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A Note on Randers Metrics of Scalar Flag Curvature

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Abstract. Some families of Randers metrics of scalar flag curvature are studied in this paper. Explicit examples that are neither locally projectively flat nor of isotropic S-curvature are given. Certain Randers metrics with Einstein α are considered and proved to be complex. Three dimensional Randers manifolds, with α having constant scalar curvature, are studied.

1 Introduction

The flag curvature is a natural extension of the Riemannian sectional curvature to Finsler manifolds. The flag curvature on a Finsler manifold (M, F) is a function $K = K(x, y, P)$ depending on a base point $x \in M$, a two plane (flag) $P \subset T_xM$, and a flagpole $y \in P\backslash\{0\}$. The metric F is said to be of *scalar flag curvature* if $K = K(x, y)$ is independent of P containing $y \in T_xM$. It is well known that a Riemannian metric is of scalar flag curvature if and only if it is of isotropic sectional curvature, and the Schur lemma ensures the constancy when the dimension is at least 3. There are many non-Riemannian Finsler metrics of scalar flag curvature whose flag curvature are not necessarily isotropic. The study of Finsler metrics of scalar flag curvature is an important project. Since each Finsler surface is of scalar flag curvature, all the manifolds considered in this paper have dimensions *greater than two*.

A Randers metric on a manifold is a Finsler metric in the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on the manifold with $\|\beta\|_\alpha < 1$. Through a group of works, the classification of Randers metrics of constant flag curvature was finished by the navigation description [1]. Recently, the authors also studied Randers metrics of sectional flag curvature and proved that they are in fact of constant flag curvature ([5]). The importance of Randers metrics of scalar flag curvature can be recognized by [6], which proves that any negatively curved Finsler metric of scalar flag curvature must be of Randers type. However, the classification of Randers metrics of scalar flag curvature has not been done. Fortunately, by studying the projective Weyl curvature, Shen and Yildrim ([7]) found a system of PDEs that are sufficient and necessary for a Randers metric to be of scalar flag curvature. They also completely determined Randers metrics of weakly isotropic flag curvature by using S-curvature, which plays a special role in the classification of Randers metrics of constant flag curvature.

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Locally projectively flat Finsler metrics form an important class of metrics of scalar flag curvature. It is known that a Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is locally projectively flat and β is closed. There are Randers metrics of scalar flag curvature that are not locally projectively flat, but these examples are all of constant S-curvature. There has been no explicit example of a Randers metric of scalar flag curvature that is not locally projectively flat and not of isotropic S-curvature, until we find the following in this paper.

Theorem 1.1 *Let α be the Bergman metric on $D = \{x \in \mathbb{R}^{2n} = \mathbb{C}^n : |x| < 1\}$ with constant holomorphic curvature -4, $f = \frac{1}{2}\ln(1 - |x|^2)$ the potential of α, and J the complex structure. Then for any nonzero number ϵ, the metric*

$$F(x, y) = \alpha(y) + df(\epsilon y - Jy)$$

is a Randers metric of scalar flag curvature that is not locally projectively flat and not of isotropic S-curvature.

In fact, we also have an abstract existence result of such metrics (see Lemma 2.6), which indicates that the class of Randers metrics of scalar flag curvature is much larger. It is interesting that α is Einstein in the above example. The following result tells us it will never happen in the case of odd dimensions.

Theorem 1.2 *Let $(M^n, \alpha + \beta)$ be a Randers space of scalar flag curvature. If α is Einstein, then either $\alpha+\beta$ is locally projectively flat, or α is a Kähler metric with negative constant holomorphic sectional curvature.*

A more precise statement is given in Section 4, which classifies the case of Einstein α. A more general condition is that α has constant scalar curvature. Bao and Shen's metric on the Lie group $\mathbb{S}^3$ is such an example. In [3], Bejancu and Farran classified certain Randers metrics by Sasakian structures. Motivated by their work, we have the following theorem.

Theorem 1.3 *Let $(\mathbb{S}^3, \alpha + \beta)$ be a Randers sphere of scalar flag curvature. If α has constant scalar curvature, then $\alpha + \beta$ is either locally projectively flat or projectively equivalent to a Randers sphere form.*

For a complete list, please see Theorem 5.2. Since projective equivalence is a closed relation among the Finsler metrics of scalar curvature, the above theorem is a rigid type result. We also remark here that projectively related Randers metrics have been studied in [8].

2 Motivations

Randers spaces are Finsler spaces constructed from just two pieces of familiar data: a Riemannian metric and a differential 1-form, both globally defined on an underlying smooth manifold. They were introduced by Randers in 1941 in the context of general relativity and play a prominent role in Ingarden's study of electron optics.

Let $F=\alpha+\beta=\sqrt{a_{ik}(x)y^iy^k}+b_i(x)y^i$ be a Randers metric, where $a=a_{ik}dx^i\otimes dx^k$ is a Riemannian metric and $b=b_idx^i$ is a 1-form. The covariant differential of b with respect to a is denoted by $\nabla b=b_{i|k}dx^i\otimes dx^k$. Let

$$r_{ik}=\tfrac{1}{2}(b_{i|k}+b_{k|i}),\quad s_{ik}=\tfrac{1}{2}(b_{i|k}-b_{k|i})$$

be the Lie derivative and the exterior differential of b respectively. Moreover, set

$$s_k=b^is_{ik},\quad t_{ik}=s_{im}s^m_{\ k},\quad t_k=b^it_{ik}.$$

Here and from now on we use the Riemannian metric a to raise and lower the indices, *e.g.*, $s^m_{\ k}=a^{mi}s_{ik}$, *etc.*. The $(2,0)$-Riemann curvature tensor of α is defined by ${}^aR_{ik}={}^aR_{jikl}y^jy^l$, where ${}^aR_{jikl}$ is the $(4,0)$-curvature tensor. In [7], a characterization of Randers metrics to be of scalar flag curvature is given.

Theorem 2.1 *Let $F=\alpha+\beta$ be a Randers metric on an n-dimensional manifold M. F is of scalar flag curvature if and only if the Riemann curvature of α and the 1-form β satisfy*

(2.1)
$$^aR_{ik}=\Big(\lambda(x)-\frac{1}{n-1}t^m_{\ m}\Big)(\alpha^2a_{ik}-y_iy_k)+\alpha^2t_{ik}+t_{00}a_{ik}-t_{k0}y_i-t_{i0}y_k-3s_{i0}s_{k0},$$

(2.2)
$$s_{ij|k}=\frac{1}{n-1}\big(a_{ik}s^m_{\ j|m}-a_{jk}s^m_{\ i|m}\big),$$

where $\lambda(x)$ is a function on M, and the index "0" *means the contraction with y^i*, e.g., $t_{k0}=t_{ki}y^i$.

Particularly, the Ricci curvature of α satisfies

$$^a\mathrm{Ric}_{ik}=(n-1)\lambda a_{ik}+(n+1)t_{ik}. \tag{2.3}$$

Moreover, the formula of the flag curvature of such a Randers metric is also obtained in [7].

Our goal is to produce new examples that are not projectively flat or of constant S-curvature. A natural idea is to disturb a projectively flat Randers metric, but the following proposition rejects this idea.

Proposition 2.2 *Let $n\geq 3$ and $F=\alpha+\beta$ be a Randers metric, then any two of the following conditions imply the other one:*

(i) *F is of scalar flag curvature;*
(ii) *α has constant sectional curvature;*
(iii) *β is closed.*

Hence, any two of them imply F is projectively flat.

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Proof We will only prove that (i) and (ii) imply (iii). Assume ${}^aR_{ik} = \mu(\alpha^2 a_{ik} - y_i y_k)$, then (2.3) implies

$$0 = (n-1)(\lambda - \mu)a_{ik} + (n+1)t_{ik}. \tag{2.4}$$

Substituting (2.4) back into (2.1), we get

$$s_{i0}s_{k0} = \frac{\lambda - \mu}{n+1}(\alpha^2 a_{ik} - y_i y_k).$$

Hence,

$$s_0 s_0 = \frac{\lambda - \mu}{n+1}(\|b\|^2 \alpha - \beta^2).$$

Since $n \geq 3$, we can choose y such that $s_0 = 0 = \beta$. Then one can get $\lambda = \mu$, $t_{ik} = 0$, $s_{ik}s^{ik} = 0$. ■

This proposition force us to relax the curvature restriction on α in order to obtain new examples. For instance, we may assume α has constant scalar curvature. Then the relation between $d\beta$ and the scalar curvature R_a should be studied.

Lemma 2.3 *Let $F = \alpha + \beta$ be a Randers metric of scalar flag curvature. Then*

$$-\nabla(\text{trace } t) = \nabla \|s\|^2 = -\frac{4}{n}\text{div}(t), \tag{2.5}$$

where $s = s_{ik}dx^i \otimes dx^k, t = t_{ik}dx^i \otimes dx^k$.

Proof Applying (2.2), one can easily have

$$\begin{aligned} \nabla_k \|s\|^2 = 2s^{ij}s_{ij|k} &= \frac{2}{n-1}s^{ij}\left(a_{ik}s^m{}_{j|m} - a_{jk}s^m{}_{i|m}\right) \\ &= \frac{2}{n-1}\left(s_k{}^j s^m{}_{j|m} - s^i{}_k s^m{}_{i|m}\right) = -\frac{4}{n-1}s_{kj}s^{jm}{}_{|m}. \end{aligned} \tag{2.6}$$

On the other hand,

$$\begin{aligned} s_{kj}s^{jm}{}_{|m} &= (s_{kj}s^{jm})_{|m} - s_{kj|m}s^{jm} = t_k{}^m{}_{|m} - \frac{1}{n-1}\left(a_{km}s^i{}_{j|i} - a_{jm}s^i{}_{k|i}\right)s^{jm} \\ &= t_k{}^m{}_{|m} - \frac{1}{n-1}s^m{}_{j|m}s^j{}_k, \end{aligned}$$

which means

$$t_k{}^m{}_{|m} = \frac{n}{n-1}s_{kj}s^{jm}{}_{|m}. \tag{2.7}$$

Then (2.6) and (2.7) imply (2.5). ■

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Lemma 2.4 *Let $n \geq 3$ and $F = \alpha + \beta$ be a Randers metric of scalar flag curvature. Then there is a constant C, such that*

$$\lambda = \frac{R_a + C}{(n-1)(n+2)}, \quad \|s\|^2 = \frac{nC - 2R_a}{(n+1)(n+2)},$$

where R_a is the scalar curvature of a.

Proof Recalling the Ricci equation (2.3), the second Bianchi identity tells us

$$\tfrac{1}{2}\nabla R_a = \operatorname{div}({}^a\operatorname{Ric}) = (n-1)\nabla\lambda + (n+1)\operatorname{div}(t). \tag{2.8}$$

On the other hand, the trace of (2.3) gives

$$R_a = n(n-1)\lambda - (n+1)\|s\|^2,$$

so

$$\nabla R_a = n(n-1)\nabla\lambda - (n+1)\nabla\|s\|^2. \tag{2.9}$$

Using (2.5), (2.8), and (2.9), we reach

$$\nabla R_a = (n-1)(n+2)\nabla\lambda = -\tfrac{1}{2}(n+1)(n+2)\nabla\|s\|^2,$$

which proves the lemma. ■

Corollary 2.5 *Let $n \geq 3$ and $F = \alpha + \beta$ be a Randers metric of scalar flag curvature. If $R_a = \text{const}$, then either $d\beta \equiv 0$ or $d\beta \neq 0$.*

This corollary implies that many cases must be trivial. In the next section, by using Kähler metrics, a non-trivial example will be presented.

As an end to this section, let us give another way to produce Randers metrics that are of scalar curvature, but neither projectively flat or of isotropic S-curvature. It is clear that $F_t = \alpha + \beta + tdf(y)$ is of scalar flag curvature if and only if F_0 is, where f is an arbitrary function on M. If F_0 is not projectively flat in addition, then so is each F_t. We try to find suitable data such that F_t is not of isotropic S-curvature.

Lemma 2.6 *Let (M, α) be a Riemannian metric, and f be a function on M with $df \neq 0$ almost everywhere, and $\Delta f \neq 0$ at some point. Suppose*

$$F_t = \alpha + \beta_t = \alpha + \beta + tdf(y)$$

are regular Randers metrics for $t \in (-1, 1)$, then the subset

$$T_S = \{t : F_t \text{ does not have isotropic } S\text{-curvature}\}$$

is dense in $(-1, 1)$.

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Proof If not, then there is an interval (p,q) such that all $\{F_t : p < t < q\}$ have isotropic S-curvature. Then we have a smooth function $\sigma(x,t)$ such that

$$r_{00}(t) + 2s_0(t)\beta(t) = \sigma(x,t)\big(\alpha^2 - \beta^2(t)\big). \tag{2.10}$$

It is easy to find

$$b_i(t) = b_i + tf_i, \quad r_{00}(t) = r_{00} + tf_{00}, \quad s_{ik}(t) = s_{ik}, \quad s_0(t) = s_0 + tf^i s_{i0}. \tag{2.11}$$

So, the t-derivatives of (2.10) in (p,q) will tell us

$$f_{00} + 2f^i s_{i0}\beta(t) + 2s_0(t)f_0 = \sigma'(\alpha^2 - \beta^2(t)) - 2\sigma\beta(t)f_0, \tag{2.12}$$

$$4f^i s_{i0}f_0 = \sigma''(\alpha^2 - \beta^2(t)) - 4\sigma'\beta(t)f_0 - 2\sigma f_0 f_0. \tag{2.13}$$

Hence, $\sigma''(\alpha^2 - \beta^2(t))$ is divisible by f_0, then $\sigma''(x,t) = 0$, and (2.13) becomes

$$4f^i s_{i0}f_0 = -4\sigma'\beta(t)f_0 - 2\sigma f_0 f_0. \tag{2.14}$$

Taking the t-derivative again, one can get $\sigma' f_0 f_0 = 0$. Since $df \neq 0$ almost everywhere, we get $\sigma'(x,t) = 0$. Substituting it into (2.14), we get $2f^i s_{i0}f_0 = -\sigma f_0 f_0$. Choose $y_i = f_i$, and we see that $\sigma = 0$ in fact. Now, (2.12) becomes

$$f_{00} + 2f^i s_{i0}(\beta + tf_0) + 2(s_0 + tf^i s_{i0})f_0 = 0,$$

which is equivalent to

$$f_{00} + 2f^i s_{i0}\beta + 2s_0 f_0 = 0, \quad f^i s_{i0}f_0 = 0.$$

Hence,

$$f_{jk} + f^i s_{ij}b_k + f^i s_{ik}b_j + s_j f_k + s_k f_j = 0,$$

which leads to $\Delta f = 0$. This is a contradiction to the assumption on f. ■

Hence, putting F_0 the metric given in [2], many F_t's are non-trivial, though we do not know which t's are suitable. It is interesting to present explicit examples.

3 Disturbed Bergman Metrics

Let M be a complex manifold with the complex structure $J = J^i_k dx^k \otimes \partial_{x^i}$, and let $a = a_{ik}dx^i \otimes dx^k$ be a Kähler metric on M. Then the Kähler form is

$$\kappa = \kappa_{ik}dx^i \otimes dx^k, \quad \kappa_{ik} := a_{im}J^m_k = -a_{km}J^m_i.$$

In other words, $\kappa(X,Y) := a(X,JY)$.

Since a is Kählerian, the Kähler form is closed, *i.e.*, $d\kappa = 0$. Then the Poincaré lemma claims κ must be locally exact. Hence we can find some suitable 1-form β

locally such that $d\beta = 2\kappa$. Moreover, we can let $\|\beta\| < 1$ near a point x_0 by letting $\beta_{x_0} = 0$. Now,

$$s = s_{ik} dx^i \otimes dx^k = \tfrac{1}{2}(b_{i|k} - b_{k|i}) dx^i \otimes dx^k = -\tfrac{1}{2} d\beta,$$

where we use the identification $\omega \wedge \eta = \omega \otimes \eta - \eta \otimes \omega$. Then we have

$$s_{ik} = -\kappa_{ik}, \quad t_{ik} = s_{im} s^m_k = a_{ij} J^j_m \delta^m_l J^l_k = -a_{ik}.$$

Noting that the Kähler form is parallel, $\nabla\kappa = 0$, equation (2.2) holds automatically. Then equation (2.1) becomes

$${}^a R_{0ik0} = \Big(\lambda - \frac{n-2}{n-1}\Big)(\alpha^2 a_{ik} - y_i y_k) - 3\kappa_{i0}\kappa_{k0};$$

equivalently,

$$(3.1) \quad {}^a R(Y, X, X, Y) = \Big(\lambda - \frac{n-2}{n-1}\Big)\big[a(Y, Y)a(X, X) - a^2(X, Y)\big] - 3a^2(X, JY).$$

Setting $\lambda = -\frac{1}{n-1}$, it turns out to be an interesting equation:

$${}^a R(Y, X, X, Y) = -\big(a(Y, Y)a(X, X) - a^2(X, Y) + 3a^2(X, JY)\big),$$

which is equivalent to saying a has constant holomorphic curvature -4.

Lemma 3.1 *Let α be the Bergman metric with constant holomorphic curvature -4. Let β be a local 1-form such that $\|\beta\|_a < 1$, $d\beta = 2\kappa$. Then the Randers metric $F = \alpha + \beta$ is of scalar flag curvature.*

Since $d\beta \neq 0$ and the sectional curvature of α is not constant, the above resulting metric $F = \alpha + \beta$ is not projectively flat. Explicitly, let $D = \{z \in \mathbb{C}^N : |z| < 1\}$, and $\{z^A\}$ the standard coordinate on D. The Bergman metric on D with constant holomorphic curvature -4 is

$$h = h_{A\bar{B}} dz^A \otimes d\bar{z}^B, \quad h_{A\bar{B}} = \frac{(1 - |z|^2)\delta_{AB} + z^B \bar{z}^A}{(1 - |z|^2)^2}.$$

One can easily find

$$(3.2) \qquad h^{A\bar{B}} = (1 - |z|^2)(\delta_{AB} - z^A \bar{z}^B).$$

The Kähler form of h is

$$\kappa = \mathrm{Im}(h) = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\ln(1 - |z|^2) = \frac{1}{4} dd^c \ln(1 - |z|^2),$$

where $d^c = \sqrt{-1}(\bar{\partial} - \partial)$. That means the Kähler form is globally exact. Put

$$f = \tfrac{1}{2}\ln(1 - |z|^2),$$

and let

$$a = \mathrm{Re}(h), \quad b = d^c f. \tag{3.3}$$

It is clear that

$$d^c f = \frac{\sqrt{-1}}{2}\frac{(\bar{z}^A dz^A - z^A d\bar{z}^A)}{(1-|z|^2)}, \quad df = -\frac{1}{2}\frac{(\bar{z}^A dz^A + z^A d\bar{z}^A)}{(1-|z|^2)}.$$

By (3.2) and (3.3), we see

$$\|b + \epsilon df\|^2 = (1+\epsilon^2)|z|^2.$$

Finally, since f is the potential of h, $\Delta f < 0$. So we have the following proposition.

Proposition 3.2 *Let a be the Bergman metric on $D = \{x \in \mathbb{R}^{2N} : |x| < 1\}$ with constant holomorphic curvature -4, and $f = \frac{1}{2}\ln(1-|x|^2)$ be the potential of a. Then for any ϵ, the metric*

$$F_\epsilon(x,y) = \sqrt{a(y,y)} + d^c f(y) + \epsilon df(y), \quad |x| < \frac{1}{\sqrt{1+\epsilon^2}}$$

is a regular Randers metric of scalar flag curvature that is not projectively flat. Moreover, for suitable ϵ, the metric does not have isotropic S-curvature.

Next, we shall study which ϵ is suitable. One may easily find

$$b_k = -J_k^i f_i, \quad s_{ik} = -\kappa_{ik} = a_{ks}J_i^s, \quad s_k = -f_k, \quad t_{ik} = -a_{ik},$$
$$a_{kj} = -\tfrac{1}{2}(f_{si}J_k^s J_j^i + f_{jk}), \quad r_{ik} = -\tfrac{1}{2}(f_{si}J_k^s + f_{sk}J_i^s).$$

Then $r_{00} + 2s_0\beta = -\nabla^2_{y,Jy} f + 2\nabla_y f \nabla_{Jy} f$. It is clear that the vectors in $T^{1,0}M$ and TM are related by

$$v = v^A\partial_{z^A} = \tfrac{1}{2}(y - \sqrt{-1}Jy), \qquad y = v + \bar{v}, \quad Jy = \sqrt{-1}(v - \bar{v}).$$

Then

$$\sqrt{-1}(r_{00} + 2s_0\beta) = \nabla^2_{v,v} f - \nabla^2_{\bar{v},\bar{v}} f - 2\nabla_v f \nabla_v f + 2\nabla_{\bar{v}} f \nabla_{\bar{v}} f.$$

The connection coefficients of the Bergman metric are

$$\Gamma^D_{AC} = h^{D\bar{B}}\partial_C h_{A\bar{B}} = \frac{\bar{z}^C\delta_{AD} + \bar{z}^A\delta_{CD}}{1-|z|^2}.$$

The derivatives of f are

$$f_A = -\frac{\bar{z}^A}{2(1-|z|^2)}, \quad \partial_C\partial_A f = -\frac{\bar{z}^A\bar{z}^C}{2(1-|z|^2)^2}.$$

Hence the Hessian of f is

$$f_{AC} = \partial_C \partial_A f - f_D \Gamma^D_{AC} = \frac{\bar{z}^A \bar{z}^C}{2(1-|z|^2)^2} = 2 f_A f_C.$$

Hence $r_{00} + 2s_0\beta = 0$, which also means the S-curvature of F_0 is zero. Now, let us consider F_ϵ. If it has isotropic S-curvature, then

$$r_{00}(\epsilon) + 2s_0(\epsilon)\beta(\epsilon) = \sigma\big(\alpha^2 - \beta^2(\epsilon)\big).$$

Noting $f^i s_{i0} = \beta$, $s_i = -f_i$, and (2.11), we see

$$\epsilon f_{00} - 2\epsilon f_0^2 + 2\epsilon\beta^2 + 2\epsilon^2 \beta f_0 = \sigma\alpha^2 - \sigma\beta^2 - 2\sigma\epsilon\beta f_0 - \sigma\epsilon^2 f_0^2.$$

Setting $y \to Jy$, it turns to

$$\epsilon f_{J0,J0} - 2\epsilon\beta^2 + 2\epsilon f_0^2 - 2\epsilon^2 f_0\beta = \sigma\alpha^2 - \sigma f_0^2 + 2\sigma\epsilon f_0\beta - \sigma\epsilon^2\beta^2.$$

Hence $2(\sigma+\epsilon)\alpha^2 = \sigma(1+\epsilon^2)(\beta^2 + f_0^2)$. If $n = 2N > 2$, then we can choose y such that $\beta = f_0 = 0$, then $\epsilon(\beta^2 + f_0^2) = 0$. Finally, we get the following theorem.

Theorem 3.3 *Let a be the Bergman metric on $D = \{x \in \mathbb{R}^{2N} : |x| < 1\}(N > 1)$ with constant holomorphic curvature -4, $f = \frac{1}{2}\ln(1-|x|^2)$ be the potential of a, and J be the complex structure. Then the metric*

$$F_\epsilon(x,y) = \sqrt{a(y,y)} + df(\epsilon y - Jy), \epsilon \neq 0$$

is regular near the origin, is of scalar flag curvature, and is neither projectively flat nor of isotropic S-curvature.

One can easily find that this is a nontrivial example satisfying Corollary 2.5. Moreover, α is Einstein in this example. This indicates that we should study the Randers metrics with Einstein α that contain projectively flat Randers manifolds.

4 Einstein α

Lemma 4.1 *If α is Einstein and $n = 2m+1$, then $\alpha + \beta$ is projectively flat.*

Proof By (2.3) we see

$$\frac{R_a}{n} a_{ik} = (n-1)\lambda a_{ik} + (n+1)t_{ik}. \tag{4.1}$$

Note that in odd dimensions, the matrix (s_{ij}) is degenerate and hence $\det(t_{ij}) = 0$. Then $R_a = n(n-1)\lambda$ and $t_{ij} = 0$. ■

Lemma 4.2 *If α is Einstein and $\alpha + \beta$ is not projectively flat, then α is an almost Hermitian metric and $\frac{1}{\mu} s^i{}_j \partial_i \otimes dx^j$ is the almost complex structure with suitable number μ.*

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Proof Lemma 3.1 and (4.1) mean that $s^i{}_m s^m{}_j = -\mu^2 \delta^i_j$ for some nonzero constant μ. Then $J^i_j = \frac{1}{\mu} s^i{}_j$ is an almost complex structure. Since s is skew-symmetric, we see $a(X, JX) = 0$. Then one can deduce $a(X, Y) = a(JX, JY)$ from $a(X + JY, J(X + JY)) = 0$. That means α is almost Hermitian with respect to J. ∎

Lemma 4.3 *J is integrable, and α is Kählerian.*

Proof We only need to show $\nabla J = 0$, *i.e.*, $s_{ij|k} = 0$, which is indeed implied by (2.6) and (2.2). ∎

Theorem 4.4 *If $\alpha + \beta$ is of scalar flag curvature, α is Einstein and $d\beta \neq 0$, then α is a Kähler metric of constant holomorphic sectional curvature $-4\mu^2$, and $-\frac{1}{2\mu} d\beta$ is the Kähler form.*

Proof By (3.1), one can get

$$
{}^aR(Y, X, X, Y) = \Big(\lambda - \frac{n-2}{n-1}\mu^2\Big)\big[a(Y, Y)a(X, X) - a^2(X, Y)\big] - 3\mu^2 a^2(X, JY). \tag{4.2}
$$

Then

$$
{}^aR(JX, X, X, JX) = \Big(\lambda - \frac{n-2}{n-1}\mu^2 - 3\mu^2\Big) a^2(X, X),
$$

which means the holomorphic curvature of a is $\lambda - \frac{4n-5}{n-1}\mu^2$. Hence the sectional curvature must be

$$
{}^aR(Y, X, X, Y) = \frac{1}{4}\Big(\lambda - \frac{4n-5}{n-1}\mu^2\Big)\big(a(Y, Y)a(X, X) - a^2(X, Y) + 3a^2(X, JY)\big). \tag{4.3}
$$

Combining (4.2) and (4.3), one can reach

$$
\Big(\lambda + \frac{1}{n-1}\mu^2\Big)\Big(a(Y, Y)a(X, X) - a^2(X, Y) - a^2(X, JY)\Big) = 0.
$$

Then $\lambda = -\frac{1}{n-1}\mu^2$ since the dimension is greater than two. Finally, α has constant holomorphic sectional curvature $-4\mu^2$. ∎

This theorem implies that the examples in Proposition 3.2 are the most important ones. Other examples can only be obtained by multiplying a positive constant and adding a closed 1-form. Hence the classification of Einstein α is finished. The next question is what will happen if α only has constant scalar curvature.

5 Dimension Three

In dimension 3, the conformal Weyl tensor vanishes and the Ricci tensor uniquely determines the Riemann curvature tensor. Then a Randers metric is of scalar curvature if and only if it satisfies

(5.1) $$s_{ij|k} = \tfrac{1}{2}\left(a_{ik}s^m_{\ j|m} - a_{jk}s^m_{\ i|m}\right), \quad {}^a\mathrm{Ric}_{ik} = 2\lambda a_{ik} + 4t_{ik}.$$

Moreover, Lemma 2.4 gives

(5.2) $$\lambda = \frac{R_a + C}{10}, \quad t^m_{\ m} = \frac{2R_a - 3C}{20},$$

where C is a constant.

Since s_{ik} is skew-symmetric, $\det(s_{ik}) = 0$ in odd dimension. Hence we can find an orthonomal basis $\{e_i\}$ such that $s_{1k} = 0$, *i.e.*,

(5.3) $$(s_{ik}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu \\ 0 & -\mu & 0 \end{pmatrix} \qquad (t_{ik}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mu^2 & 0 \\ 0 & 0 & -\mu^2 \end{pmatrix},$$

where $\mu = \mu(x)$ is a scalar function. By (5.2) and (5.3), we see that the second equation of (5.1) is

$$({}^a\mathrm{Ric}_{ik}) = \frac{1}{10}\begin{pmatrix} 2R_a + 2C & & \\ & 4R_a - C & \\ & & 4R_a - C \end{pmatrix}.$$

In order to consider the equation of s_{ik}, let us denote the dual frame of $\{e_i\}$ by $\{\omega^i\}$ and the Riemann connection of a by

$$\nabla\omega^i = -\omega^k \otimes \omega^i_k = -\Gamma^i_{kj}\omega^k \otimes \omega^j, \quad \Gamma^i_{kj} = -\Gamma^k_{ij}.$$

Since $s = \mu\omega^2 \otimes \omega^3 - \mu\omega^3 \otimes \omega^2$, we see

$$\nabla s = \mu_i\omega^2 \otimes \omega^3 \otimes \omega^i - \mu_i\omega^3 \otimes \omega^2 \otimes \omega^i - \mu\Gamma^2_{ij}\omega^i \otimes \omega^3 \otimes \omega^j - \mu\Gamma^3_{ij}\omega^2 \otimes \omega^i \otimes \omega^j + \mu\Gamma^3_{ij}\omega^i \otimes \omega^2 \otimes \omega^j + \mu\Gamma^2_{ij}\omega^3 \otimes \omega^i \otimes \omega^j.$$

Then the first equation of (5.1) is equivalent to the following table.

$s_{12\|3} = s_{13\|2} = s_{23\|1} = 0$	$s_{12\|2} = s_{13\|3}$	$s_{21\|1} = s_{23\|3}$	$s_{31\|1} = s_{32\|2}$
$\mu\Gamma^3_{13} = \mu\Gamma^2_{12} = \mu_1 = 0$	$\mu\Gamma^3_{12} = -\mu\Gamma^2_{13}$	$\mu\Gamma^3_{11} = -\mu_3$	$\mu\Gamma^2_{11} = -\mu_2$

Lemma 5.1 *Let $(M^3, \alpha + \beta)$ be a Randers space of scalar curvature. If $R_a =$ const and $\mu \neq 0$, then e_1 is Killing, and $d\omega$ and s are linearly dependent.*

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Proof By Lemma 2.4, μ must be a nonzero constant. Then the above table tells us $\langle \nabla_{e_2} e_1, e_2 \rangle = \langle \nabla_{e_3} e_1, e_3 \rangle = 0$, $\langle \nabla_{e_2} e_1, e_3 \rangle = -\langle \nabla_{e_3} e_1, e_2 \rangle$ and $\nabla_{e_1} e_1 = 0$, which are equivalent to saying that e_1 is a Killing field that is globally defined. The Bochner technique tells us

$$|\nabla e_1|^2 = \frac{1}{2}\Delta |e_1|^2 + {}^a \operatorname{Ric}(e_1, e_1) = \text{const},$$

then $\Gamma^3_{12} = -\Gamma^2_{13} = \text{const}$ and $d\omega^1 = \omega^k \wedge \omega^1_k = 2\Gamma^3_{12}\omega^2 \wedge \omega^3$, which ends the proof. ■

If $d\omega^1 = 0$, then e_1 is parallel, and the de Rham decomposition tells us $M = \mathbb{R} \times N^2$. Moreover, s is the volume form of N up to a constant, and N must be of constant Gaussian curvature. If $d\omega^1 \neq 0$, then (a, ω^1, e_1) is a K-contact structure, and $d\beta = c\, d\omega^1$ for a constant c. Hence $\alpha + \beta$ is projectively related to $\alpha + c\omega^1$, and the latter is also of scalar flag curvature. In addition, since ω^1 is a Killing form with unit norm, $\alpha + c\omega^1$ must have vanishing S-curvature, hence a result in [7] implies $\alpha + c\omega^1$ has constant flag curvature. Consequently, we have the following theorem.

Theorem 5.2 *Let (M^3, F) be a Randers space of scalar curvature. If $R_a = \text{const}$, then it must be one of the following:*

(i) *F is projectively flat.*
(ii) *(M, α) is locally isometric to $\mathbb{R} \times N^2$, where N is a surface with constant curvature; $d\beta$ is the volume form of N multiplied with a constant.*
(iii) *There exists a closed form ω such that $F + \omega$ has constant flag curvature and vanishing S-curvature.*

By the main theorem in [3], we know that (α, ω^1) must be a Sasakian space form if $M = \mathbb{S}^3$.

Corollary 5.3 *Let $(\mathbb{S}^3, F)$ be a Randers sphere of scalar curvature. If $R_a = \text{const}$, then it is projectively flat or projectively related to a Randers sphere form constructed from a Sasakian space form.*

Bao and Shen's example only belongs to the latter case in the above corollary.

6 Sasakian Structures

In this section, we will say something about Sasakian structure. It can be considered as a remark.

Let $(M^{2m+1}, \alpha, \beta)$ be a Sasakian structure. Then β is a Killing form with unit norm. Moreover,

$$b_{i|j|k} = a_{ik} b_j - a_{jk} b_i, \quad b_{i|k} b_{j|l} a^{kl} = a_{ij} - b_i b_j \tag{6.1}$$

Then $s_{ij} = b_{i|j}$ and $s_{kj|k} = 2m b_j$. Recall that $\operatorname{Ric}_{ik} = 2m\lambda a_{ik} + (2m+2)\epsilon^2 t_{ik}$; then (6.1) means

$$\operatorname{Ric}_{ik} = (2m\lambda - (2m+2)\epsilon^2) a_{ik} + (2m+2)\epsilon^2 b_i b_k.$$

Hence the Sasakian structure (α, β) is eta-Einstein with $\lambda = 1$.

Proposition 6.1 *Let (α, β) be a Sasakian structure. If the Randers metrics $F = \alpha + \epsilon\beta(-1 < \epsilon < 1)$ is of scalar flag curvature, then (α, β) is eta-Einstein with*

$$\mathrm{Ric} = \left(2m - (2m+2)\epsilon^2\right) a + (2m+2)\epsilon^2 \beta \otimes \beta.$$

In dimension three, it is also sufficient.

Proposition 6.2 *Let (M^3, α, β) be a Sasakian structure. Then the Randers metric $F = \alpha + \epsilon\beta(-1 < \epsilon < 1)$ is of scalar flag curvature if and only if (α, β) is eta-Einstein with*

$$\mathrm{Ric} = (2 - 4\epsilon^2)a + 4\epsilon^2 \beta \otimes \beta.$$

Then $\mathcal{D}$-homothety will tell us (see [4, Section 5]).

Corollary 6.3 *Let M^3 be a differentiable manifold. Then M admits a Sasakian–Einstein metric if and only if it admits a Sasakian structure (α, β) such that $F = \alpha + \epsilon\beta$ is of scalar curvature for some $-1 < \epsilon < 1$.*

The existence of Sasakian–Einstein metrics is a central problem in Sasakian geometry. The above corollary relates it to the existence of certain Finsler metrics of scalar flag curvature.

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