitly describe the Zelevinsky–Aubert duality. There are some partial results due to Mœglin [Mœg06], Matić [Mat17, Mat19], Jantzen [Jan18a] and the first author of the present paper [Ato22b]. In order to explain the novelties of the present article, let us introduce some notation.

Let G be a connected algebraic reductive group defined over F. Fix a minimal parabolic subgroup P_0 of G. We denote by Ind_P^G the normalized parabolic induction and by Jac_P^G its left adjoint functor, the Jacquet functor.

Let Π be a smooth finite-length representation of G. We consider the virtual semisimple representation

$$D_G(\Pi) = \sum_P (-1)^{\dim A_M} \left[\mathrm{Ind}_P^G(\mathrm{Jac}_P^G(\Pi))\right],$$

where $P = MN$ runs over all standard parabolic subgroups of G and A_M is the maximal split torus of the center of M. Then Aubert [Aub95] showed that if π is irreducible, there exists a sign $\epsilon \in \{\pm 1\}$ such that $\hat{\pi} = \epsilon \cdot D_G(\pi)$ is also an irreducible representation. We call the map $\pi \mapsto \hat{\pi}$ the *Zelevinsky–Aubert duality*.

This map has the following important properties.

(1) The dual of $\hat{\pi}$ is equal to π, i.e. the map $\pi \mapsto \hat{\pi}$ is an involution.
(2) If π is supercuspidal, then $\hat{\pi} = \pi$.
(3) The duality commutes with Jacquet functors (see (2.1)).

Let us now restrict ourselves to the case where $G = G_n$ is either the split special orthogonal group $\mathrm{SO}_{2n+1}(F)$ or the symplectic group $\mathrm{Sp}_{2n}(F)$ of rank n. In this case, when π (respectively τ_i) is a smooth representation of G_{n_0} (respectively $\mathrm{GL}_{d_i}(F)$), with $d_1 + \cdots + d_r + n_0 = n$, we denote by

$$\tau_1 \times \cdots \times \tau_r \rtimes \pi$$

the normalized parabolically induced representation of $\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi$ from the standard parabolic subgroup P of G_n with Levi subgroup isomorphic to $\mathrm{GL}_{d_1}(F) \times \cdots \times \mathrm{GL}_{d_r}(F) \times G_{n_0}$.

https://doi.org/10.1112/S0010437X22007904 Published online by Cambridge University Press

Given an irreducible representation π of G_n and a supercuspidal *non-self-dual* representation ρ of $\mathrm{GL}_d(F)$, there exists a unique $k \geq 0$ and a unique irreducible representation π_0 of G_{n_0}, with $n = dk + n_0$, such that

- π is a unique irreducible subrepresentation of

$$\underbrace{\rho \times \cdots \times \rho}_{k \text{ times}} \rtimes \pi_0; \tag{1.1}$$

- k is maximal, in the sense that for every irreducible representation π_0' of G_{n_0-d}, π_0 is not a subrepresentation of $\rho \rtimes \pi_0'$.

We call π_0 the *highest ρ-derivative* of π and denote it by $D_\rho^{\max}(\pi)$. An important consequence of the commutativity of the Zelevinsky–Aubert duality with Jacquet functors is that

$$D_\rho^{\max}(\pi)\hat{} = D_{\rho^\vee}^{\max}(\hat{\pi}), \tag{1.2}$$

where $\rho^\vee$ denotes the contragredient of ρ.

We can now describe the main idea of the algorithm for explicating the Zelevinsky–Aubert dual of an irreducible representation π of G_n. It is a two-step procedure as follows.

Step 1. If there exists a supercuspidal non-self-dual representation ρ of $\mathrm{GL}_d(F)$ such that $D_\rho^{\max}(\pi) \neq \pi$, then we give an explicit formula for the Langlands data of $D_\rho^{\max}(\pi)$ in terms of those of π. By induction we can compute the Langlands data of $D_\rho^{\max}(\pi)\hat{}$. We finally give an explicit formula for the Langlands data of $\hat{\pi}$ in terms of those of $D_\rho^{\max}(\pi)\hat{} = D_{\rho^\vee}^{\max}(\hat{\pi})$.

Step 2. Assume finally that for all supercuspidal representations ρ of $\mathrm{GL}_d(F)$ such that π is a subrepresentation of $\rho \times \pi_0$ for some irreducible representation π_0 of G_{n-d}, we have that ρ is self-dual. Then the following hold.

- If π is tempered, then π is 'almost supercuspidal', and we can compute its Zelevinsky–Aubert dual explicitly (see § 5.3, in particular Proposition 5.4).
- If π is not tempered, then we show that there exists a supercuspidal self-dual representation ρ of $\mathrm{GL}_d(F)$ such that π is a unique irreducible subrepresentation of

$$\underbrace{\Delta_\rho[0,-1] \times \cdots \times \Delta_\rho[0,-1]}_{k \text{ times}} \rtimes \pi_0$$

for some irreducible representation π_0 of G_{n_0}, with $n = 2dk + n_0$, and some positive integer k maximal as above, where $\Delta_\rho[0,-1]$ is a Steinberg representation (see § 2.3 for a precise definition). We call π_0 the *highest $\Delta_\rho[0,-1]$-derivative* and denote it by $D_{\Delta_\rho[0,-1]}^{\max}(\pi)$. Similar to (1.2), this derivative satisfies a formula

$$D_{\Delta_\rho[0,-1]}^{\max}(\pi)\hat{} = D_{Z_\rho[0,1]}^{\max}(\hat{\pi}),$$

where $D_{Z_\rho[0,1]}^{\max}(\hat{\pi})$ is the *highest $Z_\rho[0,1]$-derivative* of $\hat{\pi}$ (see § 3.4). As in Step 1, this allows us to compute by induction the Zelevinsky–Aubert dual of π. The precise algorithm is explained in § 4.

Let us first remark on the self-duality condition on ρ. When ρ is self-dual, a representation of the form (1.1) may have several irreducible subrepresentations and there is no simple way of distinguishing them. The same problem was already observed by Jantzen [Jan18a]. For these reasons he just considered what is called the half-integral case.

This also explains one of the differences between the case of $\mathrm{GL}_n(F)$ and the case of classical groups that we treat in this article. In the former case, induced representations of the form $\rho \times \pi_0$,

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with ρ supercuspidal, always have a unique irreducible subrepresentation. The second difference is that for $\mathrm{GL}_n(F)$ it is much easier to explicate the Langlands data of this subrepresentation in terms of those of π. However, the most intricate part of this article is to explicitly describe, in terms of Langlands data, the correspondence $\pi \leftrightarrow D_\tau^{\max}(\pi)$ for τ either supercuspidal non-self-dual or of the form $Z_\rho[0,1]$; see Theorems 7.1, 7.4 and 8.1. To explicate these formulas, we use matching functions as in [LM16] and A-parameters. These results are interesting in their own right. In particular, we get a combinatorial criterion for the irreducibility of parabolically induced representations of the form $\rho \rtimes \pi_0$ with ρ non-self-dual supercuspidal and π_0 irreducible; see Corollary 7.2. Moreover, the explicit formulas established in this paper are used in [Ato22a] to make Mœglin's construction of local A-packets more computable.

The paper is organized as follows. In § 2, we recall some general results on representation theory of p-adic classical groups. In § 3, we define ρ-derivatives and other derivatives, and we prove some general results about them, in particular their compatibility with the Zelevinsky–Aubert duality. In § 4 we give our algorithm for computing the Zelevinsky–Aubert dual using derivatives and socles. We will prove explicit formulas for these derivatives and socles in several situations in §§ 6–8. To do this, we review Arthur's theory of endoscopic classification in § 5 and the theory of matching functions at the beginning of § 6.

2. Notation and preliminaries

In this section we introduce some notation, in particular the functors of induction and restriction, Tadić's formula and Jantzen's decomposition.

2.1 Notation

Throughout this article, we fix a non-Archimedean locally compact field F of characteristic zero with normalized absolute value $|\cdot|$. Let G be the group of F-points of a connected reductive group defined over F, with the usual topology. We will only consider smooth representations of G, that is, representations such that the stabilizer of every vector is an open subgroup of G, and we write $\mathrm{Rep}(G)$ for the category of smooth complex representations of G of finite length. Denote by $\mathrm{Irr}(G)$ the set of equivalence classes of irreducible objects of $\mathrm{Rep}(G)$. Let $\mathscr{R}(G)$ be the Grothendieck group of $\mathrm{Rep}(G)$. The canonical map from the objects of $\mathrm{Rep}(G)$ to $\mathscr{R}(G)$ will be denoted by $\pi \mapsto [\pi]$.

For $\pi, \pi' \in \mathrm{Rep}(G)$ we write $\pi \hookrightarrow \pi'$ (respectively $\pi \twoheadrightarrow \pi'$) if there exists an injective (respectively surjective) morphism from π to π'.

Fix a minimal F-parabolic subgroup P_0 of G. A parabolic subgroup P of G is said to be *standard* if it contains P_0. Henceforth, the letter P will always denote a standard parabolic subgroup of G with an implicit standard Levi decomposition $P = MU$. Let Σ denote the set of roots of G with respect to P_0, and let Δ be a basis of Σ. For $\Theta \subset \Delta$ let P_Θ denote the standard parabolic subgroup of G corresponding to Θ and let M_Θ be a corresponding standard Levi subgroup. Let W be the Weyl group of G.

Let τ be a representation of M, regarded as a representation of P on which U acts trivially. We denote by $\mathrm{Ind}_P^G \tau$ the representation of G parabolically induced from τ. (We will always mean the normalized induction.) We view Ind_P^G as a functor. Its left adjoint, the Jacquet functor with respect to P, will be denoted by Jac_P^G.

An irreducible representation π of G is said to be *supercuspidal* if it is not a composition factor of any representation of the form $\mathrm{Ind}_P^G(\tau)$ with P a proper parabolic subgroup of G and τ a representation of M. We write $\mathscr{C}(G)$ for the subset of $\mathrm{Irr}(G)$ consisting of supercuspidal

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representations. For any $\pi \in \mathrm{Rep}(G)$, we denote by $\pi^\vee$ the contragredient of π. (The sets $\mathrm{Irr}(G)$ and $\mathscr{C}(G)$ are invariant under $^\vee$.)

Let Π be a smooth representation of G of finite length. The *socle* of Π is the largest semisimple subrepresentation of Π. It is denoted by $\mathrm{soc}(\Pi)$. We say that Π is *socle irreducible* (*SI*) if $\mathrm{soc}(\Pi)$ is irreducible and occurs with multiplicity one in $[\Pi]$.

2.2 The Zelevinsky–Aubert duality

We consider the map

$$\begin{aligned} D_G \colon \mathscr{R}(G) &\longrightarrow \mathscr{R}(G) \\ \pi &\longmapsto \sum_P (-1)^{\dim A_M} \left[\mathrm{Ind}_P^G(\mathrm{Jac}_P^G(\pi))\right], \end{aligned}$$

where $P = MN$ runs over all standard parabolic subgroups of G. Aubert [Aub95] showed that if π is irreducible, then there exists a sign $\epsilon \in \{\pm 1\}$ such that $\hat{\pi} = \epsilon \cdot D_G(\pi)$ is also an irreducible representation. We call the map

$$\begin{aligned} \mathrm{Irr}(G) &\to \mathrm{Irr}(G) \\ \pi &\mapsto \hat{\pi} \end{aligned}$$

the *Zelevinsky–Aubert duality*.

It has the following important properties.

(1) For any $\pi \in \mathrm{Irr}(G)$, the dual of $\hat{\pi}$ is equal to π, that is, the map $\pi \mapsto \hat{\pi}$ is an involution [Aub95, Théorème 1.7(3)].
(2) If $\pi \in \mathscr{C}(G)$, then $\hat{\pi} = \pi$ [Aub95, Théorème 1.7(4)].
(3) Let $\Theta \subset \Delta$ and consider the standard parabolic subgroup $P = P_\Theta$ with Levi decomposition $P = MN$. Let w_0 be the longest element in the set $\{w \in W \mid w^{-1}(\Theta) > 0\}$ and let P' be the standard parabolic subgroup with Levi subgroup $M' = w^{-1}(M)$. Then we have (cf. [Aub95, Théorème 1.7(2)])
$$\mathrm{Jac}_P^G \circ D_G = \mathrm{Ad}(w_0) \circ D_{M'} \circ \mathrm{Jac}_{P'}^G. \tag{2.1}$$

2.3 Representations of general linear groups

Set $\mathrm{Irr}^{\mathrm{GL}} \coloneqq \bigcup_{n\geq 0} \mathrm{Irr}(\mathrm{GL}_n(F))$, and let $\mathscr{C}^{\mathrm{GL}} \subset \mathrm{Irr}^{\mathrm{GL}}$ be the subset of supercuspidal representations of $\mathrm{GL}_n(F)$ for every $n > 0$. We write $\mathscr{R}^{\mathrm{GL}} \coloneqq \bigoplus_{n \geq 0} \mathscr{R}(\mathrm{GL}_n(F))$.

Let $d_1, \dots, d_r$ be some positive integers. Let $\tau_i \in \mathrm{Rep}(\mathrm{GL}_{d_i}(F))$ for $1 \leq i \leq r$. It is customary to denote the normalized parabolically induced representation by

$$\tau_1 \times \cdots \times \tau_r \coloneqq \mathrm{Ind}_P^{\mathrm{GL}_{d_1+\cdots+d_r}(F)}(\tau_1 \boxtimes \cdots \boxtimes \tau_r).$$

This product induces a $\mathbb{Z}$-graded ring structure on $\mathscr{R}^{\mathrm{GL}}$. We denote the multiplication by m. If $\tau_1 = \cdots = \tau_r = \tau$, we will write $\tau^r = \tau \times \cdots \times \tau$ (r times).

The Jacquet functor for $\mathrm{GL}_m(F)$ along the maximal parabolic subgroup $P_{(d,m-d)}$ with Levi subgroup isomorphic to $\mathrm{GL}_d(F) \times \mathrm{GL}_{m-d}(F)$ is denoted by $\mathrm{Jac}_{(d,m-d)} = \mathrm{Jac}_{P_{(d,m-d)}}^{\mathrm{GL}_m(F)}$. It induces a co-multiplication, that is, a ring homomorphism

$$\begin{aligned} m^* \colon \mathscr{R}^{\mathrm{GL}} &\longrightarrow \mathscr{R}^{\mathrm{GL}} \otimes \mathscr{R}^{\mathrm{GL}} \\ \tau &\longmapsto \sum_{n \geq 0} \left(\sum_{n_1+n_2=n} \left[\mathrm{Jac}_{(n_1,n_2)}(\tau)\right] \right). \end{aligned}$$

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We finally take

$$M^*\colon \mathscr{R}^{\mathrm{GL}} \longrightarrow \mathscr{R}^{\mathrm{GL}} \otimes \mathscr{R}^{\mathrm{GL}}$$

to be the composition $M^* = (m \otimes 1) \circ (\cdot^\vee \otimes m^*) \circ s \circ m^*$, where $s\colon \mathscr{R}^{\mathrm{GL}} \otimes \mathscr{R}^{\mathrm{GL}} \to \mathscr{R}^{\mathrm{GL}} \otimes \mathscr{R}^{\mathrm{GL}}$ denotes the transposition $s(\sum_i \tau_i \otimes \tau_i') = \sum_i \tau_i' \otimes \tau_i$.

If $\tau \in \mathrm{Irr}^{\mathrm{GL}}$, there exist $\rho_1, \dots, \rho_r \in \mathscr{C}^{\mathrm{GL}}$ such that τ is a subrepresentation of $\rho_1 \times \cdots \times \rho_r$. The set $\mathrm{scusp}(\pi) \coloneqq \{\rho_1, \dots, \rho_r\}$ is uniquely determined by π and is called the *supercuspidal support* of τ.

For $\pi \in \mathrm{Rep}(\mathrm{GL}_n(F))$ and a character χ of $F^\times$, we denote by $\pi \cdot \chi$ the representation obtained from π by twisting by the character $\chi \circ \det$. If $\rho \in \mathscr{C}^{\mathrm{GL}}$, we denote by $\mathbb{Z}_\rho = \{\rho|\cdot|^a \mid a \in \mathbb{Z}\}$ *the line of* ρ.

A *segment* $[x,y]_\rho$ is a sequence of supercuspidal representations of the form

$$\rho|\cdot|^x,\ \rho|\cdot|^{x-1}, \dots, \rho|\cdot|^y,$$

where $\rho \in \mathscr{C}^{\mathrm{GL}}$ and $x, y \in \mathbb{R}$ with $x - y \in \mathbb{Z}$ and $x \geq y$.

One can associate with a segment $[x,y]_\rho$ two irreducible representations of $\mathrm{GL}_{d(x-y+1)}(F)$. We denote by $\Delta_\rho[x,y]$ the *Steinberg representation* of $\mathrm{GL}_{d(x-y+1)}(F)$, i.e. the unique irreducible subrepresentation of

$$\rho|\cdot|^x \times \rho|\cdot|^{x-1} \times \cdots \times \rho|\cdot|^y,$$

and we also write $Z_\rho[y,x]$ for its unique irreducible quotient. For example, when $\rho = \mathbf{1}_{\mathrm{GL}_1(F)}$, we have $Z_\rho[-(n-1)/2, (n-1)/2] = \mathbf{1}_{\mathrm{GL}_n(F)}$.

The Steinberg representation $\Delta_\rho[x,y]$ is an essentially discrete series, and all essentially discrete series are of this form [Zel80, Theorem 9.3]. By convention, we take $\Delta_\rho[x, x+1] = Z_\rho[x+1, x]$ to be the trivial representation of the trivial group $\mathrm{GL}_0(F)$.

If the segments $[x_1, y_1]_{\rho_1}, \dots, [x_r, y_r]_{\rho_r}$ are such that $x_i \geq y_i$ and $x_1 + y_1 \leq \cdots \leq x_r + y_r$, then the socle (Langlands subrepresentation)

$$L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r]) \coloneqq \mathrm{soc}(\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r])$$

is irreducible. When $\rho_1 = \cdots = \rho_r$, $x_1 < \cdots < x_r$, $y_1 < \cdots < y_r$ and $x_1 \equiv \cdots \equiv x_r \bmod \mathbb{Z}$, we call it a *ladder representation*. As a special case, when $x_i = x_1 + i - 1$ and $y_i = y_1 + i - 1$ for $1 \leq i \leq r$, the ladder representation $L(\Delta_\rho[x_1, y_1], \dots, \Delta_\rho[x_r, y_r])$ is also called a *Speh representation*.

The Jacquet modules of $\Delta_\rho[x,y]$ and $Z_\rho[y,x]$ are given by

$$\mathrm{Jac}_{(d, d(x-y))}(\Delta_\rho[x,y]) = \rho|\cdot|^x \boxtimes \Delta_\rho[x-1, y],$$
$$\mathrm{Jac}_{(d, d(x-y))}(Z_\rho[y,x]) = \rho|\cdot|^y \boxtimes Z_\rho[y+1, x],$$

respectively (see [Zel80, Propositions 3.4 and 9.5]). For Jacquet modules of ladder representations, see [KL12, Theorem 2.1].

2.4 Representations of classical groups

In this paper, we let G_n be either the split special orthogonal group $\mathrm{SO}_{2n+1}(F)$ or the symplectic group $\mathrm{Sp}_{2n}(F)$ of rank n. Set $\mathrm{Irr}^G \coloneqq \bigcup_{n \geq 0} \mathrm{Irr}(G_n)$ and $\mathscr{R}^G \coloneqq \bigoplus_{n \geq 0} \mathscr{R}(G_n)$, where the union and the direct sum are taken over groups of the same type. Let $\mathscr{C}^G \subset \mathrm{Irr}^G$ be the subset of supercuspidal representations of G_n for every $n \geq 0$ of the same type.

Fix a rational Borel subgroup of G_n. Let P be the standard parabolic subgroup of G_n with Levi subgroup isomorphic to $\mathrm{GL}_{d_1}(F) \times \cdots \times \mathrm{GL}_{d_r}(F) \times G_{n_0}$. Let $\pi \in \mathrm{Rep}(G_{n_0})$ and let $\tau_i \in \mathrm{Rep}(\mathrm{GL}_{d_i}(F))$ for $1 \leq i \leq r$. We denote the normalized parabolically induced

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representation by

$$\tau_1 \times \cdots \times \tau_r \rtimes \pi := \mathrm{Ind}_P^{G_n}(\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi).$$

As in the case of general linear groups, the Jacquet functors give rise, at the level of Grothendieck groups, to a map

$$\mu^* \colon \mathscr{R}^G \longrightarrow \mathscr{R}^{\mathrm{GL}} \otimes \mathscr{R}^G$$
$$\mathscr{R}(G_n) \ni \pi \longmapsto \sum_{k=0}^{n} \left[\mathrm{Jac}_{P_k}^{G_n}(\pi)\right],$$

where P_k is the standard parabolic subgroup of G_n with Levi subgroup isomorphic to $\mathrm{GL}_k(F) \times G_{n-k}$. The geometric lemma at the level of Grothendieck groups is commonly known in this case as *Tadić's formula*.

Proposition 2.1 (Tadić's formula [Tad95]). *For $\tau \in \mathscr{R}^{\mathrm{GL}}$ and $\pi \in \mathscr{R}^G$, we have*

$$\mu^*(\tau \rtimes \pi) = M^*(\tau) \rtimes \mu^*(\pi).$$

We will also use the *MVW-functor* [MVW87]. It is a covariant functor

$$\mathrm{MVW} \colon \mathrm{Rep}(G_n) \longrightarrow \mathrm{Rep}(G_n)$$
$$\Pi \longmapsto \Pi^{\mathrm{MVW}}$$

satisfying the following properties:

- if $\pi \in \mathrm{Irr}(G_n)$, then π^{MVW} is isomorphic to $\pi^\vee$;
- $(\tau \rtimes \pi)^{\mathrm{MVW}} \cong \tau \rtimes \pi^{\mathrm{MVW}}$ for any $\pi \in \mathrm{Rep}(G_{n_0})$ and any $\tau \in \mathrm{Rep}(\mathrm{GL}_d(F))$ with $n = n_0 + d$.

The Zelevinsky–Aubert duality extends by linearity to a map $D^G : \mathscr{R}^G \to \mathscr{R}^G$. With this notation, the compatibility of the duality with Jacquet functors in (2.1) stands:

$$\mu^* \circ D^G = d^G \circ \mu^*, \tag{2.2}$$

where

$$d^G \colon \mathscr{R}^{\mathrm{GL}} \otimes \mathscr{R}^G \longrightarrow \mathscr{R}^{\mathrm{GL}} \otimes \mathscr{R}^G$$
$$\sum_i \tau_i \otimes \pi_i \longmapsto \sum_i \hat{\tau}_i^\vee \otimes \hat{\pi}_i.$$

Let $[x_1, y_1]_{\rho_1}, \ldots, [x_r, y_r]_{\rho_r}$ be some segments with $\rho_i \in \mathscr{C}(\mathrm{GL}_{d_i}(F))$ being unitary for $1 \leq i \leq r$, and let π_{temp} be an irreducible tempered representation of G_{n_0}. A parabolically induced representation of the form

$$\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r] \rtimes \pi_{\mathrm{temp}}$$

is called a *standard module* if $x_1 + y_1 \leq \cdots \leq x_r + y_r < 0$.

The Langlands classification says that any standard module is SI, and that any irreducible representation π of G_n is the unique irreducible subrepresentation (Langlands subrepresentation) of a standard module $\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r] \rtimes \pi_{\mathrm{temp}}$ with $n = n_0 + \sum_{i=1}^r d_i(x_i - y_i + 1)$, which is unique up to isomorphism. For more details, see [Kon03]. In this case, we write $\pi = L(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r]; \pi_{\mathrm{temp}})$ and refer to $(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r]; \pi_{\mathrm{temp}})$ as the *Langlands data* of π.

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2.5 The Jantzen decomposition

If $\pi \in \mathrm{Irr}(G_n)$, there exist $\rho_1, \ldots, \rho_r \in \mathscr{C}^{\mathrm{GL}}$ and $\sigma \in \mathscr{C}^G$ such that π is a subrepresentation of $\rho_1 \times \cdots \times \rho_r \rtimes \sigma$. The set

$$\mathrm{scusp}(\pi) \coloneqq \{\rho_1, \ldots, \rho_r, \rho_1^\vee, \ldots, \rho_r^\vee, \sigma\}$$

is uniquely determined by π and is called the *supercuspidal support* of π. For $\sigma \in \mathscr{C}^G$, we put $\mathrm{Irr}_\sigma \coloneqq \{\pi \in \mathrm{Irr}^G \mid \sigma \in \mathrm{scusp}(\pi)\}$.

In this paragraph, we fix a supercuspidal representation $\sigma \in \mathscr{C}^G$.

DEFINITION 2.2. Recall that $\mathbb{Z}_\rho = \{\rho|\cdot|^a \mid a \in \mathbb{Z}\}$ is the line of ρ for $\rho \in \mathscr{C}^{\mathrm{GL}}$.

- We say that ρ is *good* if $\mathbb{Z}_\rho = \mathbb{Z}_{\rho^\vee}$ and $\rho' \rtimes \sigma$ is reducible for some $\rho' \in \mathbb{Z}_\rho$.
- We say that ρ is *bad* if $\mathbb{Z}_\rho = \mathbb{Z}_{\rho^\vee}$ and $\rho' \rtimes \sigma$ is irreducible for all $\rho' \in \mathbb{Z}_\rho$.
- We say that ρ is *ugly* if $\mathbb{Z}_\rho \neq \mathbb{Z}_{\rho^\vee}$.

Every supercuspidal representation is either good, bad or ugly.

Remark 2.3. It is known that

- the notions of good and bad are independent of σ;
- if $\rho'|\cdot|^z$ is good or bad with ρ' unitary and $z \in \mathbb{R}$, then ρ' is self-dual and $z \in (1/2)\mathbb{Z}$;
- if $\rho'|\cdot|^{z_1}, \rho'|\cdot|^{z_2}$ are both good or both bad, then $z_1 - z_2 \in \mathbb{Z}$.

See Remark 5.1 below.

DEFINITION 2.4.

(1) We say that two good (respectively bad) supercuspidal representations ρ and ρ' are *line equivalent* if $\mathbb{Z}_\rho = \mathbb{Z}_{\rho'}$. We denote by $\mathscr{C}^{\mathrm{good}}$ (respectively $\mathscr{C}^{\mathrm{bad}}$) a set of representatives of good (respectively bad) representations under this equivalence relation.

(2) Similarly, we say that two ugly representations ρ and ρ' are *line equivalent* if $\mathbb{Z}_\rho \cup \mathbb{Z}_{\rho^\vee} = \mathbb{Z}_{\rho'} \cup \mathbb{Z}_{\rho'^\vee}$. We denote by $\mathscr{C}^{\mathrm{ugly}}$ a set of representatives of ugly representations under this equivalence relation.

DEFINITION 2.5. Let $\pi \in \mathrm{Irr}_\sigma$.

(1) If

$$\mathrm{scusp}(\pi) \subset \left(\bigcup_{\rho \in \mathscr{C}^{\mathrm{good}}} \mathbb{Z}_\rho \right) \cup \{\sigma\},$$

we say that π is *of good parity*. We write $\mathrm{Irr}_\sigma^{\mathrm{good}}$ for the set of such representations.

(2) If $\mathrm{scusp}(\pi) \subset \mathbb{Z}_\rho \cup \{\sigma\}$ for some bad representation ρ, we say that π is *of bad parity* (or of ρ-bad parity if we want to specify ρ). We write $\mathrm{Irr}_\sigma^{\rho\text{-bad}}$ for the set of such representations.

(3) If $\mathrm{scusp}(\pi) \subset (\mathbb{Z}_\rho \cup \mathbb{Z}_{\rho^\vee}) \cup \{\sigma\}$ for some ugly representation ρ, we say that π is *ugly* (or ρ-ugly if we want to specify ρ). We write $\mathrm{Irr}_\sigma^{\rho\text{-ugly}}$ for the set of such representations.

Ugly representations are easy to deal with owing to the following proposition, which reduces every problem to a similar problem for general linear groups.

PROPOSITION 2.6. *Let $\pi \in \mathrm{Irr}_\sigma^{\rho\text{-ugly}}$. Then there exists an irreducible representation τ of $\mathrm{GL}_m(F)$ with $\mathrm{scusp}(\tau) \subset \mathbb{Z}_\rho$ such that $\pi = \tau \rtimes \sigma$ (irreducible induction).*

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Proof. We can write

$$\pi \hookrightarrow \rho|\cdot|^{x_1} \times \cdots \times \rho|\cdot|^{x_r} \times \rho^\vee|\cdot|^{-y_1} \times \cdots \times \rho^\vee|\cdot|^{-y_s} \rtimes \sigma$$

for some $x_i, y_j \in \mathbb{Z}$. There exist irreducible subquotients τ_1 of $\rho|\cdot|^{x_1} \times \cdots \times \rho|\cdot|^{x_r}$ and τ_2 of $\rho^\vee|\cdot|^{-y_1} \times \cdots \times \rho^\vee|\cdot|^{-y_s}$ such that this inclusion factors through $\pi \hookrightarrow \tau_1 \times \tau_2 \rtimes \sigma$. As ρ is ugly, we can apply [LT20, Lemma 6.2] to $\tau_2 \rtimes \sigma$ and see that $\tau_2 \rtimes \sigma$ is irreducible. Hence $\pi \hookrightarrow \tau_1 \times \tau_2^\vee \rtimes \sigma$. Take an irreducible subquotient τ of $\tau_1 \times \tau_2^\vee$ such that $\pi \hookrightarrow \tau \rtimes \sigma$. Then by [LT20, Lemma 6.2] again, we conclude that $\tau \rtimes \sigma$ is irreducible. □

Remark 2.7. More precisely, by the Langlands classification, one can take τ_1 and τ_2 in the proof of this proposition so that

$$\tau_1 = L(\Delta_\rho[x'_1, y'_1], \ldots, \Delta_\rho[x'_{r'}, y'_{r'}]), \quad \tau_2 = L(\Delta_{\rho^\vee}[x''_1, y''_1], \ldots, \Delta_{\rho^\vee}[x''_{r''}, y''_{r''}])$$

with $x'_1 + y'_1 \leq \cdots \leq x'_{r'} + y'_{r'} \leq 0$ and $x''_1 + y''_1 \leq \cdots \leq x''_{r''} + y''_{r''} \leq 0$. Then since $\tau_2^\vee = L(\Delta_\rho[-y''_{r''}, -x''_{r''}], \ldots, \Delta_\rho[-y''_1, -x''_1])$ and $\pi = \mathrm{soc}(\tau_1 \times \tau_2^\vee \rtimes \sigma) \hookrightarrow \mathrm{soc}(\tau_1 \times \tau_2^\vee) \rtimes \sigma$, one can take τ to be

$$\tau := \mathrm{soc}(\tau_1 \times \tau_2^\vee) = L(\Delta_\rho[x'_1, y'_1], \ldots, \Delta_\rho[x'_{r'}, y'_{r'}], \Delta_\rho[-y''_{r''}, -x''_{r''}], \ldots, \Delta_\rho[-y''_1, -x''_1]).$$

Let $\pi \in \mathrm{Irr}_\sigma$. Then Jantzen [Jan97] defined representations $\pi^{\mathrm{good}} \in \mathrm{Irr}_\sigma^{\mathrm{good}}$, $\pi^{\rho\text{-bad}} \in \mathrm{Irr}_\sigma^{\rho\text{-bad}}$ and $\pi^{\rho\text{-ugly}} \in \mathrm{Irr}_\sigma^{\rho\text{-ugly}}$ as follows:

- π^{good} is the unique representation in $\mathrm{Irr}_\sigma^{\mathrm{good}}$ such that $\pi \hookrightarrow \tau \times \pi^{\mathrm{good}}$ with no good representations in $\mathrm{scusp}(\tau)$;
- if ρ is a bad supercuspidal representation, then $\pi^{\rho\text{-bad}}$ is the unique representation in $\mathrm{Irr}_\sigma^{\rho\text{-bad}}$ such that $\pi \hookrightarrow \tau \times \pi^{\rho\text{-bad}}$ with $\mathrm{scusp}(\tau) \cap \mathbb{Z}_\rho = \emptyset$;
- if ρ is an ugly supercuspidal representation, then $\pi^{\rho\text{-ugly}}$ is the unique representation in $\mathrm{Irr}_\sigma^{\rho\text{-ugly}}$ such that $\pi \hookrightarrow \tau \times \pi^{\rho\text{-ugly}}$ with $\mathrm{scusp}(\tau) \cap (\mathbb{Z}_\rho \cup \mathbb{Z}_{\rho^\vee}) = \emptyset$.

The following theorem is a special case of Jantzen's decomposition.

Theorem 2.8 [Jan97, Theorem 9.3]. *The map*

$$\Psi \colon \mathrm{Irr}_\sigma \longrightarrow \mathrm{Irr}_\sigma^{\mathrm{good}} \sqcup \Bigg(\bigsqcup_{\rho \in \mathscr{C}^{\mathrm{bad}}} \mathrm{Irr}_\sigma^{\rho\text{-bad}} \Bigg) \sqcup \Bigg(\bigsqcup_{\rho \in \mathscr{C}^{\mathrm{ugly}}} \mathrm{Irr}_\sigma^{\rho\text{-ugly}} \Bigg)$$
$$\pi \longmapsto \left(\pi^{\mathrm{good}}, \{\pi^{\rho\text{-bad}}\}_\rho, \{\pi^{\rho\text{-ugly}}\}_\rho\right)$$

is bijective. Moreover, it commutes with the Zelevinsky–Aubert duality in the sense that

$$\Psi(\hat{\pi}) = \left(\widehat{\pi^{\mathrm{good}}}, \{\widehat{\pi^{\rho\text{-bad}}}\}_\rho, \{\widehat{\pi^{\rho\text{-ugly}}}\}_\rho\right).$$

In practice, this theorem enables us to reduce the problem of making the Zelevinsky–Aubert duality explicit to the case where the representation is either ugly or of good or bad parity.

3. The theory of ρ-derivatives

Let $d > 0$ be an integer. In this section, we fix $\rho \in \mathscr{C}(\mathrm{GL}_d(F))$. We recall the definition of ρ-derivatives in [LT20] and introduce the notions of $\Delta_\rho[0, -1]$-derivative and $Z_\rho[0, 1]$-derivative. One should not confuse these notions with the Bernstein–Zelevinsky notion of derivatives.

https://doi.org/10.1112/S0010437X22007904 Published online by Cambridge University Press

3.1 Definitions

We treat first the case of general linear groups. For $\tau \in \mathrm{Rep}(\mathrm{GL}_n(F))$, define semisimple representations $L_\rho^{(k)}(\tau)$ and $R_\rho^{(k)}(\tau)$ of $\mathrm{GL}_{n-dk}(F)$ so that

$$\left[\mathrm{Jac}_{(dk,n-dk)}(\tau)\right] = \rho^k \boxtimes L_\rho^{(k)}(\tau) + \sum_i \tau_i \boxtimes \sigma_i,$$

$$\left[\mathrm{Jac}_{(n-dk,dk)}(\tau)\right] = R_\rho^{(k)}(\tau) \boxtimes \rho^k + \sum_i \sigma_i' \boxtimes \tau_i',$$

where τ_i and τ_i' are irreducible representations of $\mathrm{GL}_{dk}(F)$ which are not isomorphic to ρ^k. We call $L_\rho^{(k)}(\tau)$ (respectively $R_\rho^{(k)}(\tau)$) the *kth left ρ-derivative* (respectively the *kth right ρ-derivative*) of τ.

Definition 3.1.

(1) If $L_\rho^{(k)}(\tau) \neq 0$ but $L_\rho^{(k+1)}(\tau) = 0$, we say that $L_\rho^{(k)}(\tau)$ is the *highest left ρ-derivative*. We define the *highest right ρ-derivative* similarly.
(2) When $L_\rho^{(1)}(\tau) = 0$ (respectively $R_\rho^{(1)}(\tau) = 0$), we say that τ is *left ρ-reduced* (respectively *right ρ-reduced*).

Similarly we now treat the case of G_n. Again let $k \geq 0$, and now let P_{dk} be the standard parabolic subgroup of G_n with Levi subgroup of the form $\mathrm{GL}_{dk}(F) \times G_{n-dk}$. For $\Pi \in \mathrm{Rep}(G_n)$, define a semisimple representation $D_\rho^{(k)}(\Pi)$ of G_{n-dk} so that

$$\left[\mathrm{Jac}_{P_{dk}}^{G_n}(\Pi)\right] = \rho^k \boxtimes D_\rho^{(k)}(\Pi) + \sum_i \tau_i \boxtimes \Pi_i,$$

where τ_i is an irreducible representation of $\mathrm{GL}_{dk}(F)$ which is not isomorphic to ρ^k. We call $D_\rho^{(k)}(\Pi)$ the *kth ρ-derivative* of Π.

Definition 3.2.

(1) If $D_\rho^{(k)}(\Pi) \neq 0$ but $D_\rho^{(k+1)}(\Pi) = 0$, we say that $D_\rho^{(k)}(\Pi)$ is the *highest ρ-derivative.*
(2) When $D_\rho^{(1)}(\Pi) = 0$, we say that Π is *ρ-reduced.*

3.2 The non-self-dual case

If π is irreducible and ρ is not self-dual, then the highest ρ-derivative $D_\rho^{(k)}(\pi)$ is irreducible and π is isomorphic to the unique irreducible subrepresentation of $\rho^k \rtimes D_\rho^{(k)}(\pi)$ (see [Jan14, Lemma 3.1.3] and [Ato22b, Proposition 2.7]). Using these properties, we can show the following.

Proposition 3.3. *Let π be an irreducible representation of G_n and r a non-negative integer. If ρ is not self-dual, then $\rho^r \rtimes \pi$ is SI.*

Proof. Consider the highest ρ-derivative $D_\rho^{(k)}(\pi)$. If $\pi' \hookrightarrow \rho^r \rtimes \pi$, then $\pi' \hookrightarrow \rho^{k+r} \rtimes D_\rho^{(k)}(\pi)$. In particular, $D_\rho^{(k+r)}(\pi') = D_\rho^{(k)}(\pi)$. However, since

$$D_\rho^{(k+r)}\big(\rho^{k+r} \rtimes D_\rho^{(k)}(\pi)\big) = D_\rho^{(k)}(\pi)$$

by Tadić's formula (Proposition 2.1), we see that π' is determined uniquely. Hence $\mathrm{soc}(\rho^r \rtimes \pi)$ is irreducible and satisfies

$$D_\rho^{(k+r)}\big(\mathrm{soc}(\rho^r \rtimes \pi)\big) = D_\rho^{(k+r)}\big(\rho^r \rtimes \pi\big) = D_\rho^{(k)}(\pi).$$

These equations imply that $\mathrm{soc}(\rho^r \rtimes \pi)$ appears with multiplicity one in $[\rho^r \rtimes \pi]$. $\square$

We set

$$S_\rho^{(r)}(\pi) = \underbrace{S_\rho^{(1)} \circ \cdots \circ S_\rho^{(1)}}_{r \text{ times}}(\pi) = \mathrm{soc}(\rho^r \rtimes \pi)$$

for any $\pi \in \mathrm{Irr}(G_n)$.

3.3 The self-dual case

Recall from [Ato22b, Proposition 2.7] that the highest ρ-derivative $D_\rho^{(k)}(\pi)$ of an irreducible representation is isotypic, i.e. $D_\rho^{(k)}(\pi) = m \cdot \pi_0$ with some irreducible representation π_0 and a certain multiplicity $m > 0$. In this case, we have $\pi \hookrightarrow \rho^k \rtimes \pi_0$, but $\mathrm{soc}(\rho^k \rtimes \pi_0)$ can be reducible.

We give a criterion for $\rho^r \rtimes \pi$ being SI.

Proposition 3.4. *Suppose that ρ is self-dual. Let $\pi \in \mathrm{Irr}(G_n)$, and let r be a positive integer. The following are equivalent:*

(a) *$\rho^r \rtimes \pi$ is SI;*
(b) *$\rho^r \rtimes \pi$ is irreducible;*
(c) *$\rho^r \rtimes \pi$ has an irreducible subquotient π' such that $D_\rho^{(k+r)}(\pi') = 2^r \cdot D_\rho^{(k)}(\pi)$, where $D_\rho^{(k)}(\pi)$ is the highest ρ-derivative of π.*

Proof. We use here the MVW-functor; see § 2.4. As we assume that ρ is self-dual, if an irreducible representation π' satisfies $\pi' \hookrightarrow \rho^r \rtimes \pi$, by taking the MVW-functor and the contragredient functor we have $\rho^r \rtimes \pi \twoheadrightarrow \pi'$.

Now we assume that $\mathrm{soc}(\rho^r \rtimes \pi)$ is irreducible but $\rho^r \rtimes \pi$ is reducible. The above remark implies that the quotient $(\rho^r \rtimes \pi)/\mathrm{soc}(\rho^r \rtimes \pi)$ has an irreducible quotient isomorphic to $\mathrm{soc}(\rho^r \rtimes \pi)$. This means that $\mathrm{soc}(\rho^r \rtimes \pi)$ appears with multiplicity greater than one in $[\rho^r \rtimes \pi]$. Hence (a) implies (b). As the opposite implication is obvious, (a) and (b) are equivalent.

Note that $D_\rho^{(k+r)}(\rho^r \rtimes \pi) = 2^r \cdot D_\rho^{(k)}(\pi)$. In particular, (b) implies (c). On the other hand, let π' be an irreducible subquotient of $\rho^r \rtimes \pi$ such that $D_\rho^{(k+r)}(\pi') = 2^r \cdot D_\rho^{(k)}(\pi)$. Then π' must be a subrepresentation of $\rho^r \rtimes \pi$, and $(\rho^r \rtimes \pi)/\pi'$ has no irreducible quotient. Hence $\pi' = \rho^r \rtimes \pi$ so that $\rho^r \rtimes \pi$ is irreducible. □

3.4 $\Delta_\rho[0,-1]$-derivatives and $Z_\rho[0,1]$-derivatives

In the case where ρ is self-dual, ρ-derivatives are difficult. Therefore, we define some other derivatives in this paragraph. These will be key ingredients in making the Zelevinsky–Aubert duality explicit. In this subsection we assume that $\rho \in \mathscr{C}(\mathrm{GL}_d(F))$ is self-dual.

Let $\Pi \in \mathrm{Rep}(G_n)$. Define the $\Delta_\rho[0,-1]$-*derivative* $D_{\Delta_\rho[0,-1]}^{(k)}(\Pi)$ and the $Z_\rho[0,1]$-*derivative* $D_{Z_\rho[0,1]}^{(k)}(\Pi)$ by the semisimple representations of G_{n-2dk} satisfying

$$\left[\mathrm{Jac}_{P_{2dk}}^{G_n}(\pi)\right] = \Delta_\rho[0,-1]^k \boxtimes D_{\Delta_\rho[0,-1]}^{(k)}(\pi) + Z_\rho[0,1]^k \boxtimes D_{Z_\rho[0,1]}^{(k)}(\pi) + \sum_i \tau_i \boxtimes \pi_i,$$

where $\tau_i \in \mathrm{Irr}(\mathrm{GL}_{2dk}(F))$ such that $\tau_i \not\cong \Delta_\rho[0,-1]^k, Z_\rho[0,1]^k$.

Typically, when the supercuspidal representation ρ is clear from the context, we will write $[0,-1]$-*derivative* for short instead of $\Delta_\rho[0,-1]$-derivative, and $[0,1]$-*derivative* instead of $Z_\rho[0,1]$-derivative. We also write $D_{[0,-1]}^{(k)}(\Pi) := D_{\Delta_\rho[0,-1]}^{(k)}(\Pi)$ and $D_{[0,1]}^{(k)}(\Pi) := D_{Z_\rho[0,1]}^{(k)}(\Pi)$. Similar to Definition 3.2, we define the notion of *highest* $[0,-1]$-*derivative* (respectively *highest* $[0,1]$-*derivative*) and the property of being $\Delta_\rho[0,-1]$-*reduced* (respectively $Z_\rho[0,1]$-*reduced*).

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Lemma 3.5. *Fix $\rho \in \mathscr{C}(\mathrm{GL}_d(F))$ and $\epsilon \in \{\pm 1\}$. Let $\pi \in \mathrm{Irr}(G_n)$. Suppose that π is $\rho|\cdot|^{\epsilon}$-reduced. Let $D_\rho^{(k_0)}(\pi) = m \cdot \pi_0$ be the highest ρ-derivative of π (with multiplicity $m > 0$) and let $\pi_1 = D_{\rho|\cdot|^{\epsilon}}^{(k_1)}(\pi_0)$ be the highest $\rho|\cdot|^{\epsilon}$-derivative of π_0. Then the following hold:*

(1) $k_0 \geq k_1$;
(2) $D_{[0,\epsilon]}^{(k_1)}(\pi)$ *is the highest* $[0,\epsilon]$*-derivative;*
(3) $D_{[0,\epsilon]}^{(k_1)}(\pi)$ *is* $\rho|\cdot|^{\epsilon}$*-reduced.*

Proof. Note that $\pi \hookrightarrow \rho^{k_0} \times (\rho|\cdot|^{\epsilon})^{k_1} \rtimes \pi_1$. If $k_1 > k_0$, then no irreducible subquotient of $\rho^{k_0} \times (\rho|\cdot|^{\epsilon})^{k_1}$ is left $\rho|\cdot|^{\epsilon}$-reduced. Since π is $\rho|\cdot|^{\epsilon}$-reduced, we must have $k_0 \geq k_1$ and

$$\pi \hookrightarrow \begin{cases} Z_\rho[0,1]^{k_1} \times \rho^{k_0-k_1} \rtimes \pi_1 & \text{if } \epsilon = 1, \\ \Delta_\rho[0,-1]^{k_1} \times \rho^{k_0-k_1} \rtimes \pi_1 & \text{if } \epsilon = -1. \end{cases}$$

Now we claim that π_1 is ρ-reduced. This is trivial when $k_1 = 0$. If $k_1 > 0$ and π_1 is not ρ-reduced, since π_0 is ρ-reduced, we can find a representation $\pi_1' \neq 0$ such that

$$\pi_0 \hookrightarrow \begin{cases} \Delta_\rho[1,0] \rtimes \pi_1' & \text{if } \epsilon = 1, \\ Z_\rho[-1,0] \rtimes \pi_1' & \text{if } \epsilon = -1. \end{cases}$$

Since $\pi \hookrightarrow \rho^{k_0} \rtimes \pi_0$, this implies that $D_{\rho|\cdot|^{\epsilon}}^{(1)}(\pi) \neq 0$, which is a contradiction, so we obtain the claim.

Since π_1 is ρ-reduced and $\rho|\cdot|^{\epsilon}$-reduced, we see that $D_{[0,\epsilon]}^{(1)}(\rho^{k_0-k_1} \rtimes \pi_1) = 0$ by Tadić's formula (Proposition 2.1). Hence $D_{[0,\epsilon]}^{(k_1)}(\pi)$ is the highest $[0,\epsilon]$-derivative. Since it is a subrepresentation of $[\rho^{k_0-k_1} \rtimes \pi_1]$, we see that $D_{[0,\epsilon]}^{(k_1)}(\pi)$ is $\rho|\cdot|^{\epsilon}$-reduced. ☐

In the next proposition, we will use the following simple lemma on representations of general linear groups.

Lemma 3.6. *Let $k > 0$ and let $\tau \in \mathrm{Rep}(\mathrm{GL}_{2dk}(F))$. Suppose that*

- *τ is left $\rho|\cdot|^{-1}$-reduced (respectively left $\rho|\cdot|^{1}$-reduced);*
- *$[\tau]$ contains $\Delta_\rho[0,-1]^k$ (respectively $Z_\rho[0,1]^k$).*

Then there is a surjection $\tau \twoheadrightarrow \Delta_\rho[0,-1]^k$ (respectively $\tau \twoheadrightarrow Z_\rho[0,1]^k$).

Proof. We may assume that all irreducible constituents of τ have the same supercuspidal support. They are all left $\rho|\cdot|^{-1}$-reduced (respectively left $\rho|\cdot|^{1}$-reduced), as is τ. By [Zel80, Example 11.3], the irreducible representations of $\mathrm{GL}_{2dk}(F)$ which have the same supercuspidal support as $\Delta_\rho[0,-1]^k$ (respectively $Z_\rho[0,1]^k$) are of the form $\Delta_\rho[0,-1]^a \times Z_\rho[-1,0]^b$ (respectively $\Delta_\rho[1,0]^a \times Z_\rho[0,1]^b$) for some $a, b \geq 0$ with $a + b = k$. Among them, $\Delta_\rho[0,-1]^k$ (respectively $Z_\rho[0,1]^k$) is characterized as the only left $\rho|\cdot|^{-1}$-reduced (respectively left $\rho|\cdot|^{1}$-reduced) representation. Therefore, we have $\tau \twoheadrightarrow \Delta_\rho[0,-1]^k$ (respectively $\tau \twoheadrightarrow Z_\rho[0,1]^k$). ☐

Now we can prove the irreducibility of the highest $[0,\pm 1]$-derivatives of $\rho|\cdot|^{\pm 1}$-reduced irreducible representations.

Proposition 3.7. *Let $\pi \in \mathrm{Irr}(G_n)$. Suppose that π is $\rho|\cdot|^{-1}$-reduced (respectively $\rho|\cdot|^{1}$-reduced). Then the highest $[0,-1]$-derivative $D_{[0,-1]}^{(k)}(\pi)$ (respectively the highest $[0,1]$-derivative $D_{[0,1]}^{(k)}(\pi)$) is irreducible. Moreover, $\Delta_\rho[0,-1]^r \rtimes \pi$ (respectively $Z_\rho[0,1]^r \rtimes \pi$) is SI.*

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Proof. We prove the assertions only for $[0,1]$. By the previous lemma, there exists an irreducible subrepresentation of $\pi_{[0,1]}$ of the highest $[0,1]$-derivative $D^{(k)}_{[0,1]}(\pi)$ such that

$$\mathrm{Jac}^{G_n}_{P_{2dk}}(\pi) \twoheadrightarrow Z_\rho[0,1]^k \boxtimes \pi_0$$

or, equivalently,

$$\pi \hookrightarrow Z_\rho[0,1]^k \rtimes \pi_0.$$

Since π is $\rho|\cdot|^1$-reduced, so is π_0. Hence, by Tadić's formula (Proposition 2.1) for

$$[\mathrm{Jac}^{G_n}_{P_{2dk}}(Z_\rho[0,1]^k \rtimes \pi_0)],$$

we see that

$$D^{(k)}_{[0,1]}(Z_\rho[0,1]^k \rtimes \pi_0) = \pi_0.$$

Therefore, $0 \neq D^{(k)}_{[0,1]}(\pi) \subset \pi_0$ so that $D^{(k)}_{[0,1]}(\pi) = \pi_0$. Moreover, this implies that $Z_\rho[0,1]^k \rtimes \pi_0$ is SI.

When π' is an irreducible subrepresentation of $Z_\rho[0,1]^r \rtimes \pi$, we have $\pi' \subset \mathrm{soc}(Z_\rho[0,1]^{k+r} \rtimes \pi_0)$. In particular, π' is unique and appears with multiplicity one in $[Z_\rho[0,1]^{k+r} \rtimes \pi_0]$ and hence in $[Z_\rho[0,1]^r \rtimes \pi]$. Therefore, $Z_\rho[0,1]^r \rtimes \pi$ is SI. □

For simplicity, we set

$$S^{(r)}_{[0,1]}(\pi) = S^{(r)}_{Z_\rho[0,1]}(\pi) \coloneqq \mathrm{soc}(Z_\rho[0,1]^r \rtimes \pi)$$

for an irreducible representation π of G_n which is $\rho|\cdot|^1$-reduced.

The highest $[0,-1]$-derivatives are easy in a special case.

Proposition 3.8. *Let $\pi = L(\Delta_{\rho_1}[x_1,y_1],\dots,\Delta_{\rho_r}[x_r,y_r];\pi_{\mathrm{temp}})$ be an irreducible representation of G_n. Suppose that π is $\rho|\cdot|^z$-reduced for all $z \neq 0$ and that there exists $i \in \{1,\dots,r\}$ such that $\rho_i \cong \rho$. Then $\min\{x_i \mid \rho_i \cong \rho\} = 0$, and the highest $[0,-1]$-derivative $D^{(k)}_{[0,-1]}(\pi)$ of π is given by*

$$D^{(k)}_{[0,-1]}(\pi) = L(\Delta_{\rho_1}[z_1,y_1],\dots,\Delta_{\rho_r}[z_r,y_r];\pi_{\mathrm{temp}})$$

with

$$z_i = \begin{cases} -2 & \text{if } \rho_i \cong \rho,\ x_i = 0,\\ x_i & \text{otherwise.}\end{cases}$$

In particular,

$$k = \left|\{i \in \{1,\dots,r\} \mid \rho_i \cong \rho,\ x_i = 0\}\right| \geq 1.$$

Proof. With $x \coloneqq \min\{x_i \mid \rho_i \cong \rho\}$, we see that π is not $\rho|\cdot|^x$-reduced. Hence we must have $x = 0$. Moreover, we note that if $\rho_i \cong \rho$ and $x_i = 0$, then $y_i \leq -1$ since $x_i + y_i < 0$.

We remark that $D^{(l)}_\rho(\pi_{\mathrm{temp}})$ is tempered since ρ is self-dual (see [Ato20, Theorem 4.2(1) and (4)]), so $D^{(l)}_\rho(\pi_{\mathrm{temp}})$ is $\rho|\cdot|^{-1}$-reduced by Casselman's criterion (see e.g. [Kon03, Lemma 2.4]). Hence by Lemma 3.5, with k as in the statement, $D^{(k)}_{[0,-1]}(\pi)$ is the highest $[0,-1]$-derivative.

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Set $\tau := L(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r])$. Then $\pi \hookrightarrow \tau \rtimes \pi_{\mathrm{temp}}$. Since $\min\{x_i \mid \rho_i \cong \rho\} = 0$ and $y_i < 0$, we see that $\tau \hookrightarrow \Delta_\rho[0, -1]^k \times \tau'$ with $\tau' := L(\Delta_{\rho_1}[z_1, y_1], \ldots, \Delta_{\rho_r}[z_r, y_r])$. Hence

$$\pi \hookrightarrow \Delta_\rho[0, -1]^k \times \tau' \rtimes \pi_{\mathrm{temp}}.$$

By the Frobenius reciprocity, we have a non-zero map

$$\mathrm{Jac}^{G_n}_{P_{2dk}}(\pi) \to \Delta_\rho[0, -1]^k \boxtimes (\tau' \rtimes \pi_{\mathrm{temp}}),$$

which must factor through a non-zero map

$$\Delta_\rho[0, -1]^k \boxtimes D^{(k)}_{[0,-1]}(\pi) \to \Delta_\rho[0, -1]^k \boxtimes (\tau' \rtimes \pi_{\mathrm{temp}}).$$

Since $D^{(k)}_{[0,-1]}(\pi)$ is irreducible by Proposition 3.7 and since $\tau' \rtimes \pi_{\mathrm{temp}}$ is SI, we deduce that

$$D^{(k)}_{[0,-1]}(\pi) = \mathrm{soc}(\tau' \rtimes \pi_{\mathrm{temp}}).$$

This completes the proof. $\square$

3.5 The Zelevinsky–Aubert duality and derivatives

We deduce the following compatibility between derivatives and duality.

Proposition 3.9. *Let $\pi \in \mathrm{Irr}(G_n)$ and $\rho \in \mathscr{C}(\mathrm{GL}_d(F))$.*

(1) *If $D^{(k)}_\rho(\pi)$ is the highest ρ-derivative, then*

$$D^{(k)}_\rho(\pi)\hat{} = D^{(k)}_{\rho^\vee}(\hat{\pi}).$$

(2) *If ρ is self-dual, π is $\rho|\cdot|^{-1}$-reduced and $D^{(k)}_{\Delta_\rho[0,-1]}(\pi)$ is the highest $\Delta_\rho[0,-1]$-derivative, then*

$$D^{(k)}_{\Delta_\rho[0,-1]}(\pi)\hat{} = D^{(k)}_{Z_\rho[0,1]}(\hat{\pi}).$$

Proof. This is a consequence of the commutativity of the Jacquet functor with the duality; see (2.2). $\square$

4. The algorithm

In this section we give an algorithm for computing the Zelevinsky–Aubert dual of an irreducible representation π. Thanks to Jantzen's decomposition (see § 2.5), we can reduce π to the case where π is either ugly or of good or bad parity. Then we proceed as follows.

Algorithm 4.1. Assume that we can compute $\hat{\pi}_0$ for all irreducible representations of G_{n_0} for $n_0 < n$. Let π be an irreducible representation of G_n.

(1) If there exists $\rho \in \mathscr{C}^{\mathrm{GL}}$ such that ρ is not self-dual and such that $D^{(k)}_\rho(\pi)$ is the highest ρ-derivative with $k \geq 1$, then

$$\hat{\pi} = S^{(k)}_{\rho^\vee}\big(D^{(k)}_\rho(\pi)\hat{}\,\big).$$

(2) Otherwise, and if π is not tempered, one can find $\rho \in \mathscr{C}^{\mathrm{GL}}$ such that ρ is self-dual and $D^{(k)}_{\Delta_\rho[0,-1]}(\pi)$ is the highest $\Delta_\rho[0,-1]$-derivative with $k \geq 1$. Then

$$\hat{\pi} = S^{(k)}_{Z_\rho[0,1]}\big(D^{(k)}_{\Delta_\rho[0,-1]}(\pi)\hat{}\,\big).$$

(3) Otherwise, and if π is tempered, one can use an explicit formula for $\hat{\pi}$ (Proposition 5.4 below).

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In order to run the algorithm, we need the following formulas.

- Explicit formulas for the highest ρ-derivative $D_\rho^{(k)}(\pi)$ and for the socle $S_\rho^{(k)}(\pi)$ for any $\rho \in \mathscr{C}^{\mathrm{GL}}$ which is not self-dual: these are given in Proposition 6.1 if ρ is ugly or if the exponent of ρ is negative, and in Theorem 7.1 (respectively Theorem 7.4) if the exponent of ρ is positive and ρ is in the good (respectively bad) case.
- Explicit formulas for the $\Delta_\rho[0,-1]$-derivative $D^{(k)}_{\Delta_\rho[0,-1]}(\pi)$ and the socle $S^{(k)}_{Z_\rho[0,1]}(\pi)$ when ρ is self-dual and π is non-tempered and $\rho|\cdot|^z$-reduced for all $z \neq 0$: these are established in Proposition 3.8 for the $\Delta_\rho[0,-1]$-derivative and in Theorem 8.1 for the socle, respectively.
- An explicit formula for $\hat{\pi}$ when π is tempered such that π is $\rho|\cdot|^z$-reduced for all $z \neq 0$: this is given in Proposition 5.4.

In the rest of the paper, we will prove all these formulas.

5. The endoscopic classification

In §§ 7.1 and 8.3 below, we will give explicit formulas for several derivatives and socles in the good-parity case. In these formulas, certain special irreducible representations π_A play an important and mysterious role. These special representations π_A are of *Arthur type*, and the mystery comes from Arthur's theory of the endoscopic classification [Art13]. In this section, we review his theory.

5.1 *A*-parameters

We denote by W_F the Weil group of F. A homomorphism

$$\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$$

is called an *A-parameter for* $\mathrm{GL}_n(F)$ if

- $\psi(\mathrm{Frob}) \in \mathrm{GL}_n(\mathbb{C})$ is semisimple and all its eigenvalues have absolute value 1, where Frob is a fixed (geometric) Frobenius element;
- $\psi|W_F$ is smooth, i.e. has an open kernel;
- $\psi|\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ is algebraic.

The local Langlands correspondence for $\mathrm{GL}_d(F)$ asserts that there is a canonical bijection between the set of irreducible unitary supercuspidal representations of $\mathrm{GL}_d(F)$ and the set of irreducible d-dimensional representations of W_F of bounded image. We identify these two sets and use the symbol ρ for their elements.

Any such irreducible representation of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ is of the form $\rho \boxtimes S_a \boxtimes S_b$, where S_a is the unique irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension a. We write $\rho \boxtimes S_a = \rho \boxtimes S_a \boxtimes S_1$ and $\rho = \rho \boxtimes S_1 \boxtimes S_1$ for short. For an A-parameter ψ, the multiplicity of $\rho \boxtimes S_a \boxtimes S_b$ in ψ is denoted by $m_\psi(\rho \boxtimes S_a \boxtimes S_b)$. When $\psi = \bigoplus_{i\in I} \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}$ is an A-parameter of $\mathrm{GL}_n(F)$, we define τ_ψ by the product of Speh representations (see § 2.3)

$$\tau_\psi := \bigtimes_{i\in I} L\left(\Delta_{\rho_i}\left[\frac{a_i-b_i}{2}, -\frac{a_i+b_i}{2}+1\right], \ldots, \Delta_{\rho_i}\left[\frac{a_i+b_i}{2}-1, -\frac{a_i-b_i}{2}\right]\right).$$

Now we consider a split odd special orthogonal group $\mathrm{SO}_{2n+1}(F)$ or a symplectic group $\mathrm{Sp}_{2n}(F)$. We call ψ an *A-parameter for* $\mathrm{SO}_{2n+1}(F)$ if it is an A-parameter for $\mathrm{GL}_{2n}(F)$ of symplectic type, i.e.

$$\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C}).$$

Similarly, ψ is called an *A-parameter for* $\mathrm{Sp}_{2n}(F)$ if it is an A-parameter for $\mathrm{GL}_{2n+1}(F)$ of orthogonal type with the trivial determinant, i.e.

$$\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2n+1}(\mathbb{C}).$$

For $G_n = \mathrm{SO}_{2n+1}(F)$ (respectively $G_n = \mathrm{Sp}_{2n}(F)$), we let $\Psi(G_n)$ be the set of $\widehat{G_n}$-conjugacy classes of A-parameters for G_n, where $\widehat{G_n} = \mathrm{Sp}_{2n}(\mathbb{C})$ (respectively $\widehat{G_n} = \mathrm{SO}_{2n+1}(\mathbb{C})$). We say that

- $\psi \in \Psi(G_n)$ is *tempered* if the restriction of ψ to the second $\mathrm{SL}_2(\mathbb{C})$ is trivial;
- $\psi \in \Psi(G_n)$ is *of good parity* if ψ is a sum of irreducible self-dual representations of the same type as ψ.

We denote by $\Psi_{\mathrm{temp}}(G_n) \coloneqq \Phi_{\mathrm{temp}}(G_n)$ (respectively $\Psi_{\mathrm{gp}}(G_n)$) the subset of $\Psi(G_n)$ consisting of tempered A-parameters (respectively A-parameters of good parity). Also, we put $\Phi_{\mathrm{gp}}(G_n) \coloneqq \Phi_{\mathrm{temp}}(G_n) \cap \Psi_{\mathrm{gp}}(G_n)$. Set $\Psi_*(G) \coloneqq \bigcup_{n\geq 0} \Psi_*(G_n)$ and $\Phi_*(G) \coloneqq \bigcup_{n \geq 0} \Phi_*(G_n)$ for $* \in \{\emptyset, \mathrm{temp}, \mathrm{gp}\}$.

For $\psi \in \Psi(G)$, a component group $\mathcal{S}_\psi$ is defined. We recall the definition only in the case where $\psi \in \Psi_{\mathrm{gp}}(G)$. Hence we can write $\psi = \bigoplus_{i=1}^r \psi_i$, where ψ_i is an irreducible self-dual representation of the same type as ψ. We define an *enhanced component group* $\mathcal{A}_\psi$ as

$$\mathcal{A}_\psi \coloneqq \bigoplus_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z})\alpha_{\psi_i}.$$

Specifically, $\mathcal{A}_\psi$ is a free $\mathbb{Z}/2\mathbb{Z}$-module of rank r with a basis $\{\alpha_{\psi_i}\}$ associated with the irreducible components $\{\psi_i\}$. Define the *component group* $\mathcal{S}_\psi$ as the quotient of $\mathcal{A}_\psi$ by the subgroup generated by the elements

- $z_\psi \coloneqq \sum_{i=1}^r \alpha_{\psi_i}$; and
- $\alpha_{\psi_i} + \alpha_{\psi_{i'}}$ such that $\psi_i \cong \psi_{i'}$.

Let $\widehat{\mathcal{S}_\psi}$ and $\widehat{\mathcal{A}_\psi}$ be the Pontryagin duals of $\mathcal{S}_\psi$ and $\mathcal{A}_\psi$, respectively. Via the canonical surjection $\mathcal{A}_\psi \twoheadrightarrow \mathcal{S}_\psi$, we may regard $\widehat{\mathcal{S}_\psi}$ as a subgroup of $\widehat{\mathcal{A}_\psi}$. For $\eta \in \widehat{\mathcal{A}_\psi}$, we write $\eta(\alpha_{\psi_i}) = \eta(\psi_i)$.

Let $\mathrm{Irr}_{\mathrm{unit}}(G_n)$ (respectively $\mathrm{Irr}_{\mathrm{temp}}(G_n)$) be the set of equivalence classes of irreducible unitary (respectively tempered) representations of G_n. For $\psi \in \Psi(G_n)$, Arthur [Art13, Theorem 2.2.1] defined a multiset Π_ψ over $\mathrm{Irr}_{\mathrm{unit}}(G_n)$, which is called the *A-packet* for G_n associated with ψ. It has the following properties.

- The multiset Π_ψ is actually a (multiplicity-free) subset of $\mathrm{Irr}_{\mathrm{unit}}(G_n)$ (Mœglin [Mœg11]).
- There exists a map $\Pi_\psi \to \widehat{\mathcal{S}_\psi}$, $\pi \mapsto \langle \cdot\,, \pi \rangle_\psi$. If $\phi \in \Phi_{\mathrm{temp}}(G)$, it is a bijection. When $\pi \in \Pi_\phi$ corresponds to $\eta \in \widehat{\mathcal{S}_\phi}$, we write $\pi = \pi(\phi, \eta)$.
- There is a canonical decomposition into a disjoint union

$$\mathrm{Irr}_{\mathrm{temp}}(G_n) = \bigsqcup_{\phi \in \Phi_{\mathrm{temp}}(G_n)} \Pi_\phi.$$

- If $\psi = \psi_1 \oplus \psi_0 \oplus \psi_1^\vee$ for some irreducible representation ψ_1, then there exists a canonical injection $\mathcal{S}_{\psi_0} \hookrightarrow \mathcal{S}_\psi$, and

$$\tau_{\psi_1} \rtimes \pi_0 \cong \bigoplus_{\substack{\pi \in \Pi_\psi \\ \langle \cdot\,, \pi\rangle_\psi | \mathcal{S}_{\psi_0} = \langle \cdot\,, \pi_0 \rangle_{\psi_0}}} \pi$$

  for every $\pi_0 \in \Pi_{\psi_0}$ (see [Art13, Proposition 2.4.3]).

Remark 5.1. Let $\rho \in \mathscr{C}^{\mathrm{GL}}$ be unitary and $x \geq 0$ a real number. Then the following statements are equivalent.

(1) For any $\pi(\phi, \eta)$ with $\phi \in \Phi_{\mathrm{gp}}(G)$ and $\eta \in \widehat{\mathcal{S}_\phi}$, there exists $m \in \mathbb{Z}$ such that $\rho|\cdot|^{x+m} \rtimes \pi(\phi, \eta)$ is reducible.
(2) For some $\pi(\phi, \eta)$ with $\phi \in \Phi_{\mathrm{gp}}(G)$ and $\eta \in \widehat{\mathcal{S}_\phi}$, there exists $m \in \mathbb{Z}$ such that $\rho|\cdot|^{x+m} \rtimes \pi(\phi, \eta)$ is reducible.
(3) We have that $x \in (1/2)\mathbb{Z}$ and $\rho \boxtimes S_{2x+1}$ is self-dual of the same type as elements of $\Phi_{\mathrm{gp}}(G)$, i.e.
   - $x \in \mathbb{Z}$ and ρ is self-dual of the same type as elements of $\Phi_{\mathrm{gp}}(G)$; or
   - $x \in (1/2)\mathbb{Z} \setminus \mathbb{Z}$ and ρ is self-dual of the opposite type to elements of $\Phi_{\mathrm{gp}}(G)$.

This follows, for example, from [MW12, Théorème (i)] and [Jan18b, Theorem 4.7]. In particular, $\rho|\cdot|^x$ is good in the sense of Definition 2.2 if and only if $\rho \boxtimes S_{2x+1}$ is self-dual of the same type as elements of $\Phi_{\mathrm{gp}}(G)$. Also, an irreducible representation $\pi = L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r]; \pi_{\mathrm{temp}})$ is of good parity if and only if $\pi_{\mathrm{temp}} = \pi(\phi, \eta)$ with $\phi \in \Phi_{\mathrm{gp}}(G)$ and $\rho_i \boxtimes S_{2|x_i|+1}$ is self-dual of the same type as ϕ for all $i = 1, \dots, r$.

5.2 A special example

Now we consider a special A-parameter of the form

$$\psi = \phi \oplus (\rho \boxtimes S_{2x} \boxtimes S_2)^t$$

for $t \geq 1$, $\phi \in \Phi_{\mathrm{gp}}(G)$ and $x \in (1/2)\mathbb{Z}$ with $x > 0$ such that $\rho \boxtimes S_{2x+1}$ is self-dual of the same type as ϕ.

For $l \in \mathbb{Z}/2\mathbb{Z}$ and for η in a certain subset $\widehat{\mathcal{S}_{\psi,l}}$ of $\widehat{\mathcal{S}_\psi}$ (depending on l), we define $\pi(\psi, l, \eta)$ as follows. When $l = 1$, we set $\widehat{\mathcal{S}_{\psi,1}} \coloneqq \widehat{\mathcal{S}_\phi} = \{\eta \in \widehat{\mathcal{S}_\psi} \mid \eta(\rho \boxtimes S_{2x} \boxtimes S_2) = 1\}$ and

$$\pi(\psi, 1, \eta) \coloneqq L(\Delta_\rho[x-1, -x]^t; \pi(\phi, \eta)).$$

When $l = 0$ and $x \geq 1$, we let $\widehat{\mathcal{S}_{\psi,0}}$ be the subset of $\widehat{\mathcal{S}_\psi}$ consisting of η that satisfy

- $\eta(\rho \boxtimes S_{2x} \boxtimes S_2) = \eta(\rho \boxtimes S_{2x-1})$ if $\rho \boxtimes S_{2x-1} \subset \phi$;
- $\eta(\rho \boxtimes S_{2x} \boxtimes S_2) = (-1)^t \eta(\rho \boxtimes S_{2x+1})$ if $\rho \boxtimes S_{2x+1} \subset \phi$;
- $\eta(z_\phi) = (-1)^t$.

When $l = 0$ and $x = 1/2$, we let $\widehat{\mathcal{S}_{\psi,0}}$ be the subset of $\widehat{\mathcal{S}_\psi}$ consisting of η that satisfy

- $\eta(\rho \boxtimes S_1 \boxtimes S_2) = -1$;
- $\eta(\rho \boxtimes S_2) = (-1)^t$ if $\rho \boxtimes S_2 \subset \phi$;
- $\eta(z_\phi) = (-1)^t$.

For $\eta \in \widehat{\mathcal{S}_{\psi,0}}$, we define

$$\pi(\psi, 0, \eta) \coloneqq L\big(\Delta_\rho[x-1, -x]^{t-1}; \pi(\phi + \rho \boxtimes (S_{2x-1} + S_{2x+1}), \eta)\big).$$

Here, we regard η as a character of the component group of $\phi + \rho \boxtimes (S_{2x-1} + S_{2x+1})$ by setting

$$\begin{cases} \eta(\rho \boxtimes S_{2x-1}) = (-1)^t \eta(\rho \boxtimes S_{2x+1}) = \eta(\rho \boxtimes S_{2x} \boxtimes S_2) & \text{if } x \geq 1, \\ \eta(\rho \boxtimes S_2) = (-1)^t & \text{if } x = 1/2. \end{cases}$$

By specifying Mœglin's construction of Π_ψ, we have the following.

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Proposition 5.2. *Let $\psi = \phi \oplus (\rho \boxtimes S_{2x} \boxtimes S_2)^t \in \Psi_{\mathrm{gp}}(G)$ with $t \geq 1$. Then*

$$\Pi_\psi = \left\{\pi(\psi, l, \eta) \;\middle|\; l \in \mathbb{Z}/2\mathbb{Z},\ \eta \in \widehat{\mathcal{S}_{\psi,l}}\right\}.$$

Moreover, the map $\Pi_\psi \to \widehat{\mathcal{S}_\psi}$ is given by $\langle \cdot\,, \pi(\psi, l, \eta)\rangle_\psi = \varepsilon_{l,\eta}$, where

$$\varepsilon_{l,\eta}(\rho \boxtimes S_d) = \eta(\rho \boxtimes S_d),$$
$$\varepsilon_{l,\eta}(\rho \boxtimes S_{2x} \boxtimes S_2) = \begin{cases} (-1)^{l-1} & \text{if } x \geq 1, \\ \eta(\rho \boxtimes S_1 \boxtimes S_2) & \text{if } x = 1/2. \end{cases}$$

Proof. The A-packet Π_ψ was constructed by Mœglin explicitly; see [Xu17a, § 8] for details. For $x \geq 1$, its construction was computed in [Ato22b, Proposition 3.13]. The same calculation can be applied to $x = 1/2$. By [Xu17a, Corollary 8.10], the map $\Pi_\psi \to \widehat{\mathcal{S}_\psi}$ is given by $\langle \cdot\,, \pi(\psi, l, \eta)\rangle_\psi = \varepsilon_{l,\eta} \cdot \epsilon_\psi^{M/W}$ for some character $\epsilon_\psi^{M/W} \in \widehat{\mathcal{S}_\psi}$. By definition (see [Xu17a, Definitions 5.2, 5.5 and 8.1]), one can easily see that $\epsilon_\psi^{M/W} = \mathbf{1}$ in our case. □

Using this description, we obtain the formula for the highest $\rho|\cdot|^x$-derivatives and socles.

Theorem 5.3. *Fix $\phi \in \Phi_{\mathrm{gp}}(G)$ and write $m = m_\phi(\rho \boxtimes S_{2x+1})$ and $m' = m_\phi(\rho \boxtimes S_{2x-1})$. Consider $\psi = \phi \oplus (\rho \boxtimes S_{2x} \boxtimes S_2)^t \in \Psi_{\mathrm{gp}}(G)$ with $t \geq 0$. Let $\pi(\psi, l, \eta) \in \Pi_\psi$ be such that $\eta(\rho \boxtimes S_{2x-1})\eta(\rho \boxtimes S_{2x+1}) = (-1)^t$ if $mm' \neq 0$. Here, if $x = 1/2$, we formally understand that $m' = 1$ and $\eta(\rho \boxtimes S_0) = 1$. Let s be a non-negative integer such that $s = 0$ if $x = 1/2$. Then the highest $\rho|\cdot|^x$-derivative of $\mathrm{soc}((\rho|\cdot|^{-x})^s \rtimes \pi(\psi, l, \eta))$ is given by*

$$D_{\rho|\cdot|^x}^{(m+\max\{s-m',0\})}\big(\mathrm{soc}\big((\rho|\cdot|^{-x})^s \rtimes \pi(\psi, l, \eta)\big)\big)$$
$$= \mathrm{soc}\big((\rho|\cdot|^{-x})^{\min\{s,m'\}} \rtimes \pi\big(\psi - (\rho \boxtimes S_{2x+1})^m + (\rho \boxtimes S_{2x-1})^m, l+m, \eta\big)\big),$$

where we set $\eta(\rho \boxtimes S_{2x-1}) = (-1)^t \eta(\rho \boxtimes S_{2x+1})$ if needed. In particular,

$$S_{\rho|\cdot|^x}^{(1)}\big(\mathrm{soc}\big((\rho|\cdot|^{-x})^s \rtimes \pi(\psi, l, \eta)\big)\big)$$
$$= \begin{cases} \mathrm{soc}\big((\rho|\cdot|^{-x})^s \rtimes \pi(\psi - \rho \boxtimes S_{2x-1} + \rho \boxtimes S_{2x+1}, l-1, \eta)\big) & \text{if } s < m', \\ \mathrm{soc}\big((\rho|\cdot|^{-x})^{s+1} \rtimes \pi(\psi, l, \eta)\big) & \text{if } s \geq m', \end{cases}$$

where we set $\eta(\rho \boxtimes S_{2x+1}) = (-1)^t \eta(\rho \boxtimes S_{2x-1})$.

Proof. When $x \geq 1$ (respectively $x = 1/2$), the formula for the highest $\rho|\cdot|^x$-derivatives was obtained in [Ato22b, Theorem 4.1] (respectively in [Jan18a, Theorem 3.3]). It implies the formula for socles. □

5.3 Zelevinsky–Aubert duals of certain tempered representations

The initial step of our algorithm for computing the Zelevinsky–Aubert duals (Algorithm 4.1(3)) is to compute $\hat{\pi}$ for tempered π such that π is ρ'-reduced for every non-self-dual $\rho' \in \mathscr{C}^{\mathrm{GL}}$. If $\pi = \pi(\phi, \eta)$ for $\phi \in \Phi_{\mathrm{gp}}(G)$, then π satisfies this condition if and only if

($\ast$) if $\rho \boxtimes S_d \subset \phi$ with $d \geq 2$, then $m_\phi(\rho \boxtimes S_d) = 1$, $\rho \boxtimes S_{d-2} \subset \phi$ and $\eta(\rho \boxtimes S_d) \neq \eta(\rho \boxtimes S_{d-2})$.

See [Ato20, Theorem 4.2]. Here, we formally understand that $\rho \boxtimes S_0 \subset \phi$ and $\eta(\rho \boxtimes S_0) = +1$ if ρ is self-dual of the opposite type to ϕ.

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Proposition 5.4. *Let $\pi = \pi(\phi, \eta)$ with $\phi \in \Phi_{\mathrm{gp}}(G)$. Assume that π satisfies the above condition $(*)$. Write*

$$\{\rho \mid m_\phi(\rho) > 0,\, m_\phi(\rho) \equiv 0 \bmod 2\} = \{\rho_1, \dots, \rho_r\}$$

and set

$$y_i := \max\left\{ \frac{d_i - 1}{2} \;\middle|\; \rho_i \boxtimes S_{d_i} \subset \phi \right\}.$$

Suppose that $y_1 \geq \cdots \geq y_t > 0 = y_{t+1} = \cdots = y_r$. Then

$$\hat{\pi} = L\big(\Delta_{\rho_1}[0, -y_1], \dots, \Delta_{\rho_t}[0, -y_t]; \pi(\phi', \eta')\big),$$

where

$$\phi' = \phi - \bigoplus_{i=1}^{t} \rho_i \boxtimes (S_1 + S_{2y_i+1})$$

and

$$\eta'(\rho \boxtimes S_d) = \begin{cases} -\eta(\rho \boxtimes S_d) & \text{if } \rho \in \{\rho_1, \dots, \rho_r\}, \\ \eta(\rho \boxtimes S_d) & \text{otherwise.} \end{cases}$$

Proof. Set

$$\{\rho \mid m_\phi(\rho) > 0,\, m_\phi(\rho) \equiv 1 \bmod 2\} = \{\rho'_1, \dots, \rho'_{r'}\}.$$

Write $m_\phi(\rho_i) = 2k_i > 0$ and $m_\phi(\rho'_j) = 2k'_j + 1$. Then, by [Ato20, Theorem 4.2], we have

$$\left(\circ_{j=1}^{r'} D_{\rho'_j}^{(k'_j)}\right) \circ \left(\circ_{i=1}^{r} D_{\rho_i|\cdot|^{y_i}}^{(1)} \circ \cdots \circ D_{\rho_i|\cdot|^1}^{(1)} \circ D_{\rho_i}^{(k_i)}\right)(\pi) \neq 0.$$

This is $\pi(\phi'', \eta'')$ up to multiplicity, where

$$\phi'' = \phi - \left(\bigoplus_{j=1}^{r'} {\rho'_j}^{2k'_j}\right) - \left(\bigoplus_{i=1}^{r} \rho_i \boxtimes (S_1^{2k_i - 1} + S_{2y_i+1})\right)$$

and

$$\eta''(\rho \boxtimes S_d) = \begin{cases} -\eta(\rho \boxtimes S_d) & \text{if } \rho \in \{\rho_1, \dots, \rho_t\}, \\ \eta(\rho \boxtimes S_d) & \text{if } \rho \notin \{\rho_1, \dots, \rho_r\}. \end{cases}$$

For $t < i \leq r$, we note that $\rho_i \not\subset \phi''$ so that $\eta''(\rho_i \boxtimes S_d)$ does not appear. In particular, $\pi(\phi'', \eta'')$ is supercuspidal. By [Ato22b, Theorem 2.13], with ϕ' as in the statement, we have

$$\hat{\pi} = L(\Delta_{\rho_1}[0, -y_1], \dots, \Delta_{\rho_t}[0, -y_t]; \pi(\phi', \eta'))$$

for some $\eta' \in \mathcal{A}_{\phi'}$ such that $\eta'' = \eta' | \mathcal{A}_{\phi''}$ via the canonical inclusion $\mathcal{A}_{\phi''} \hookrightarrow \mathcal{A}_{\phi'}$. Since $\mathcal{S}_{\phi'}$ is generated by $\mathcal{S}_{\phi''}$ and the image of $\{\alpha_{\rho_i} \mid i > t\}$, the remaining task is to determine $\eta'(\rho_{i_0})$ for $i_0 > t$. To do this, by replacing π with

$$\left(\circ_{j=1}^{r'} D_{\rho'_j}^{(k'_j)}\right) \circ \left(\circ_{\substack{1 \leq i \leq r \\ i \neq i_0}} D_{\rho_i|\cdot|^{y_i}}^{(1)} \circ \cdots \circ D_{\rho_i|\cdot|^1}^{(1)} \circ D_{\rho_i}^{(k_i)}\right)(\pi),$$

we may assume that $\pi \subset \rho^k \rtimes \sigma$ with σ supercuspidal such that $\rho \rtimes \sigma$ is semisimple of length two. If we write $\rho \rtimes \sigma = \pi_+ \oplus \pi_-$, then $\rho^{k-1} \rtimes \pi_\pm$ is irreducible and its Zelevinsky–Aubert dual is given by $\rho^{k-1} \rtimes \hat{\pi}_\pm$. By [Aub95, Corollaire 1.10], we know that $\hat{\pi}_\pm = \pi_\mp$. Hence we see that $\eta'(\rho_{i_0}) = -\eta(\rho_{i_0})$, as desired. □

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If π is tempered, of ρ-bad parity and $\rho|\cdot|^z$-reduced for all $z \neq 0$, then π must be of the form $\pi = \rho^m \rtimes \sigma$ for some $m \geq 0$ and σ supercuspidal. In particular, we have $\hat{\pi} = \pi$. Similarly, if π is tempered, ugly and ρ'-reduced for all non-self-dual $\rho' \in \mathscr{C}^{\mathrm{GL}}$, then π must be supercuspidal so that $\hat{\pi} = \pi$.

6. Best matching functions: the ugly and negative cases

To give formulas for derivatives and socles, following [LM16, § 5.3] we introduce the notion of best matching functions. We then use these functions to explicate the ugly and the negative case.

6.1 Best matching functions

Let A and B be totally ordered finite sets with respect to $\geq_A$ and $\geq_B$, respectively. For $a \in A$, write $A_{>a} \coloneqq \{a' \in A \mid a' >_A a\}$. We consider a relation $\rightsquigarrow$ between B and A such that

$$\forall\, a_1 \geq_A a_2 \in A \quad \text{and} \quad \forall\, b_1 \geq_B b_2 \in B,$$
$$b_1 \rightsquigarrow a_1 \text{ and } b_2 \rightsquigarrow a_1 \text{ and } b_2 \rightsquigarrow a_2 \implies b_1 \rightsquigarrow a_2.$$

We say that such a relation is *traversable*. In this case, we define a subset A^0 of A and an injective map $f\colon A^0 \to B$ recursively by

$$a \in A^0 \iff \exists\, b \in B \setminus f(A^0 \cap A_{>a}) \text{ such that } b \rightsquigarrow a,$$
$$\text{in which case } f(a) \coloneqq \min\{b \in B \setminus f(A^0 \cap A_{>a}) \mid b \rightsquigarrow a\}.$$

Let $B^0 \coloneqq f(A^0)$ be the image of f. We call the bijection $f\colon A^0 \to B^0$ the *best matching function* between A and B. By [LM16, Lemma 5.7], the domain A^0 is equal to A if and only if Hall's criterion is satisfied, i.e. for any subset $A' \subset A$,

$$\big|\{b \in B \mid b \rightsquigarrow a \text{ for some } a \in A'\}\big| \geq |A'|.$$

When one of A or B is the empty set, note that we have $A^0 = B^0 = \emptyset$. We set $A^{\mathrm{c}} = A \setminus A^0$ and $B^{\mathrm{c}} = B \setminus B^0$.

6.2 Derivatives and socles in the ugly and negative cases

Fix $\rho \in \mathscr{C}^{\mathrm{GL}}$ and $x \in \mathbb{R}$. In this subsection, we give explicit formulas using the best matching functions for the highest $\rho|\cdot|^x$-derivatives $D^{(k)}_{\rho|\cdot|^x}(\pi)$ and the socles $S^{(1)}_{\rho|\cdot|^x}(\pi) = \mathrm{soc}(\rho|\cdot|^x \rtimes \pi)$ in the case where $\rho|\cdot|^x$ is ugly or where ρ is self-dual and x is negative.

Let $\pi \in \mathrm{Irr}(G_n)$. By Remark 2.7 and the Langlands classification, we can write $\pi = \mathrm{soc}(L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r]) \rtimes \pi_{\mathrm{temp}})$, where

- if $\rho|\cdot|^x$ is ugly, then $\rho_i = \rho$ for all $i = 1, \dots, r$, $x_1 + y_1 \leq \cdots \leq x_r + y_r$ and $\pi_{\mathrm{temp}} = \sigma$ is supercuspidal;
- if ρ is self-dual and x is negative, then $x_1 + y_1 \leq \cdots \leq x_r + y_r < 0$ and π_{temp} is tempered.

To unify the notation, let us call $(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r]; \pi_{\mathrm{temp}})$ the *inducing data*.

Define an ordered set $A_{\rho|\cdot|^x}$ by

$$A_{\rho|\cdot|^x} \coloneqq \{i \in \{1, \dots, r\} \mid \rho_i \cong \rho,\ x_i = x\}$$

with

$$a \geq a' \iff y_a \geq y_{a'}.$$

We define a relation $\rightsquigarrow$ between $A_{\rho|\cdot|^x}$ and $A_{\rho|\cdot|^{x-1}}$ by

$$A_{\rho|\cdot|^x} \ni a' \rightsquigarrow a \in A_{\rho|\cdot|^{x-1}} \iff y_{a'} > y_a.$$

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Namely, $a' \rightsquigarrow a$ if and only if $L(\Delta_\rho[x_a, y_a], \Delta_\rho[x_{a'}, y_{a'}])$ is a ladder representation. Note that this relation is traversable. Let $f\colon A^0_{\rho|\cdot|^{x-1}} \to A^0_{\rho|\cdot|^x}$ be the best matching function. In the next proposition, we obtain explicit formulas for the highest $\rho|\cdot|^x$-derivative $D^{(k)}_{\rho|\cdot|^x}(\pi)$ and the socle $S^{(1)}_{\rho|\cdot|^x}(\pi)$.

Proposition 6.1. *Suppose that $\rho|\cdot|^x$ is ugly or that ρ is self-dual and x is negative. With notation as above, the highest $\rho|\cdot|^x$-derivative $D^{(k)}_{\rho|\cdot|^x}(\pi)$ is the unique irreducible subrepresentation of $L(\Delta_{\rho_1}[x'_1, y_1], \dots, \Delta_{\rho_r}[x'_r, y_r]) \rtimes \pi_{\mathrm{temp}}$, where*

$$x'_i = \begin{cases} x-1 & \text{if } i \in A^{\mathrm{c}}_{\rho|\cdot|^x}, \\ x_i & \text{otherwise.} \end{cases}$$

In particular, $k = |A^{\mathrm{c}}_{\rho|\cdot|^x}|$. Moreover, the following hold.

(a) *If $A^{\mathrm{c}}_{\rho|\cdot|^{x-1}} \neq \emptyset$, then the inducing data of $S^{(1)}_{\rho|\cdot|^x}(\pi)$ can be obtained from those of π by replacing $x_a = x-1$ with x, where a is the minimum element of $A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}$.*

(b) *If $A^{\mathrm{c}}_{\rho|\cdot|^{x-1}} = \emptyset$, then the inducing data of $S^{(1)}_{\rho|\cdot|^x}(\pi)$ can be obtained from those of π by inserting $\rho|\cdot|^x = \Delta_\rho[x, x]$.*

Proof. Since $\rho|\cdot|^x$ is ugly or ρ is self-dual and x negative, we have

$$D^{(k)}_{\rho|\cdot|^x}(\pi) = \mathrm{soc}\big(L^{(k)}_{\rho|\cdot|^x}(L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r])) \rtimes \pi_{\mathrm{temp}}\big),$$
$$S^{(1)}_{\rho|\cdot|^x}(\pi) = \mathrm{soc}\big(\mathrm{soc}(\rho|\cdot|^x \times L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r])) \rtimes \pi_{\mathrm{temp}}\big).$$

Therefore, the proposition is essentially a problem for general linear groups, which was treated in [LM16, Theorem 5.11]. □

7. Explicit formulas for derivatives and socles: the positive case

In this section, we give explicit formulas for the highest derivatives and the socles of several parabolically induced representations in the positive case. The main results are Theorem 7.1, where we describe derivatives and socles in the good-parity case, and Theorem 7.4, in which the bad-parity case is treated. In Corollary 7.2 we deduce a result on irreducibility of certain parabolic inductions.

Throughout this section we fix $\rho \in \mathscr{C}^{\mathrm{GL}}$ self-dual and $x \in (1/2)\mathbb{Z}$ with $x > 0$.

7.1 The good-parity case

In this subsection, we assume that $\pi \in \mathrm{Irr}(G_n)$ is of good parity and that $\rho \boxtimes S_{2x+1}$ is self-dual of the same type as elements in $\Phi_{\mathrm{gp}}(G)$. Write $\pi = L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_{r'}}[x_{r'}, y_{r'}]; \pi(\phi, \eta))$ as a Langlands subrepresentation so that $x_1 + y_1 \leq \dots \leq x_{r'} + y_{r'} < 0$ and $\phi \in \Phi_{\mathrm{gp}}(G)$. Set

$$t = \big|\{i \in \{1, \dots, r'\} \mid \Delta_{\rho_i}[x_i, y_i] \cong \Delta_\rho[x-1, -x]\}\big|$$

and $r = r' - t$. Then we can rewrite

$$\pi = \mathrm{soc}\big(L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r]) \rtimes \pi_A\big),$$

where we set $\pi_A \coloneqq L(\Delta_\rho[x-1, -x]^t; \pi(\phi, \eta))$.

If $m_\phi(\rho \boxtimes S_{2x+1}) \neq 0$, $m_\phi(\rho \boxtimes S_{2x-1}) \neq 0$ and $\eta(\rho \boxtimes S_{2x+1})\eta(\rho \boxtimes S_{2x-1}) = (-1)^{t+1}$, set

$$\psi \coloneqq \phi - \rho \boxtimes (S_{2x+1} + S_{2x-1}) + (\rho \boxtimes S_{2x} \boxtimes S_2)^{t+1}$$

and $l \coloneqq 0$. Otherwise, set $\psi \coloneqq \phi + (\rho \boxtimes S_{2x} \boxtimes S_2)^t$ and $l \coloneqq 1$. Then $\pi_A = \pi(\psi, l, \eta) \in \Pi_\psi$ by Proposition 5.2. Set $m \coloneqq m_\psi(\rho \boxtimes S_{2x+1})$ and $m' \coloneqq m_\psi(\rho \boxtimes S_{2x-1})$. Then the highest $\rho|\cdot|^x$-derivative of $\mathrm{soc}((\rho|\cdot|^{-x})^s \rtimes \pi_A)$ is described in Theorem 5.3.

Note that $x_i \geq y_i$ for all $i = 1, \ldots, r$. Define ordered sets

$$A_{\rho|\cdot|^x} \coloneqq \{i \in \{1, \ldots, r\} \mid \rho_i \cong \rho,\ x_i = x\},$$
$$B_{\rho|\cdot|^x} \coloneqq \{i \in \{1, \ldots, r\} \mid \rho_i \cong \rho,\ y_i = -x\}$$

with

$$a \geq a' \iff y_a \geq y_{a'} \quad \text{for } a, a' \in A_{\rho|\cdot|^x},$$
$$b \geq b' \iff x_b \leq x_{b'} \quad \text{for } b, b' \in B_{\rho|\cdot|^x}.$$

Notice that any two of $A_{\rho|\cdot|^{x-1}}, A_{\rho|\cdot|^x}, B_{\rho|\cdot|^{x-1}}$ and $B_{\rho|\cdot|^x}$ have no intersection. Define relations $\rightsquigarrow$ between $A_{\rho|\cdot|^x}$ and $A_{\rho|\cdot|^{x-1}}$ and between $B_{\rho|\cdot|^x}$ and $B_{\rho|\cdot|^{x-1}}$ by

$$A_{\rho|\cdot|^x} \ni a' \rightsquigarrow a \in A_{\rho|\cdot|^{x-1}} \iff y_{a'} > y_a,$$
$$B_{\rho|\cdot|^x} \ni b' \rightsquigarrow b \in B_{\rho|\cdot|^{x-1}} \iff x_{b'} < x_b,$$

respectively. Note that these relations are traversable. Let $f\colon A^0_{\rho|\cdot|^{x-1}} \to A^0_{\rho|\cdot|^x}$ and $g\colon B^0_{\rho|\cdot|^{x-1}} \to B^0_{\rho|\cdot|^x}$ be the best matching functions. Write $B^{\mathrm{c}}_{\rho|\cdot|^x} = \{i_1, \ldots, i_s\}$ with $i_1 < \cdots < i_s$. Notice that $s > 0$ only if $x > 1$.

THEOREM 7.1. *With notation as above, suppose that $x > 0$, $x \in (1/2)\mathbb{Z}$ and $\rho \boxtimes S_{2x+1}$ is self-dual of the same type as ϕ. Then the highest $\rho|\cdot|^x$-derivative $D^{(k)}_{\rho|\cdot|^x}(\pi)$ is the unique irreducible subrepresentation of $L(\Delta_{\rho_1}[x'_1, y'_1], \ldots, \Delta_{\rho_r}[x'_r, y'_r]) \rtimes \pi'_A$, where*

$$x'_i = \begin{cases} -1 & \text{if } i \in A^{\mathrm{c}}_{\rho|\cdot|^x}, \\ x_i & \text{otherwise}, \end{cases}$$

$$y'_i = \begin{cases} -(x-1) & \text{if } i = i_j,\ j > m' + \max\{|A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}| - m, 0\}, \\ y_i & \text{otherwise}, \end{cases}$$

and $\pi'_A = \pi(\psi', l', \eta)$ with

$$\psi' = \psi - (\rho \boxtimes S_{2x+1})^{\max\{m - |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|, 0\}} + (\rho \boxtimes S_{2x-1})^{\max\{m - |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|, 0\}}$$

and

$$l' = l + \max\{m - |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|, 0\}.$$

In particular,

$$k = |A^{\mathrm{c}}_{\rho|\cdot|^x}| + \max\bigl\{m + \max\{|B^{\mathrm{c}}_{\rho|\cdot|^x}| - m', 0\} - |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|, 0\bigr\}.$$

Moreover, the following hold.

(a) *If $m + \max\{|B^{\mathrm{c}}_{\rho|\cdot|^x}| - m', 0\} < |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|$, then the Langlands data of $S^{(1)}_{\rho|\cdot|^x}(\pi)$ can be obtained from those of π by replacing $x_a = x - 1$ with x, where a is the minimum element of $A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}$.*

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(b) *If $|B^{\mathrm{c}}_{\rho|\cdot|^x}| < m'$ and $m \geq |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|$, the Langlands data of $S^{(1)}_{\rho|\cdot|^x}(\pi)$ can be obtained from those of π by replacing $\pi_A = \pi(\psi, l, \eta)$ with*

$$S^{(1)}_{\rho|\cdot|^x}(\pi_A) = \pi\big(\psi - (\rho \boxtimes S_{2x-1}) + (\rho \boxtimes S_{2x+1}), l-1, \eta\big).$$

(c) *If $|B^{\mathrm{c}}_{\rho|\cdot|^x}| \geq m'$, $m + |B^{\mathrm{c}}_{\rho|\cdot|^x}| - m' \geq |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|$ and $B^{\mathrm{c}}_{\rho|\cdot|^{x-1}} \neq \emptyset$, the Langlands data of $S^{(1)}_{\rho|\cdot|^x}(\pi)$ can be obtained from those of π by replacing $y_b = -(x-1)$ with $-x$, where b is the minimum element of $B^{\mathrm{c}}_{\rho|\cdot|^{x-1}}$.*

(d) *If $|B^{\mathrm{c}}_{\rho|\cdot|^x}| \geq m'$, $m + |B^{\mathrm{c}}_{\rho|\cdot|^x}| - m' \geq |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|$ and $B^{\mathrm{c}}_{\rho|\cdot|^{x-1}} = \emptyset$, then the Langlands data of $S^{(1)}_{\rho|\cdot|^x}(\pi)$ can be obtained from those of π by inserting $\rho|\cdot|^{-x} = \Delta_\rho[-x,-x]$.*

Proof. To obtain the formula for the highest derivative, we use Jantzen's algorithm [Jan18a, § 3.3] together with [LM16, Theorem 5.11] and Theorem 5.3.

(1) Recall that

$$\pi = \mathrm{soc}\big(L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r]) \rtimes \pi_A\big)$$

with $\pi_A = L(\Delta_\rho[x-1,-x]^t; \pi(\phi, \eta))$ and $\Delta_{\rho_i}[x_i, y_i] \not\cong \Delta_\rho[x-1,-x]$ for all $i = 1, \dots, r$.

(2) By [LM16, Theorem 5.11], we can compute the highest right $\rho|\cdot|^{-x}$-derivative

$$R^{(s)}_{\rho|\cdot|^{-x}}(L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r])) = L(\Delta_{\rho_1}[x_1, y''_1], \dots, \Delta_{\rho_r}[x_r, y''_r]),$$

where

$$y''_i = \begin{cases} -(x-1) & \text{if } i \in B^{\mathrm{c}}_{\rho|\cdot|^x}, \\ y_i & \text{otherwise.} \end{cases}$$

In particular, $s = |B^{\mathrm{c}}_{\rho|\cdot|^x}|$. Claim 1 in [Jan18a, § 3.3] says that

$$\pi = \mathrm{soc}\big(L(\Delta_{\rho_1}[x_1, y''_1], \dots, \Delta_{\rho_r}[x_r, y''_r]) \rtimes \pi_1\big)$$

with $\pi_1 \coloneqq \mathrm{soc}((\rho|\cdot|^{-x})^s \rtimes \pi_A)$.

(3) By Theorem 5.3, the highest $\rho|\cdot|^x$-derivative $\pi_2 \coloneqq D^{(k_1)}_{\rho|\cdot|^x}(\pi_1)$ of π_1 is

$$\pi_2 = \mathrm{soc}\big((\rho|\cdot|^{-x})^{\min\{s,m'\}} \rtimes \pi(\psi - (\rho \boxtimes S_{2x+1})^m + (\rho \boxtimes S_{2x-1})^m, l+m, \eta)\big)$$

with $k_1 = m + \max\{s - m', 0\}$. Claim 2 in [Jan18a, § 3.3] says that

$$\pi = \mathrm{soc}\big(L(\Delta_{\rho_1}[x_1, y''_1], \dots, \Delta_{\rho_r}[x_r, y''_r], (\rho|\cdot|^x)^{k_1}) \rtimes \pi_2\big).$$

(4) We will apply [LM16, Theorem 5.11] to compute the highest left $\rho|\cdot|^x$-derivative of $L(\Delta_{\rho_1}[x_1, y''_1], \dots, \Delta_{\rho_r}[x_r, y''_r], (\rho|\cdot|^x)^{k_1})$. To do this, we have to replace $A_{\rho|\cdot|^x}$ with $A_{\rho|\cdot|^x} \cup \{r+1, \dots, r+k_1\}$, where we set $\Delta_{\rho_i}[x_i, y_i] = \rho|\cdot|^x$ for $i = r+1, \dots, r+k_1$. Note that any $a' \in \{r+1, \dots, r+k_1\}$ is bigger than any element of $A_{\rho|\cdot|^x}$ with respect to the order of $A_{\rho|\cdot|^x} \cup \{r+1, \dots, r+k_1\}$, and $a' \rightsquigarrow a$ for every $a \in A_{\rho|\cdot|^{x-1}}$. Hence the image of the resulting best matching function is

$$A^0_{\rho|\cdot|^x} \cup \big\{r+i \;\big|\; 1 \leq i \leq \min\{k_1, |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|\}\big\}.$$

Therefore, with $k_2 = \min\{k_1, |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|\}$ and $k = |A^{\mathrm{c}}_{\rho|\cdot|^x}| + k_1 - k_2$, the highest left $\rho|\cdot|^x$-derivative is

$$\begin{aligned} &L^{(k)}_{\rho|\cdot|^x}\big(L(\Delta_{\rho_1}[x_1, y''_1], \dots, \Delta_{\rho_r}[x_r, y''_r], (\rho|\cdot|^x)^{k_1})\big) \\ &\quad = L(\Delta_{\rho_1}[x'_1, y''_1], \dots, \Delta_{\rho_r}[x'_r, y''_r], (\rho|\cdot|^x)^{k_2}), \end{aligned}$$

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where x'_i is as in the statement of this theorem. Then the highest $\rho|\cdot|^x$-derivative of π is

$$D^{(k)}_{\rho|\cdot|^x}(\pi) = \mathrm{soc}\big(L(\Delta_{\rho_1}[x'_1, y''_1], \dots, \Delta_{\rho_r}[x'_r, y''_r], (\rho|\cdot|^x)^{k_2}) \rtimes \pi_2\big).$$

(5) Claim 3 in [Jan18a, § 3.3] says that

$$D^{(k)}_{\rho|\cdot|^x}(\pi) = \mathrm{soc}\big(L(\Delta_{\rho_1}[x'_1, y''_1], \dots, \Delta_{\rho_r}[x'_r, y''_r]) \rtimes S^{(k_2)}_{\rho|\cdot|^x}(\pi_2)\big).$$

By Theorem 5.3, we have

$$S^{(k_2)}_{\rho|\cdot|^x}(\pi_2) = \mathrm{soc}((\rho|\cdot|^{-x})^{s'} \rtimes \pi'_A),$$

where π'_A is as in the statement of this theorem and $s' = \min\{s, m'\} + \max\{k_2 - m, 0\}$. Note that $s' \leq s$.

(6) Finally, note that

- if $s' = s$, then $m' + \max\{|A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}| - m, 0\} \geq s$, so that $y'_i = y_i$ for all $i = 1, \dots, r$;
- if $s' < s$, then $s > m'$ and $k_1 = m + s - m' > k_2 = |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|$, so that $s' = m' + \max\{|A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}| - m, 0\}$.

By [LM16, Theorem 5.11], we have

$$\begin{aligned} &\mathrm{soc}\big(L(\Delta_{\rho_1}[x'_1, y''_1], \dots, \Delta_{\rho_r}[x'_r, y''_r]) \times (\rho|\cdot|^{-x})^{s'}\big) \\ &= L(\Delta_{\rho_1}[x'_1, y'_1], \dots, \Delta_{\rho_r}[x'_r, y'_r]), \end{aligned}$$

where y'_i is as in the statement of this theorem. Claim 4 in [Jan18a, § 3.3] says that

$$D^{(k)}_{\rho|\cdot|^x}(\pi) = \mathrm{soc}\big(L(\Delta_{\rho_1}[x'_1, y'_1], \dots, \Delta_{\rho_r}[x'_r, y'_r]) \rtimes \pi'_A\big).$$

This gives the Langlands data of $D^{(k)}_{\rho|\cdot|^x}(\pi)$.

Recall that $S^{(1)}_{\rho|\cdot|^x}(\pi)$ is an irreducible representation determined by the relation

$$D^{(k+1)}_{\rho|\cdot|^x}\big(S^{(1)}_{\rho|\cdot|^x}(\pi)\big) = D^{(k)}_{\rho|\cdot|^x}(\pi).$$

One can easily check this equation for the representations given in (a), (b), (c) and (d). $\square$

As an application of Proposition 6.1 and Theorem 7.1, we have a combinatorial irreducibility criterion for $\rho|\cdot|^x \rtimes \pi$ as follows.

Corollary 7.2. *With notation as above, suppose that $x > 0$, $x \in (1/2)\mathbb{Z}$ and $\rho \boxtimes S_{2x+1}$ is self-dual of the same type as ϕ. Then the parabolically induced representation $\rho|\cdot|^x \rtimes \pi$ is irreducible if and only if all of the following conditions hold:*

- $A^{\mathrm{c}}_{\rho|\cdot|^{-x-1}} = \emptyset$;
- $|B^{\mathrm{c}}_{\rho|\cdot|^x}| \geq m_\psi(\rho \boxtimes S_{2x-1})$;
- $m_\psi(\rho \boxtimes S_{2x+1}) + |B^{\mathrm{c}}_{\rho|\cdot|^x}| - m_\psi(\rho \boxtimes S_{2x-1}) \geq |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|$;
- $B^{\mathrm{c}}_{\rho|\cdot|^{x-1}} = \emptyset$.

Proof. Since $\rho|\cdot|^x$ is not self-dual, by Proposition 3.3, $\rho|\cdot|^x \rtimes \pi$ is SI so that both $S^{(1)}_{\rho|\cdot|^x}(\pi)$ and $S^{(1)}_{\rho|\cdot|^{-x}}(\pi)$ occur with multiplicity one in $[\rho|\cdot|^x \rtimes \pi]$. Hence $\rho|\cdot|^x \rtimes \pi$ is irreducible if and only if $S^{(1)}_{\rho|\cdot|^x}(\pi) \cong S^{(1)}_{\rho|\cdot|^{-x}}(\pi)$. By Proposition 6.1 and Theorem 7.1, this is equivalent to the case where the Langlands data of $S^{(1)}_{\rho|\cdot|^{-x}}(\pi)$ and $S^{(1)}_{\rho|\cdot|^x}(\pi)$ are obtained from those of π by inserting $\rho|\cdot|^{-x}$. $\square$

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As a special case, when $\pi = \pi(\phi, \eta)$ is tempered, since $A_{\rho^\vee|\cdot|^{-x-1}}, A_{\rho|\cdot|^{x-1}}, A_{\rho|\cdot|^x}, B_{\rho|\cdot|^{x-1}}$ and $B_{\rho|\cdot|^x}$ are all the empty set, we see that $\rho|\cdot|^x \rtimes \pi$ if and only if $m_\psi(\rho \boxtimes S_{2x-1}) = 0$, which is equivalent to

- $\phi \not\supset \rho \boxtimes S_{2x-1}$; or
- $m_\phi(\rho \boxtimes S_{2x-1}) = 1$, $m_\phi(\rho \boxtimes S_{2x+1}) > 0$ and $\eta(\rho \boxtimes S_{2x-1}) \neq \eta(\rho \boxtimes S_{2x+1})$.

This special case was already known to Jantzen [Jan18b, Theorem 4.7].

7.2 The bad-parity case

We now treat the bad-parity case. Specifically, we assume that $\rho \boxtimes S_{2x+1}$ is self-dual of the opposite type to elements in $\Phi_{\mathrm{gp}}(G)$, and we take $\pi \in \mathrm{Irr}(G_n)$ such that $\mathrm{scusp}(\pi) \subset \mathbb{Z}_{\rho|\cdot|^x} \cup \{\sigma\}$ for some $\sigma \in \mathscr{C}^G$.

We remark that Jantzen's algorithm [Jan18a, § 3.3] for computing the highest $\rho|\cdot|^x$-derivatives can be applied to the bad-parity case. According to this algorithm (see (2) in the proof of Theorem 7.1), we have to deal with a $\rho|\cdot|^x$-bad representation of the form

$$\pi_1 = L\big((\rho|\cdot|^{-x})^s, \Delta_\rho[x-1, -x]^t; \pi(\phi, \eta)\big)$$

with $\phi \in \Phi_{\mathrm{temp}}(G_n)$ and $s, t \geq 0$. Here, we may assume that $s = 0$ if $x = 1/2$ since $\rho|\cdot|^{-1/2} = \Delta_\rho[-1/2, -1/2]$. By the assumption of bad parity, if we write $\sigma = \pi(\phi_\sigma, \eta_\sigma)$, then $\phi = \phi_\sigma \oplus (\bigoplus_{i=1}^r (\rho \boxtimes S_{2x_i+1})^{m_i})$ with $x_i \in x + \mathbb{Z}$ so that $\mathcal{S}_\phi \cong \mathcal{S}_{\phi_\sigma}$, and $\eta = \eta_\sigma$. Moreover, the multiplicity m_i is even for all i. The following result is an extension of [Jan18a, Propositions 8.5 and 8.6].

Proposition 7.3. *With notation as above, when $x = 1/2$, we assume here that $s = 0$. Set $m \coloneqq m_\phi(\rho \boxtimes S_{2x+1})$ and $m' \coloneqq m_\phi(\rho \boxtimes S_{2x-1})$, both of which are even. Take $\kappa \in \{0, 1\}$ such that $t \equiv \kappa \bmod 2$. Then the highest $\rho|\cdot|^x$-derivative $D^{(k)}_{\rho|\cdot|^x}(\pi_1)$ is equal to*

$$L\big((\rho|\cdot|^{-x})^{\min\{s, m'+\kappa\}}, \Delta_\rho[x-1, -x]^{t-\kappa}; \pi(\phi - (\rho \boxtimes S_{2x+1})^m + (\rho \boxtimes S_{2x-1})^{m+2\kappa}, \eta)\big)$$

with $k = m + \kappa + \max\{s - m' - \kappa, 0\}$.

Proof. If we write $\pi_0 \coloneqq \pi(\phi - (\rho \boxtimes S_{2x+1})^m - (\rho \boxtimes S_{2x-1})^{m'}, \eta)$, then

$$\pi(\phi, \eta) = \Delta_\rho[x-1, -(x-1)]^{m'/2} \times \Delta_\rho[x, -x]^{m/2} \rtimes \pi_0$$

is an irreducible induction. Moreover,

$$\Delta_\rho[x-1, -x] \times \Delta_\rho[x-1, -(x-1)]^{m'/2} \times \Delta_\rho[x, -x]^{m/2} \rtimes \pi_0$$

is always irreducible by [MW12, Théorème (i)]. Also, any subquotient of $\Delta_\rho[x-1, -x] \times \Delta_\rho[x, -(x-1)]$ is $\Delta_\rho[x-1, -(x-1)] \times \Delta_\rho[x, -x]$ or $L_0 \coloneqq L(\Delta_\rho[x-1, -x], \Delta_\rho[x, -(x-1)])$, both of which commute with all of $\Delta_\rho[x-1, -(x-1)]$, $\Delta_\rho[x, -x]$ and $\Delta_\rho[x-1, -x]$ (see for example [Tad14, Theorem 1.1]).

First we assume that t is even. By considering the Langlands data, we have

$$\begin{aligned}
&\mathrm{soc}\big(\Delta_\rho[x-1, -x]^t \times \Delta_\rho[x-1, -(x-1)]^{m'/2} \times \Delta_\rho[x, -x]^{m/2} \rtimes \pi_0\big) \\
&\hookrightarrow L_0^{t/2} \times \Delta_\rho[x-1, -(x-1)]^{m'/2} \times \Delta_\rho[x, -x]^{m/2} \rtimes \pi_0 \\
&\hookrightarrow \Delta_\rho[x-1, -x]^t \times \Delta_\rho[x-1, -(x-1)]^{m'/2} \times \Delta_\rho[x, -x]^{m/2} \rtimes \pi_0.
\end{aligned}$$

Since the middle induced representation is unitary and the last induced representation is a standard module and so is SI, we see that the first inclusion map is an isomorphism. In particular,

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π_1 is equal to the socle of

$$
\begin{aligned}
&(\rho|\cdot|^{-x})^s \times L_0^{t/2} \times \Delta_\rho[x-1,-(x-1)]^{m'/2} \times \Delta_\rho[x,-x]^{m/2} \rtimes \pi_0 \\
&\quad \cong L_0^{t/2} \times (\rho|\cdot|^{-x})^s \times \Delta_\rho[x-1,-(x-1)]^{m'/2} \times \Delta_\rho[x,-x]^{m/2} \rtimes \pi_0.
\end{aligned}
$$

Therefore, we may replace $(\rho|\cdot|^{-x})^s \times \Delta_\rho[x-1,-(x-1)]^{m'/2}$ with

$$
(\rho|\cdot|^{-x})^{\max\{s-m'/2,0\}} \times L_1^{\min\{s,m'/2\}} \times \Delta_\rho[x-1,-(x-1)]^{\max\{m'/2-s,0\}}, \tag{$*$}
$$

where $L_1 \coloneqq L(\rho|\cdot|^{-x}, \Delta_\rho[x-1,-(x-1)])$. Moreover, since $\rho|\cdot|^{-x} \times \Delta_\rho[x,-x]^{m/2} \rtimes \pi_0$ is irreducible by [MW12, Théorème (i)], if $s \geq m'/2$, then we may replace $(*)$ with

$$
(\rho|\cdot|^{-x})^{\max\{s-m',0\}} \times L_2^{\min\{s-m'/2,m'/2\}} \times L_1^{\max\{m'-s,0\}}, \tag{$**$}
$$

where $L_2 \coloneqq L(\rho|\cdot|^{-x}, \Delta_\rho[x-1,-(x-1)], \rho|\cdot|^{x})$. Note that if $x \geq 1$, then by [LM16, Proposition 5.15(3)] the ladder representations L_0, L_1 and L_2 commute with all of

$$
\Delta_\rho[x,-x], \quad \Delta_\rho[x-1,-x], \quad \Delta_\rho[x,-(x-1)], \quad \Delta_\rho[x-1,-(x-1)].
$$

Therefore, with

$$
k = m + \max\{s-m',0\},
$$

the $\rho|\cdot|^x$-derivative $D^{(k)}_{\rho|\cdot|^x}(\pi)$ is the highest and is a subrepresentation of

$$
\begin{cases}
L_0^{t/2} \times L_1^{s} \times \Delta_\rho[x-1,-(x-1)]^{m'/2-s+m/2} \rtimes \pi_0 & \text{if } s \leq m'/2, \\
L_0^{t/2} \times L_2^{s-m'/2} \times L_1^{m'-s} \times \Delta_\rho[x-1,-(x-1)]^{m/2} \rtimes \pi_0 & \text{if } m'/2 < s \leq m', \\
L_0^{t/2} \times L_2^{m'/2} \times \Delta_\rho[x-1,-(x-1)]^{m/2} \rtimes \pi_0 & \text{if } s > m'.
\end{cases}
$$

Since $L_2 \times L_1 \cong L_1 \times L_2$ by [LM16, Corollary 6.2] and since $L_1 \rtimes \sigma$ is irreducible by [LT20, Theorem 1.2], this representation is a subrepresentation of

$$
\begin{cases}
(\rho|\cdot|^{-x})^{s} \times \Delta_\rho[x-1,-x]^t \times \Delta_\rho[x-1,-(x-1)]^{(m'+m)/2} \rtimes \pi_0 & \text{if } s \leq m', \\
(\rho|\cdot|^{-x})^{m'} \times \Delta_\rho[x-1,-x]^t \times \Delta_\rho[x-1,-(x-1)]^{(m'+m)/2} \rtimes \pi_0 & \text{if } s > m'.
\end{cases}
$$

Since $\Delta_\rho[x-1,-(x-1)]^{(m'+m)/2} \rtimes \pi_0 = \pi(\phi - (\rho \boxtimes S_{2x+1})^m + (\rho \boxtimes S_{2x-1})^m, \eta)$, we obtain the case where t is even.

Next, assume that t is odd. By considering the Langlands data, we have

$$
\begin{aligned}
&\mathrm{soc}\big(\Delta_\rho[x-1,-x]^t \times \Delta_\rho[x-1,-(x-1)]^{m'/2} \times \Delta_\rho[x,-x]^{m/2} \rtimes \pi_0\big) \\
&\quad \hookrightarrow L_0^{(t-1)/2} \times \Delta_\rho[x-1,-x] \times \Delta_\rho[x-1,-(x-1)]^{m'/2} \times \Delta_\rho[x,-x]^{m/2} \rtimes \pi_0 \\
&\quad \cong L_0^{(t-1)/2} \times \Delta_\rho[x,-(x-1)] \times \Delta_\rho[x-1,-(x-1)]^{m'/2} \times \Delta_\rho[x,-x]^{m/2} \rtimes \pi_0.
\end{aligned}
$$

Note that the middle induced representation is SI since it is a subrepresentation of a standard module. On the other hand, by taking the MVW-functor and the contragredient functor, we see that the unique irreducible subrepresentation of the middle induced representation is also an irreducible quotient of the last induced representation. By the last isomorphism, this means that $L_0^{(t-1)/2} \times \Delta_\rho[x,-(x-1)] \times \Delta_\rho[x-1,-(x-1)]^{m'/2} \times \Delta_\rho[x,-x]^{m/2} \rtimes \pi_0$ is irreducible. Therefore, by the same argument as in the case where t is even, with $k = m + 1 + \max\{s-m'-1,0\}$,

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the $\rho|\cdot|^x$-derivative $D^{(k)}_{\rho|\cdot|^x}(\pi)$ is highest and is a subrepresentation of

$$\begin{cases}(\rho|\cdot|^{-x})^s \times \Delta_\rho[x-1,-x]^{t-1} \times \Delta_\rho[x-1,-(x-1)]^{(m'+m)/2+1} \rtimes \pi_0 & \text{if } s \leq m'+1, \\ (\rho|\cdot|^{-x})^{m'+1} \times \Delta_\rho[x-1,-x]^{t-1} \times \Delta_\rho[x-1,-(x-1)]^{(m'+m)/2+1} \rtimes \pi_0 & \text{if } s > m'+1.\end{cases}$$

Since $\Delta_\rho[x-1,-(x-1)]^{(m'+m)/2+1} \rtimes \pi_0 = \pi(\phi - (\rho \boxtimes S_{2x+1})^m + (\rho \boxtimes S_{2x-1})^{m+2}, \eta)$, we obtain the case where t is odd. □

Now we consider the general case. Let $\pi = L(\Delta_\rho[x_1,y_1], \dots, \Delta_\rho[x_{r'}, y_{r'}]; \pi(\phi,\eta))$ with $x_1 + y_1 \leq \dots \leq x_{r'} + y_{r'} < 0$ and $\phi \in \Phi_{\mathrm{temp}}(G)$. If we define $t, r \geq 0$ with $t + r = r'$ as in § 7.1, we can rewrite

$$\pi = \mathrm{soc}\big(L(\Delta_\rho[x_1,y_1], \dots, \Delta_\rho[x_r,y_r]) \rtimes \pi_A\big),$$

where

- $x_1 + y_1 \leq \dots \leq x_r + y_r < 0$;
- $\pi_A \coloneqq L(\Delta_\rho[x-1,-x]^t; \pi(\phi,\eta))$;
- $[x_i,y_i] \neq [x-1,-x]$ for all $i = 1, \dots, r$.

Set $m \coloneqq m_\phi(\rho \boxtimes S_{2x+1})$ and $m' \coloneqq m_\phi(\rho \boxtimes S_{2x-1})$, both of which are even. Take $\kappa \in \{0,1\}$ such that $t \equiv \kappa \bmod 2$.

Define

$$A_{\rho|\cdot|^x} \coloneqq \{i \in \{1,\dots,r\} \mid x_i = x\},$$
$$B_{\rho|\cdot|^x} \coloneqq \{i \in \{1,\dots,r\} \mid y_i = -x\}.$$

As in the previous paragraph, we regard $A_{\rho|\cdot|^x}$ and $A_{\rho|\cdot|^{x-1}}$ (respectively $B_{\rho|\cdot|^x}$ and $B_{\rho|\cdot|^{x-1}}$) as ordered sets and take the traversal relation $\rightsquigarrow$. Let $f\colon A^0_{\rho|\cdot|^{x-1}} \to A^0_{\rho|\cdot|^x}$ (respectively $g\colon B^0_{\rho|\cdot|^{x-1}} \to B^0_{\rho|\cdot|^x}$) be the best matching function. Write $B^{\mathrm{c}}_{\rho|\cdot|^x} = \{i_1, \dots, i_s\}$ with $i_1 < \dots < i_s$. Note that $s > 0$ only if $x > 1$.

Theorem 7.4. *With notation as above, suppose that $x > 0$, $x \in (1/2)\mathbb{Z}$ and $\rho \boxtimes S_{2x+1}$ is self-dual of the opposite type to elements in $\Phi_{\mathrm{gp}}(G)$. Then the highest $\rho|\cdot|^x$-derivative $D^{(k)}_{\rho|\cdot|^x}(\pi)$ is the unique irreducible subrepresentation of $L(\Delta_{\rho_1}[x'_1,y'_1], \dots, \Delta_{\rho_r}[x'_r,y'_r]) \rtimes \pi'_A$, where*

$$x'_i = \begin{cases} x-1 & \text{if } i \in A^{\mathrm{c}}_{\rho|\cdot|^x}, \\ x_i & \text{otherwise,}\end{cases}$$

$$y'_i = \begin{cases} -(x-1) & \text{if } i = i_j,\ j > m' + \kappa + \max\{|A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}| - m - \kappa, 0\}, \\ y_i & \text{otherwise}\end{cases}$$

and

- *if $m + \kappa \leq |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|$, then $\pi'_A = \pi_A$;*
- *if $m + \kappa > |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|$, then*

$$\pi'_A = \begin{cases} L\big(\Delta_\rho[x-1,-x]^{t-\kappa}; \pi(\phi - (\rho \boxtimes S_{2x+1})^{m-v} + (\rho \boxtimes S_{2x-1})^{m-v+2\kappa}, \eta)\big), \\ L\big(\Delta_\rho[x-1,-x]^{t-\kappa+1}; \pi(\phi - (\rho \boxtimes S_{2x+1})^{m-v+1} + (\rho \boxtimes S_{2x-1})^{m-v-1+2\kappa}, \eta)\big)\end{cases}$$

  according to whether $v = |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|$ is even or odd.

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In particular,

$$k = |A^{\mathrm{c}}_{\rho|\cdot|^x}| + \max\big\{m+\kappa+\max\{|B^{\mathrm{c}}_{\rho|\cdot|^x}| - m' - \kappa, 0\} - |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|, 0\big\}.$$

Moreover, the following hold.

(a) *If $m+\kappa+\max\{|B^{\mathrm{c}}_{\rho|\cdot|^x}| - m' - \kappa, 0\} < |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|$, then the Langlands data of $S^{(1)}_{\rho|\cdot|^x}(\pi)$ can be obtained from those of π by replacing $x_a = x-1$ with x, where a is the minimum element of $A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}$.*

(b) *If $|B^{\mathrm{c}}_{\rho|\cdot|^x}| < m'+\kappa$ and $m+\kappa \geq |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|$, the Langlands data of $S^{(1)}_{\rho|\cdot|^x}(\pi)$ can be obtained from those of π by replacing π_A with*

$$S^{(1)}_{\rho|\cdot|^x}(\pi_A) = \begin{cases} L\big(\Delta_\rho[x-1,-x]^{t+1}; \pi(\phi - (\rho\boxtimes S_{2x-1})^2, \eta)\big) & \text{if } \kappa = 0, \\ L\big(\Delta_\rho[x-1,-x]^{t-1}; \pi(\phi + (\rho\boxtimes S_{2x+1})^2, \eta)\big) & \text{if } \kappa = 1. \end{cases}$$

(c) *If $|B^{\mathrm{c}}_{\rho|\cdot|^x}| \geq m'+\kappa$, $m + |B^{\mathrm{c}}_{\rho|\cdot|^x}| - m' \geq |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|$ and $B^{\mathrm{c}}_{\rho|\cdot|^{x-1}} \neq \emptyset$, the Langlands data of $S^{(1)}_{\rho|\cdot|^x}(\pi)$ can be obtained from those of π by replacing $y_b = -(x-1)$ with $-x$, where b is the minimum element of $B^{\mathrm{c}}_{\rho|\cdot|^{x-1}}$.*

(d) *If $|B^{\mathrm{c}}_{\rho|\cdot|^x}| \geq m'+\kappa$, $m + |B^{\mathrm{c}}_{\rho|\cdot|^x}| - m' \geq |A^{\mathrm{c}}_{\rho|\cdot|^{x-1}}|$ and $B^{\mathrm{c}}_{\rho|\cdot|^{x-1}} = \emptyset$, then the Langlands data of $S^{(1)}_{\rho|\cdot|^x}(\pi)$ can be obtained from those of π by inserting $\rho|\cdot|^{-x} = \Delta_\rho[-x,-x]$.*

Proof. By a similar argument to that for Theorem 7.1, we obtain the assertions by applying Jantzen's algorithm [Jan18a, § 3.3] together with [LM16, Theorem 5.11] and Proposition 7.3. □

As a consequence, one can obtain an analogous criterion to Corollary 7.2 for the irreducibility of $\rho|\cdot|^x \rtimes \pi$. We leave the details to the reader.

8. Explicit formulas for derivatives and socles: a non-cuspidal case

Fix $\rho \in \mathscr{C}^{\mathrm{GL}}$ self-dual. In this section, we consider $\pi \in \mathrm{Irr}(G_n)$ of good or ρ-bad parity such that

(a) π is $\rho|\cdot|^1$-reduced; and
(b) π is $\rho|\cdot|^z$-reduced for all $z < 0$.

Recall that if an irreducible representation π is $\rho|\cdot|^1$-reduced, Proposition 3.7 says that $Z_\rho[0,1]^k \rtimes \pi$ is SI. In this subsection, we determine the highest $[0,1]$-derivative $\pi' = D^{(k)}_{[0,1]}(\pi)$ of π, and we show how to recover the Langlands data of π in terms of those of π'.

8.1 A reduction step

In this subsection, we reduce the computation to a particular case that will be treated at the end of the section.

We write $\pi = L(\Delta_{\rho_1}[x_1,y_1], \dots, \Delta_{\rho_r}[x_r,y_r], \Delta_\rho[0,-1]^t; \pi(\phi,\eta))$ as a Langlands subrepresentation, where

- $\phi \in \Phi_{\mathrm{temp}}(G)$;
- $t \geq 0$;
- $x_1 + y_1 \leq \dots \leq x_r + y_r < 0$;
- $\Delta_{\rho_i}[x_i,y_i] \not\cong \Delta_\rho[0,-1]$ for $i = 1, \dots, r$.

We know by the assumption (b) that $x_i \geq 0$ if $\rho_i \cong \rho$. Also, by the last condition above, we have $y_i \neq -1$ if $\rho_i \cong \rho$. Set $\pi_A \coloneqq L(\Delta_\rho[0,-1]^t; \pi(\phi,\eta))$.

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To rephrase the assumption (a), we recall Jantzen's algorithm [Jan18a, § 3.3]. Let $\pi'_A \coloneqq D^{(l)}_{\rho|\cdot|^1}(\pi_A)$ be the highest $\rho|\cdot|^1$-derivative of π_A. It can be computed thanks to Theorem 5.3 and Proposition 7.3. Then Claim 2 in [Jan18a, § 3.3] says that

$$\pi \hookrightarrow L\big(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r], (\rho|\cdot|^1)^l\big) \rtimes \pi'_A.$$

According to Jantzen's algorithm, π is $\rho|\cdot|^1$-reduced if and only if $L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r], (\rho|\cdot|^1)^l)$ is left $\rho|\cdot|^1$-reduced. For $i = r+1, \dots, r+l$, we set $\Delta_{\rho_i}[x_i, y_i] = \rho|\cdot|^1$. Define

$$A_\rho \coloneqq \{i \in \{1, \dots, r+l\} \mid \rho_i \cong \rho,\ x_i = 0\},$$
$$A_{\rho|\cdot|^1} \coloneqq \{i \in \{1, \dots, r+l\} \mid \rho_i \cong \rho,\ x_i = 1\}.$$

As in § 6.2, we regard these sets as totally ordered sets, and we define a traversable relation $\rightsquigarrow$ between $A_{\rho|\cdot|^1}$ and A_ρ. Let $f\colon A^0_\rho \to A^0_{\rho|\cdot|^1}$ be the best matching function. Then by [LM16, Theorem 5.11], $L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r], (\rho|\cdot|^1)^l)$ is left $\rho|\cdot|^1$-reduced if and only if $A^{\mathrm{c}}_{\rho|\cdot|^1} = \emptyset$. Let $D^{(k_A)}_{[0,1]}(\pi'_A)$ be the highest $[0,1]$-derivative of π'_A. We will explicitly compute it in Propositions 8.3 and 8.4 below.

Theorem 8.1. *Let $\pi \in \mathrm{Irr}(G_n)$ be of good or ρ-bad parity and satisfy the assumptions (a) and (b). We use the above notation. Then the highest $[0,1]$-derivative $D^{(k)}_{[0,1]}(\pi)$ is the unique irreducible subrepresentation of*

$$L(\Delta_{\rho_1}[x'_1, y_1], \dots, \Delta_{\rho_r}[x'_r, y_r]) \rtimes D^{(k_A)}_{[0,1]}(\pi'_A),$$

where

$$x'_i = \begin{cases} -1 & \text{if } i \in A^0_\rho, \\ 0 & \text{if } i \in A_{\rho|\cdot|^1}, \\ x_i & \text{otherwise.} \end{cases}$$

In particular, $k = k_A + r_1$ with $r_1 \coloneqq |A_{\rho|\cdot|^1}| = |A^0_\rho|$.

Proof. Since $x_i \geq 0$ if $\rho_i \cong \rho$, we see that $\Delta_{\rho_i}[x_i, y_i] \times Z_\rho[0,1] \cong Z_\rho[0,1] \times \Delta_{\rho_i}[x_i, y_i]$ for all $i = 1, \dots, r+l$ (see for example [Tad14, Theorem 1.1]). Hence

$$\begin{aligned} \pi &\hookrightarrow L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r], (\rho|\cdot|^1)^l) \rtimes \pi'_A \\ &\hookrightarrow L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r], (\rho|\cdot|^1)^l) \times Z_\rho[0,1]^{k_A} \rtimes D^{(k_A)}_{[0,1]}(\pi'_A) \\ &\cong Z_\rho[0,1]^{k_A} \times L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r], (\rho|\cdot|^1)^l) \rtimes D^{(k_A)}_{[0,1]}(\pi'_A). \end{aligned}$$

We claim that

$$L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r], (\rho|\cdot|^1)^l) \hookrightarrow Z_\rho[0,1]^{r_1} \times L(\Delta_{\rho_1}[x'_1, y_1], \dots, \Delta_{\rho_r}[x'_r, y_r]).$$

To see this, by [LM16, Proposition 5.6] it is enough to show that

$$\begin{aligned} &L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r], (\rho|\cdot|^1)^l) \\ &\quad = \mathrm{soc}\big(\rho^{r_1+k'} \times \mathrm{soc}\big((\rho|\cdot|^1)^{r_1} \times L^{(k')}_\rho(L(\Delta_{\rho_1}[x'_1, y_1], \dots, \Delta_{\rho_r}[x'_r, y_r]))\big)\big), \end{aligned}$$

where $L^{(k')}_\rho(L(\Delta_{\rho_1}[x'_1, y_1], \dots, \Delta_{\rho_r}[x'_r, y_r]))$ is the highest left ρ-derivative. By our assumptions and by the definition of x'_i, we see that $k' = r_0 - r_1$ with $r_0 = |A_\rho|$ and that

$$L^{(r_0-r_1)}_\rho\big(L(\Delta_{\rho_1}[x'_1, y_1], \dots, \Delta_{\rho_r}[x'_r, y_r])\big) = L\big(\Delta_{\rho_1}[x^{(1)}_1, y_1], \dots, \Delta_{\rho_r}[x^{(1)}_r, y_r]\big)$$

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with

$$x_i^{(1)} = \begin{cases} -1 & \text{if } i \in A_\rho^{\mathrm{c}}, \\ x_i' & \text{otherwise} \end{cases}$$
$$= \begin{cases} -1 & \text{if } i \in A_\rho, \\ 0 & \text{if } i \in A_{\rho|\cdot|^1}, \\ x_i & \text{otherwise.} \end{cases}$$

Since $x_i^{(1)} \neq 1$ if $\rho_i \cong \rho$, we have

$$\begin{aligned} &\mathrm{soc}\big((\rho|\cdot|^1)^{r_1} \times L_\rho^{(r_0-r_1)}(L(\Delta_{\rho_1}[x_1', y_1], \dots, \Delta_{\rho_r}[x_r', y_r]))\big) \\ &\quad = L(\Delta_{\rho_1}[x_1^{(2)}, y_1], \dots, \Delta_{\rho_{r+l}}[x_{r+l}^{(2)}, y_{r+l}]) \end{aligned}$$

with

$$x_i^{(2)} = \begin{cases} -1 & \text{if } i \in A_\rho, \\ 1 & \text{if } i \in A_{\rho|\cdot|^1}, \\ x_i & \text{otherwise.} \end{cases}$$

In particular, we note that $\Delta_{\rho_i}[x_i^{(2)}, y_i] \cong \rho|\cdot|^1$ for $i > r$. Since $x_i^{(2)} \neq 0$ if $\rho_i \cong \rho$, we have

$$\mathrm{soc}\big(\rho^{r_0} \rtimes L(\Delta_{\rho_1}[x_1^{(2)}, y_1], \dots, \Delta_{\rho_{r+l}}[x_{r+l}^{(2)}, y_{r+l}])\big) = L\big(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_{r+l}}[x_{r+l}, y_{r+l}]\big).$$

Hence we obtain the claim.

By the claim, we have

$$\pi \hookrightarrow Z_\rho[0,1]^{k_A+r_1} \times L(\Delta_{\rho_1}[x_1', y_1], \dots, \Delta_{\rho_r}[x_r', y_r]) \rtimes D_{[0,1]}^{(k_A)}(\pi_A').$$

Moreover, by Tadić's formula (Proposition 2.1) together with the facts that

- $L(\Delta_{\rho_1}[x_1', y_1], \dots, \Delta_{\rho_r}[x_r', y_r])$ is left $\rho|\cdot|^1$-reduced;
- $L(\Delta_{\rho_1}[x_1', y_1], \dots, \Delta_{\rho_r}[x_r', y_r])$ is right ρ-reduced and right $\rho|\cdot|^{-1}$-reduced; and
- $D_{[0,1]}^{(k_A)}(\pi_A')$ is $Z_\rho[0,1]$-reduced and $\rho|\cdot|^1$-reduced

we see that $L(\Delta_{\rho_1}[x_1', y_1], \dots, \Delta_{\rho_r}[x_r', y_r]) \rtimes D_{[0,1]}^{(k_A)}(\pi_A')$ is $Z_\rho[0,1]$-reduced and $\rho|\cdot|^1$-reduced. Therefore, $D_{[0,1]}^{(k_A+r_1)}(\pi)$ is the highest $[0,1]$-derivative, and

$$D_{[0,1]}^{(k_A+r_1)}(\pi) \hookrightarrow L(\Delta_{\rho_1}[x_1', y_1], \dots, \Delta_{\rho_r}[x_r', y_r]) \rtimes D_{[0,1]}^{(k_A)}(\pi_A').$$

Since the induced representation in the right-hand side is a subrepresentation of a standard module, it is SI. In particular, $D_{[0,1]}^{(k_A+r_1)}(\pi)$ is the unique irreducible subrepresentation of this induced representation. □

We give now the converse of Theorem 8.1. Namely, when π is of good or ρ-bad parity and satisfies the assumptions (a) and (b), we will recover the Langlands data of π from those of $D_{[0,1]}^{(k)}(\pi)$.

Write $D_{[0,1]}^{(k)}(\pi) = L(\Delta_{\rho_1}[x_1', y_1], \dots, \Delta_{\rho_r}[x_r', y_r], (\rho|\cdot|^{-1})^s, \Delta_\rho[0,-1]^t; \pi(\phi', \eta'))$ as a Langlands subrepresentation, where

- $\phi' \in \Phi_{\mathrm{temp}}(G)$;
- $s, t \geq 0$;

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- $x_1' + y_1 \leq \cdots \leq x_r' + y_r < 0$;
- $\Delta_{\rho_i}[x_i', y_i] \not\cong \rho|\cdot|^{-1}, \Delta_\rho[0,-1]$ for $i = 1, \ldots, r$.

Set $\pi_A'' := L((\rho|\cdot|^{-1})^s, \Delta_\rho[0,-1]^t; \pi(\phi', \eta'))$. Define

$$B_{\rho|\cdot|^{-1}} := \{i \in \{1, \ldots, r\} \mid \rho_i \cong \rho,\, x_i' = -1\},$$
$$B_\rho := \{i \in \{1, \ldots, r\} \mid \rho_i \cong \rho,\, x_i' = 0\}$$

with the best matching function $f' \colon B^0_{\rho|\cdot|^{-1}} \to B^0_\rho$. By Theorem 8.1, we see that $x_i' \neq 1$ if $\rho_i \cong \rho$. Also, if we set $r_1 := |B_{\rho|\cdot|^{-1}}|$, $k_A := k - r_1$ and $l := r_1 - |B^0_\rho|$, then we have $k_A \geq 0$ and $l \geq 0$.

Corollary 8.2. *Let $\pi \in \mathrm{Irr}(G_n)$ be of good or ρ-bad parity and satisfy the assumptions (a) and (b). Then π is the unique irreducible subrepresentation of*

$$L(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r]) \rtimes \pi_A,$$

where

$$x_i = \begin{cases} 0 & \text{if } i \in B_{\rho|\cdot|^{-1}}, \\ 1 & \text{if } i \in B^0_\rho, \\ x_i' & \text{otherwise} \end{cases}$$

and

$$\pi_A := S^{(l)}_{\rho|\cdot|^1} \circ S^{(k_A)}_{[0,1]}(\pi_A'').$$

Proof. This follows from Theorem 8.1. $\square$

8.2 The representation π_A in the bad-parity case

We use the same notation as in the previous subsection. It remains to give an explicit formula for the highest $[0,1]$-derivative of π_A' and show how to recover the Langlands data of π_A' from those of its highest $[0,1]$-derivative.

We treat the bad-parity case first, which is much simpler. Recall that $\pi_A = L(\Delta_\rho[0,-1]^t; \pi(\phi, \eta))$ with $\phi \in \Phi_{\mathrm{temp}}(G)$. Let $\pi_A' := D^{(l)}_{\rho|\cdot|^1}(\pi_A)$ be the highest $\rho|\cdot|^1$-derivative of π_A. By Proposition 7.3, $\pi_A' = L(\Delta_\rho[0,-1]^{t-\kappa}; \pi(\phi', \eta'))$ with $\kappa \in \{0,1\}$, $t \equiv \kappa \bmod 2$ and $\phi' \in \Phi_{\mathrm{temp}}(G)$ which does not contain $\rho \boxtimes S_3$. In particular, $t - \kappa$ is even. Hence what we have to prove is the following.

Proposition 8.3. *Let $\pi = L(\Delta_\rho[0,-1]^t; \pi(\phi, \eta))$ be of ρ-bad parity with t even and $\phi \in \Phi_{\mathrm{temp}}(G)$ such that $\phi \not\supset \rho \boxtimes S_3$. Then the highest $[0,1]$-derivative of π is*

$$D^{(t)}_{[0,1]}(\pi) = \pi(\phi, \eta).$$

Proof. Write $m := m_\phi(\rho)$, which is even. Since

$$\begin{aligned} \pi &\hookrightarrow \rho^{t+m/2} \rtimes L((\rho|\cdot|^{-1})^t; \pi(\phi - \rho^m, \eta)) \\ &\cong \rho^{t+m/2} \times (\rho|\cdot|^{-1})^t \rtimes \pi(\phi - \rho^m, \eta) \\ &\cong \rho^{t+m/2} \times (\rho|\cdot|^{1})^t \rtimes \pi(\phi - \rho^m, \eta), \end{aligned}$$

we see that $D^{(t)}_{[0,1]}(\pi)$ is the highest $[0,1]$-derivative and

$$D^{(t)}_{[0,1]}(\pi) \hookrightarrow \rho^{m/2} \rtimes \pi(\phi - \rho^m, \eta) = \pi(\phi, \eta).$$

Since the right-hand side is irreducible, this inclusion is an isomorphism. $\square$

By this proposition, it is easy to recover π from its highest $[0,1]$-derivative.

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8.3 The representation π_A in the good-parity case

To finish our algorithm we need to consider the case where $\pi = L(\Delta_\rho[0,-1]^t; \pi(\phi,\eta))$ with $\phi \in \Phi_{\mathrm{gp}}(G)$ and $\eta \in \widehat{\mathcal{S}_\phi}$, and ρ is self-dual of the same type as ϕ. Furthermore, we assume that π is $\rho|\cdot|^1$-reduced, which is equivalent to the statement that if $\rho \boxtimes S_3 \subset \phi$, then $m_\phi(\rho) > 0$, $m_\phi(\rho \boxtimes S_3) = 1$ and $\eta(\rho)\eta(\rho \boxtimes S_3) \neq (-1)^t$. We determine the highest $[0,1]$-derivative of π.

Proposition 8.4. *Let $\pi = L(\Delta_\rho[0,-1]^t; \pi(\phi,\eta))$ with $\phi \in \Phi_{\mathrm{gp}}(G)$ and $\eta \in \widehat{\mathcal{S}_\phi}$. Suppose that ρ is self-dual of the same type as ϕ and that π is $\rho|\cdot|^1$-reduced. Write $m \coloneqq m_\phi(\rho)$.*

(1) *If $\rho \boxtimes S_3 \subset \phi$ and m is odd, then the highest $[0,1]$-derivative of π is*

$$D^{(t)}_{[0,1]}(\pi) = \begin{cases} \pi(\phi,\eta) & \text{if } t \equiv 0 \bmod 2, \\ L(\rho|\cdot|^{-1}; \pi(\phi + \rho - \rho \boxtimes S_3, \eta)) & \text{if } t \equiv 1 \bmod 2. \end{cases}$$

(2) *If $\rho \boxtimes S_3 \subset \phi$ and m is even, then the highest $[0,1]$-derivative of π is*

$$D^{(t+1)}_{[0,1]}(\pi) = \pi(\phi - \rho \boxtimes (S_1 + S_3), \eta_{t+1}).$$

(3) *If $\rho \boxtimes S_3 \not\subset \phi$ and m is odd, then the highest $[0,1]$-derivative of π is*

$$\begin{cases} D^{(0)}_{[0,1]}(\pi) = \pi(\phi,\eta) & \text{if } t = 0, \\ D^{(t-1)}_{[0,1]}(\pi) = L(\rho|\cdot|^{-1}; \pi(\phi + \rho^2, \eta)) & \text{if } t > 0,\ t \equiv 0 \bmod 2, \\ D^{(t-1)}_{[0,1]}(\pi) = L(\Delta_\rho[0,-1]; \pi(\phi,\eta)) & \text{if } t > 0,\ t \equiv 1 \bmod 2. \end{cases}$$

(4) *If $\rho \boxtimes S_3 \not\subset \phi$ and m is even, then the highest $[0,1]$-derivative of π is*

$$D^{(t)}_{[0,1]}(\pi) = \pi(\phi, \eta_t).$$

Here, in (2) and (4) we set

$$\eta_t(\rho' \boxtimes S_d) = \begin{cases} (-1)^t \eta(\rho) & \text{if } \rho' \boxtimes S_d \cong \rho, \\ \eta(\rho' \boxtimes S_d) & \text{otherwise.} \end{cases}$$

Proof. We note that $\pi \hookrightarrow \rho^{t+u} \times L((\rho|\cdot|^{-1})^t; \pi(\phi - \rho^{2u}, \eta))$ in all cases, where $m = 2u+1$ or $m = 2u$. We will apply Theorem 7.1 to $L((\rho|\cdot|^{-1})^t; \pi(\phi - \rho^{2u}, \eta))$ and $x = 1$ in each case.

To show (1), write $m = 2u+1$. By Theorem 7.1, we have

$$\pi \hookrightarrow \rho^{t+u} \times (\rho|\cdot|^1)^t \rtimes \begin{cases} \pi(\phi - \rho^{2u}, \eta) & \text{if } t \equiv 0 \bmod 2, \\ L(\rho|\cdot|^{-1}; \pi(\phi - \rho^{2u-1} - \rho \boxtimes S_3, \eta)) & \text{if } t \equiv 1 \bmod 2. \end{cases}$$

Note that $\rho^u \rtimes \pi(\phi - \rho^{2u}, \eta) = \pi(\phi,\eta)$ and $\rho^u \rtimes L(\rho|\cdot|^{-1}; \pi(\phi - \rho^{2u-1} - \rho \boxtimes S_3, \eta)) = L(\rho|\cdot|^{-1}; \pi(\phi + \rho - \rho \boxtimes S_3, \eta))$ are both irreducible by [Art13, Proposition 2.4.3] and Mœglin's construction (see [Xu17a, § 8]). Hence

$$\pi \hookrightarrow Z_\rho[0,1]^t \rtimes \begin{cases} \pi(\phi,\eta) & \text{if } t \equiv 0 \bmod 2, \\ L(\rho|\cdot|^{-1}; \pi(\phi + \rho - \rho \boxtimes S_3, \eta)) & \text{if } t \equiv 1 \bmod 2. \end{cases}$$

This shows (1).

To show (2), write $m = 2u$. Note that $u > 0$ and $\eta(\rho \boxtimes S_3) = (-1)^{t+1}\eta(\rho)$. Hence

$$\pi \hookrightarrow \rho^{t+u} \times (\rho|\cdot|^1)^{t+1} \rtimes \pi(\phi - \rho^{2u-1} - \rho \boxtimes S_3, \eta_{t+1}).$$

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This implies that

$$\begin{aligned}\pi &\hookrightarrow Z_\rho[0,1]^{t+1} \times \rho^{u-1} \rtimes \pi(\phi - \rho^{2u-1} - \rho \boxtimes S_3, \eta_{t+1}) \\ &= Z_\rho[0,1]^{t+1} \rtimes \pi(\phi - \rho - \rho \boxtimes S_3, \eta_{t+1}),\end{aligned}$$

which shows (2).

To show (3), note that when $t = 0$, it is clear that π is $Z_\rho[0,1]$-reduced (Lemma 3.5). Suppose that $t > 0$. Write $m = 2u + 1$. Since

$$\pi \hookrightarrow \rho^{t+u} \times (\rho|\cdot|^1)^{t-1} \rtimes L\big(\rho|\cdot|^{-1}; \pi(\phi - \rho^{2u}, \eta)\big),$$

we have

$$\pi \hookrightarrow Z_\rho[0,1]^{t-1} \times \rho^{u+1} \rtimes L\big(\rho|\cdot|^{-1}; \pi(\phi - \rho^{2u}, \eta)\big).$$

By [Art13, Proposition 2.4.3] and Mœglin's construction (see [Xu17a, § 8]), we have

$$\rho^{u+1} \rtimes L\big(\rho|\cdot|^{-1}; \pi(\phi - \rho^{2u}, \eta)\big) = L\big(\rho|\cdot|^{-1}; \pi(\phi + \rho^2, \eta)\big) \oplus L\big(\Delta_\rho[0,-1]; \pi(\phi, \eta)\big).$$

In particular, $D^{(t-1)}_{[0,1]}(\pi)$ is the highest $[0,1]$-derivative and is isomorphic to one of the two direct summands in the right-hand side. Now we note that $L(\Delta_\rho[0,-1], \Delta_\rho[1,0]) \cong \mathrm{soc}(Z_\rho[0,1] \times Z_\rho[-1,0])$. When t is odd, by [Art13, Proposition 2.4.3] we have

$$\pi \hookrightarrow L\big(\Delta_\rho[0,-1], \Delta_\rho[1,0]\big)^{(t-1)/2} \rtimes L\big(\Delta_\rho[0,-1]; \pi(\phi, \eta)\big).$$

Since $L(\Delta_\rho[0,-1]; \pi(\phi,\eta))$ is $\rho|\cdot|^1$-reduced and $Z_\rho[0,1]$-reduced, by considering Tadić's formula (Proposition 2.1) we see that

$$D^{(t-1)}_{[0,1]}\big(L(\Delta_\rho[0,-1], \Delta_\rho[1,0])^{(t-1)/2} \rtimes L(\Delta_\rho[0,-1]; \pi(\phi,\eta))\big) = L\big(\Delta_\rho[0,-1]; \pi(\phi,\eta)\big),$$

which implies that $D^{(t-1)}_{[0,1]}(\pi) = L(\Delta_\rho[0,-1]; \pi(\phi,\eta))$. When $t = 2$, by [Art13, Proposition 2.4.3], we have

$$\begin{aligned}\pi &\hookrightarrow L(\Delta_\rho[0,-1], \Delta_\rho[1,0]) \rtimes \pi(\phi,\eta) \\ &\cong \mathrm{soc}(Z_\rho[0,1] \times Z_\rho[-1,0]) \rtimes \pi(\phi,\eta) \\ &\hookrightarrow Z_\rho[0,1] \times \rho|\cdot|^{-1} \rtimes \pi(\phi + \rho^2, \eta),\end{aligned}$$

which implies that $D^{(1)}_{[0,1]}(\pi) = L(\rho|\cdot|^{-1}; \pi(\phi+\rho^2,\eta))$. When $t > 2$ is even, we have

$$\begin{aligned}\pi &\hookrightarrow L(\Delta_\rho[0,-1], \Delta_\rho[1,0])^{(t-2)/2} \rtimes L(\Delta_\rho[0,-1]^2; \pi(\phi,\eta)) \\ &\hookrightarrow Z_\rho[0,1] \times L(\Delta_\rho[0,-1], \Delta_\rho[1,0])^{(t-2)/2} \rtimes L\big(\rho|\cdot|^{-1}; \pi(\phi+\rho^2,\eta)\big).\end{aligned}$$

Here, we note that $Z_\rho[0,1] \times L(\Delta_\rho[0,-1], \Delta_\rho[1,0])$ is irreducible by [Tad14, Theorem 1.1]. Since $L(\rho|\cdot|^{-1}; \pi(\phi+\rho^2,\eta))$ is $\rho|\cdot|^1$-reduced and $Z_\rho[0,1]$-reduced, by considering Tadić's formula (Proposition 2.1) we see that

$$\begin{aligned}&D^{(t-1)}_{[0,1]}\big(Z_\rho[0,1] \times L(\Delta_\rho[0,-1], \Delta_\rho[1,0])^{(t-2)/2} \rtimes L(\rho|\cdot|^{-1}; \pi(\phi+\rho^2,\eta))\big) \\ &= L\big(\rho|\cdot|^{-1}; \pi(\phi+\rho^2,\eta)\big),\end{aligned}$$

which implies that $D^{(t-1)}_{[0,1]}(\pi) = L(\rho|\cdot|^{-1}; \pi(\phi+\rho^2,\eta))$. Thus we obtain (3).

To show (4), write $m = 2u$. Since

$$\pi \hookrightarrow \rho^{t+u} \times (\rho|\cdot|^1)^t \rtimes \pi(\phi - \rho^{2u}, \eta),$$

we have

$$\pi \hookrightarrow Z_\rho[0,1]^t \times \rho^u \rtimes \pi(\phi - \rho^{2u}, \eta).$$

In particular, this shows (4) when $u = 0$. Hereafter we assume that $u > 0$. Then

$$\rho^u \rtimes \pi(\phi - \rho^{2u}, \eta) = \pi(\phi, \eta_t) \oplus \pi(\phi, \eta_{t+1}).$$

To show $\pi \hookrightarrow Z_\rho[0,1]^t \rtimes \pi(\phi, \eta_t)$, we use an argument inspired by Mœglin's construction of A-packets.

Write $\phi = \rho^m \oplus (\bigoplus_{i=1}^r \rho_i \boxtimes S_{d_i})$ with $d_1 \leq \cdots \leq d_r$ and $d_i > 3$ if $\rho_i \cong \rho$. Choose $\phi_> = (\bigoplus_{j=1}^m \rho \boxtimes S_{2x_j+1}) \oplus (\bigoplus_{i=1}^r \rho_i \boxtimes S_{d'_i})$ such that

- $x_j \in \mathbb{Z}$ with $x_j > 1$;
- $d'_i \equiv d_i \bmod 2$ with $d'_i \geq d_i$;
- $2x_1 + 1 < \cdots < 2x_m + 1 < d'_1 < \cdots < d'_r$.

Define $\eta_> \in \widehat{\mathcal{S}_{\phi_>}}$ by $\eta_>(\rho \boxtimes S_{2x_j+1}) = (-1)^t \eta(\rho)$ and $\eta_>(\rho_i \boxtimes S_{d'_i}) = \eta(\rho_i \boxtimes S_{d_i})$. Then $\pi(\phi, \eta_t) = J_2 \circ J_1(\pi(\phi_>, \eta_>))$ with

$$J_1 = \mathrm{Jac}_{\rho|\cdot|^{x_m}, \dots, \rho|\cdot|^1} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{x_1}, \dots, \rho|\cdot|^1},$$

$$J_2 = \mathrm{Jac}_{\rho_t|\cdot|^{(d'_r-1)/2}, \dots, \rho_t|\cdot|^{(d_r+1)/2}} \circ \cdots \circ \mathrm{Jac}_{\rho_1|\cdot|^{(d'_1-1)/2}, \dots, \rho_1|\cdot|^{(d_1+1)/2}},$$

where we set $\mathrm{Jac}_{\rho|\cdot|^x, \dots, \rho|\cdot|^y} = D^{(1)}_{\rho|\cdot|^y} \circ \cdots \circ D^{(1)}_{\rho|\cdot|^x}$. Since $\phi_>$ contains neither ρ nor $\rho \boxtimes S_3$, by the argument in the previous paragraph we have

$$\mathrm{soc}(Z_\rho[0,1]^t \rtimes \pi(\phi_>, \eta_>)) = L(\Delta_\rho[0,-1]^t; \pi(\phi_>, \eta_>)).$$

By Theorem 7.1, using the assumption that $m \equiv 0 \bmod 2$, we see that

$$J_2 \circ J_1(L(\Delta_\rho[0,-1]^t; \pi(\phi_>, \eta_>))) = L(\Delta_\rho[0,-1]^t; \pi(\phi, \eta)) = \pi.$$

On the other hand, since

$$\pi(\phi_>, \eta_>) \hookrightarrow \Delta_\rho[x_1, 1] \times \cdots \times \Delta_\rho[x_m, 1] \rtimes J_1(\pi(\phi_>, \eta_>))$$

by [Xu17b, Lemma 5.7], and since $Z_\rho[0,1] \times \Delta_\rho[x,1] \cong \Delta_\rho[x,1] \times Z_\rho[0,1]$ if $x \geq 1$, we see that

$$\begin{aligned}
&J_2 \circ J_1(\mathrm{soc}(Z_\rho[0,1]^t \rtimes \pi(\phi_>, \eta_>))) \\
&\quad \hookrightarrow J_2 \circ J_1(Z_\rho[0,1]^t \rtimes \pi(\phi_>, \eta_>)) \\
&\quad \hookrightarrow J_2 \circ J_1(\Delta_\rho[x_1,1] \times \cdots \times \Delta_\rho[x_m,1] \times Z_\rho[0,1]^t \rtimes J_1(\pi(\phi_>, \eta_>))) \\
&\quad = J_2(Z_\rho[0,1]^t \rtimes J_1(\pi(\phi_>, \eta_>))).
\end{aligned}$$

Finally, since $(d_i + 1)/2 > 2$ if $\rho_i \cong \rho$, we have

$$J_2\big(Z_\rho[0,1]^t \rtimes J_1(\pi(\phi_>, \eta_>))\big) = Z_\rho[0,1]^t \rtimes J_2 \circ J_1\big(\pi(\phi_>, \eta_>)\big) = Z_\rho[0,1]^t \rtimes \pi(\phi, \eta_t).$$

Therefore we conclude that $\pi \hookrightarrow Z_\rho[0,1]^t \rtimes \pi(\phi, \eta_t)$. This completes the proof of (4). □

Finally, we state the converse of Proposition 8.4 in terms of A-parameters.

Corollary 8.5. *Let $\pi = L(\Delta_\rho[0,1]^t; \pi(\phi, \eta))$ be the same as in Proposition 8.4, and let $D^{(k)}_{[0,1]}(\pi)$ be the highest $[0,1]$-derivative of π. Suppose that $k > 0$. Then one can* *write* $D^{(k)}_{[0,1]}(\pi) = L((\rho|\cdot|^{-1})^{s'}, \Delta_\rho[0,1]^{t'}; \pi(\phi', \eta'))$ *with* $s' + t' + m_{\phi'}(\rho \boxtimes S_3) \leq 1$. *Moreover, with* $m' := m_{\phi'}(\rho)$, *the following hold.*

(1) *If* $s' = 1$, *then* $m' \geq 2$, $k \equiv 1 \bmod 2$ *and*

$$\pi = \pi(\phi' - \rho^2 + (\rho \boxtimes S_2 \boxtimes S_2)^{k+1}, m', \eta').$$

(2) *If* $t' = 1$, *then* $m' \equiv 1 \bmod 2$, $k \equiv 0 \bmod 2$ *and*

$$\pi = \pi(\phi' + (\rho \boxtimes S_2 \boxtimes S_2)^{k+1}, 1, \eta').$$

(3) *If* $m_{\phi'}(\rho \boxtimes S_3) = 1$, *then* $m' \equiv 1 \bmod 2$, $k \equiv 0 \bmod 2$ *and*

$$\pi = \pi(\phi' + (\rho \boxtimes S_2 \boxtimes S_2)^{k}, 1, \eta').$$

(4) *If* $s' + t' + m_{\phi'}(\rho \boxtimes S_3) = 0$, *then*

$$\pi = \pi(\phi' + (\rho \boxtimes S_2 \boxtimes S_2)^{k}, m'+1, \eta'_k),$$

where $\eta'_k(\rho) = (-1)^k \eta'(\rho)$.

Proof. This follows from Proposition 8.4. □

9. Some examples of Zelevinsky–Aubert duality

By the results in the previous sections, we have completed Algorithm 4.1 for computing the Zelevinsky–Aubert duality. In this section, we give some examples. Here we set $\rho := \mathbf{1}_{\mathrm{GL}_1(F)}$ and drop ρ from the notation. For example, we write $\Delta[x,y] := \Delta_\rho[x,y]$ and $Z[y,x] := Z_\rho[y,x]$. When $\phi = \bigoplus_{i=1}^r S_{d_i} \in \Phi_{\mathrm{gp}}(G)$ and $\eta(S_{d_i}) = \eta_i \in \{\pm 1\}$, we write $\pi(\phi,\eta) = \pi(d_1^{\eta_1}, \dots, d_r^{\eta_r})$.

9.1 Example 1

Let us compute the Zelevinsky–Aubert dual of

$$L(\Delta[0,-2], \Delta[0,-1]; \pi(3^+)) \in \mathrm{Irr}(\mathrm{Sp}_{12}(F)).$$

Note that it is of good parity, and it is $|\cdot|^z$-reduced for $z \neq 0$ by Theorem 7.1. By Algorithm 4.1, we have the following commutative diagram.

$$\begin{tikzcd}
L(\Delta[0,-2], \Delta[0,-1]; \pi(3^+)) \arrow[r, mapsto, "\pi \mapsto \hat{\pi}"] \arrow[d, mapsto, "D^{(2)}_{\Delta[0,-1]}"'] & L(\Delta[0,-2], \Delta[0,-1]; \pi(3^+)) \\
L(|\cdot|^{-2}; \pi(3^+)) \arrow[r, mapsto, "\pi \mapsto \hat{\pi}"] \arrow[d, mapsto, "D^{(1)}_{|\cdot|^{-2}}"'] & L(\Delta[-1,-2]; \pi(1^+)) \arrow[u, mapsto, "S^{(2)}_{Z[0,1]}"'] \\
\pi(3^+) \arrow[r, mapsto, "\pi \mapsto \hat{\pi}"] \arrow[d, mapsto, "D^{(1)}_{|\cdot|^{1}}"'] & L(|\cdot|^{-1}; \pi(1^+)) \arrow[u, mapsto, "S^{(1)}_{|\cdot|^{2}}"'] \\
\pi(1^+) \arrow[r, mapsto, "\pi \mapsto \hat{\pi}"] & \pi(1^+) \arrow[u, mapsto, "S^{(1)}_{|\cdot|^{-1}}"']
\end{tikzcd}$$

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For the computation of $S^{(2)}_{Z[0,1]}$, by Corollaries 8.2 and 8.5 and Theorem 5.3, we have

$$\begin{aligned} S^{(2)}_{Z[0,1]}\big(L(\Delta[-1,-2];\pi(1^+))\big) &= \mathrm{soc}\big(\Delta[0,-2] \rtimes S^{(1)}_{|\cdot|^1} \circ S^{(1)}_{Z[0,1]}(\pi(1^+))\big) \\ &= \mathrm{soc}\big(\Delta[0,-2] \rtimes S^{(1)}_{|\cdot|^1}(\pi(1^-,1^-,3^+))\big) \\ &= L\big(\Delta[0,-2],\Delta[0,-1];\pi(3^+)\big). \end{aligned}$$

In conclusion, we see that $L(\Delta[0,-2],\Delta[0,-1];\pi(3^+))$ is fixed by the Zelevinsky–Aubert duality.

9.2 Example 2

Next, let us compute the Zelevinsky–Aubert dual of

$$\pi(1^\epsilon,1^\epsilon,3^+,5^-,5^-) \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{Sp}_{14}(F))$$

for $\epsilon \in \{\pm\}$. First, we compute derivatives as follows.

$$\begin{array}{cc}
\pi(1^+,1^+,3^+,5^-,5^-) & \pi(1^-,1^-,3^+,5^-,5^-) \\
\downarrow D^{(1)}_{|\cdot|^2} & \downarrow D^{(1)}_{|\cdot|^2} \\
L(\Delta[1,-2];\pi(1^+,1^+,3^+)) & L(\Delta[1,-2];\pi(1^-,1^-,3^+)) \\
\downarrow D^{(2)}_{|\cdot|^1} & \downarrow D^{(1)}_{|\cdot|^1} \\
L(\Delta[0,-2];\pi(1^+,1^+,1^+)) & L(\Delta[0,-2];\pi(1^-,1^-,3^+)) \\
\downarrow D^{(1)}_{|\cdot|^2} & \downarrow D^{(1)}_{\Delta[0,-1]} \\
L(\Delta[0,-1];\pi(1^+,1^+,1^+)) & L(|\cdot|^{-2};\pi(1^-,1^-,3^+)) \\
\downarrow D^{(1)}_{\Delta[0,-1]} & \downarrow D^{(1)}_{|\cdot|^{-2}} \\
\pi(1^+,1^+,1^+) & \pi(1^-,1^-,3^+)
\end{array}$$

By Proposition 5.4, we have $\hat{\pi}(1^+,1^+,1^+) = \pi(1^+,1^+,1^+)$ and $\hat{\pi}(1^-,1^-,3^+) = L(\Delta[0,-1];\pi(1^+))$. Next we compute socles as follows.

$$\begin{array}{cc}
\pi(1^+,1^+,1^+) & L(\Delta[0,-1];\pi(1^+)) \\
\downarrow S^{(1)}_{Z[0,1]} & \downarrow S^{(1)}_{|\cdot|^2} \\
\pi(1^-,1^-,1^-,1^-,3^+) & L(\Delta[0,-2];\pi(1^+)) \\
\downarrow S^{(1)}_{|\cdot|^{-2}} & \downarrow S^{(1)}_{Z[0,1]} \\
L(|\cdot|^{-2};\pi(1^-,1^-,1^-,1^-,3^+)) & L(\Delta[0,-2];\pi(1^-,1^-,3^+)) \\
\downarrow S^{(2)}_{|\cdot|^{-1}} & \downarrow S^{(1)}_{|\cdot|^{-1}} \\
L(\Delta[-1,-2],|\cdot|^{-1};\pi(1^-,1^-,1^-,1^-,3^+)) & L(|\cdot|^{-1},\Delta[0,-2];\pi(1^-,1^-,3^+)) \\
\downarrow S^{(1)}_{|\cdot|^{-2}} & \downarrow S^{(1)}_{|\cdot|^{-2}} \\
L(|\cdot|^{-2},\Delta[-1,-2],|\cdot|^{-1};\pi(1^-,1^-,1^-,1^-,3^+)) & L(|\cdot|^{-2},|\cdot|^{-1},\Delta[0,-2];\pi(1^-,1^-,3^+))
\end{array}$$

https://doi.org/10.1112/S0010437X22007904 Published online by Cambridge University Press

Therefore, we conclude that

$$\hat{\pi}(1^+, 1^+, 3^+, 5^-, 5^-) = L\big(|\cdot|^{-2}, \Delta[-1,-2], |\cdot|^{-1}; \pi(1^-, 1^-, 1^-, 1^-, 3^+)\big),$$
$$\hat{\pi}(1^-, 1^-, 3^+, 5^-, 5^-) = L\big(|\cdot|^{-2}, |\cdot|^{-1}, \Delta[0,-2]; \pi(1^-, 1^-, 3^+)\big).$$

Similarly, one can prove that $\hat{\pi}(3^+, 5^-, 5^-) = L(|\cdot|^{-2}, \Delta[-1,-2], |\cdot|^{-1}; \pi(1^-, 1^-, 3^+))$. Hence we see that

$$\begin{aligned}\mathbf{1}_{\mathrm{GL}_1(F)} \rtimes L\big(|\cdot|^{-2}, \Delta[-1,-2], |\cdot|^{-1}; \pi(1^-, 1^-, 3^+)\big) \\ \cong L\big(|\cdot|^{-2}, \Delta[-1,-2], |\cdot|^{-1}; \pi(1^-, 1^-, 1^-, 1^-, 3^+)\big) \\ \oplus L\big(|\cdot|^{-2}, |\cdot|^{-1}, \Delta[0,-2]; \pi(1^-, 1^-, 3^+)\big).\end{aligned}$$

In these computations we also proved, for example, that $L(\Delta[0,-2]; \pi(1^-, 1^-, 3^+))$ is fixed by the Zelevinsky–Aubert duality. This fact does not follow from results in [Ato22b]. As in this example, even if π is tempered, we need to compute $S^{(k)}_{Z[0,1]}$ in general.

Acknowledgements

We would like to thank E. Lapid and C. Mœglin for useful discussions. The first author was supported by JSPS KAKENHI grant no. 19K14494.

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Hiraku Atobe atobe@math.sci.hokudai.ac.jp
Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido 060-0810, Japan

Alberto Mínguez alberto.minguez@univie.ac.at
Fakultät für Mathematik, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria