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ON THE VOLUME OF LATTICE MANIFOLDS

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The volume of a general lattice polyhedron P in $\mathbb{R}^N$ can be determined in terms of numbers of lattice points from $N-1$ different lattices in P. Ehrhart gave a formula for the volume of "polyèdre entier" in even-dimensional spaces involving only $N/2$ lattices. The aim of this note is to comment on Ehrhart's formula and provide a similar volume formula applicable to lattice polyhedra that are N-dimensional manifolds in $\mathbb{R}^N$.

Denote by $\mathbb{Z}^N$ the fundamental lattice of points with integer coordinates in $\mathbb{R}^N$. Elements of $\mathbb{Z}^N$ are called *lattice points*. We say that a simplex $\Delta \subset \mathbb{R}^N$ is a *lattice simplex* if all its vertices belong to $\mathbb{Z}^N$. A lattice simplex Δ is called *fundamental* if $\Delta \cap \mathbb{Z}^N$ consists only of the vertices of Δ. A set $P \subset \mathbb{R}^N$ is said to be a *polyhedron*, if P is the underlying point set of a simplicial cell complex. A polyhedron P is called *lattice* if all its vertices (0-simplexes) lie in $\mathbb{Z}^N$. Any lattice polyhedron P can be represented as the union

$$P = \bigcup_{i=1}^{m} \Delta_i, \tag{1}$$

where each Δ_i is a fundamental lattice simplex and $\Delta_j \setminus \Delta_k \neq \emptyset$ for $j \neq k$ (no Δ_j is contained in another simplex). Lattice polyhedra in $\mathbb{R}^2$ are called, as usual, *lattice polygons*. A lattice polyhedron P in $\mathbb{R}^N$ is called *proper* if every Δ_i in the union (1) is N-dimensional.

Reeve [15] introduced additional lattices (often called the *rational lattices*)

$$\mathbb{Z}_n^N = \{x \in \mathbb{R}^N : nx \in \mathbb{Z}^N\}, \quad n \geqslant 1.$$

Notice that $\mathbb{Z}_1^N = \mathbb{Z}^N$.

For a given lattice polyhedron P in $\mathbb{R}^N$ denote by B_n and I_n, $n \geqslant 1$, the numbers of points of the lattice $\mathbb{Z}_n^N$ on the boundary and in the interior of P, respectively. Thus

$$B_n = B_n(P) = |\mathbb{Z}_n^N \cap \partial P| \quad \text{and} \quad I_n = I_n(P) = |\mathbb{Z}_n^N \cap \operatorname{int} P|.$$

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Ehrhart [**4**] discovered the remarkable fact that the numbers $G_n(P) = I_n(P) + B_n(P)$ of a proper lattice polyhedron P in $\mathbb{R}^N$ are given by the following polynomial

$$G_n(P) = V(P)n^N + a_{N-1}(P)n^{N-1} + \cdots + a_1(P)n + \chi(P) \tag{2}$$

in which $V(P)$ is the volume of P, the coefficients $a_{N-1}(P), \ldots, a_1(P)$ are some rational numbers, and $\chi(X)$ here and later on denotes the *Euler characteristic* of the set X. For an explicit description of all a_n's in the case of a lattice simplex we refer the reader to [**1**] and [**2**]. Also the numbers $I_n(P)$ and $D_n(P) = I_n(P) + B_n(P)/2$ are some polynomials in n of degree N with the leading coefficient $V(P)$. These polynomials are now called *Ehrhart polynomials.*

Using the Ehrhart polynomial describing the numbers $D_n(P)$ Macdonald [**12**] obtained the following formula for the volume of a proper lattice polyhedron P in $\mathbb{R}^N$

$$(N-1)N!V(P) = \sum_{k=1}^{N-1}(-1)^{k-1}\binom{N-1}{k-1}(B_{N-k} + 2I_{N-k}) + (-1)^{N-1}\left[2\chi(P) - \chi(\partial P)\right]. \tag{3}$$

The case $N = 3$ in formula (3) was earlier obtained by Reeve [**14, 15**].

Ehrhart polynomials of lattice polyhedra in $\mathbb{R}^N$ satisfy the so-called *reciprocity law.* From among several equivalent formulations of it we present here the following

$$I_n(P) = (-1)^N G_{-n}(P),$$

in which $G_{-n}(P) = V(P)(-n)^N + a_{N-1}(-n)^{N-1} + \cdots + a_1(-n) + \chi(P)$ and the coefficients $a_{N-1}, \ldots, a_1$ are the same as in formula (2). Let us notice that the law does not hold for all proper lattice polyhedra in $\mathbb{R}^N$.

Making use of the reciprocity law Ehrhart [**3, 4**] derived the following formula for the volume of "polyèdre entier" in even-dimensional spaces

$$\frac{N!}{2}V(P) = \sum_{j=0}^{(N/2)-1}(-1)^j\binom{N}{j}\left(I_{(N/2)-j} + \frac{1}{2}B_{(N/2)-j}\right) + \frac{1}{2}(-1)^{N/2}\binom{N}{N/2}\chi(P). \tag{4}$$

(The reader is warned that in both papers the formula was misprinted.) This formula employing only lattices $\mathbb{Z}_1^N, \ldots, \mathbb{Z}_{N/2}^N$ is compared by Ehrhart to Macdonald's formula (3) which uses lattices $\mathbb{Z}_1^N, \ldots, \mathbb{Z}_{N-1}^N$. However Ehrhart's formula cannot be applied to all proper lattice polyhedra. We shall illustrate this by an example below.

In [**3**] Ehrhart also gave the following special cases of formula (4) when $N = 2$ and $N = 4$. They are as follows:

$$A(P) = I_1 + \frac{1}{2}B_1 - \chi(P) \tag{5}$$

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and

$$12V(P) = I_2 + \frac{1}{2}B_2 - 4\left(I_1 + \frac{1}{2}B_1\right) + 3\chi(P). \tag{6}$$

While formula (5) admits an extension whose range of validity covers all proper lattice polygons in $\mathbb{R}^2$, see [**15, 16**], the latter cannot be easily modified to be applicable to all proper lattice polyhedra in $\mathbb{R}^4$. In fact, we shall show that there is no linear formula for the volume of proper lattice polyhedra in $\mathbb{R}^4$ in terms of B_2, B_1, I_2, I_1, $\chi(P)$ and $\chi(\partial P)$.

We need the following example.

EXAMPLE. Take two unit lattice cubes C_1 and C_2 in $\mathbb{R}^4$ and denote by P_k the union $C_1 \cup C_2$ having k-dimensional, $0 \leqslant k \leqslant 2$, intersection $C_1 \cap C_2$. One can check that $B_2(P_k) = 2(3^4-1)-3^k$, $B_1(P_k) = 2^5-2^k$, $I_2(P_k) = 2$, $I_1(P_k) = 0$, $\chi(P_k) = 1$, $\chi(\partial P_k) = -1$ and $V(P_k) = 2$.

If there existed a linear formula for the volume of proper lattice polyhedra in $\mathbb{R}^4$ in terms of B_2, B_1, I_2, I_1, $\chi(P)$ and $\chi(\partial P)$ then it would be of the form

$$V(P) = aB_2(P) + bI_2(P) + cB_1(P) + dI_1(P) + e\chi(P) + f\chi(\partial P).$$

By substituting the numbers from Example 1 in the above we would obtain

$$\begin{aligned} 2 &= 159a + 2b + 31c + e - f \\ 2 &= 157a + 2b + 30c + e - f \\ 2 &= 151a + 2b + 28c + e - f \end{aligned}$$

which, as is easy to check, is an inconsistent system.

An immediate consequence of the above considerations is the fact that formula (6) is not applicable for all proper lattice polyhedra in $\mathbb{R}^4$. This also implies that formula (4) cannot be used for all proper lattice polyhedra in $\mathbb{R}^N$. It can be shown, however, that formula (4) is applicable for all lattice polyhedra which are *N-dimensional manifolds*. Indeed, we always have

$$G_n(P) = V(P)n^N + a_{N-1}(P)n^{N-1} + \cdots + a_1(P)n + \chi(P)$$

and

$$I_n(P) = V(P)n^N + c_{N-1}(P)n^{N-1} + \cdots + c_1(P)n + \chi(P) - \chi(\partial P).$$

In view of [**13**, Corollary 1.6 and Theorem 4.6] one can see that the reciprocity law is satisfied for such polyhedra. So we have $I_n(P) = (-1)^N G_{-n}(P)$. This implies that $a_{2j-1} = -c_{2j-1}$ for $j = 1, \ldots, N/2$. Consequently, it follows that the numbers $D_n(P)$ of a lattice polyhedron P that is an N-dimensional manifold in an even-dimensional space $\mathbb{R}^N$ are given by a polynomial of the form

$$D_n(P) = I_n(P) + \frac{1}{2}B_n(P) = V(P)n^N + b_{N-2}n^{N-2} + b_{N-4}n^{N-4} + \cdots + b_2n^2 + \chi(P).$$

By substituting the values $1, 2, \ldots, N/2$ for n in the above polynomial we obtain a system of $N/2$ linear equations. Solving for $V(P)$ in that system results in formula (4).

Now we shall comment on the case when N is an odd number, $N = 2k - 1$. Again from [**13**, Corollary 1.6 and Theorem 4.6] it follows that the reciprocity law is satisfied for any lattice polyhedron P in $\mathbb{R}^N$ that is an N-dimensional manifold. Proceeding similarly as above we can show that in this case

$$D_n(P) = I_n(P) + \frac{1}{2}B_n(P) = V(P)n^N + b_{N-2}n^{N-2} + b_{N-4}n^{N-4} + \cdots + b_3 n^3 + b_1 n.$$

When we allow n to assume values $1, 2, \ldots, k = (N+1)/2$ in the above polynomial then it generates a system of k independent linear equations. Solving the system for $V(P)$ and evaluating the two resulting Vandermonde-type determinants, we obtain

$$V(P) = \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 & I_1 + \frac{1}{2}B_1 \\ 2 & 2^3 & \cdots & 2^{2k-2} & I_2 + \frac{1}{2}B_2 \\ 3 & 3^3 & \cdots & 3^{2k-2} & I_3 + \frac{1}{2}B_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ k & k^3 & \cdots & k^{2k-2} & I_k + \frac{1}{2}B_k \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2 & 2^3 & \cdots & 2^{2k-2} & 2^{2k-1} \\ 3 & 3^3 & \cdots & 3^{2k-2} & 3^{2k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ k & k^3 & \cdots & k^{2k-2} & k^{2k-1} \end{vmatrix}}$$

$$= \frac{\left(\prod_{j=1}^{k}(2j-1)!\right) \sum_{j=1}^{k}(-1)^{k+j}\frac{j}{(k-j)!\,(k+j)!}(2I_j + B_j)}{\prod_{j=1}^{k}(2j-1)!}$$

$$= \frac{\sum_{j=1}^{k}(-1)^{k+j}\binom{2k}{k-j}\, j\,(2I_j + B_j)}{(2k)!}.$$

Returning now to $N = 2k - 1$ we get the following formula for the volume of lattice

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polyhedra that are N-dimensional manifolds in odd-dimensional space $\mathbb{R}^N$;

$$(N+1)!\,V(P) = \sum_{j=1}^{(N+1)/2} (-1)^{(N+1)/2-j} \binom{N+1}{(N+1)/2-j}\, j\,(2I_j + B_j).$$

In the special cases of $N = 3$ and $N = 5$ the formula reads

$$12V(P) = 2I_2 + B_2 - 2(2I_1 + B_1) \tag{7}$$

and

$$240V(P) = 5(2I_1 + B_1) - 4(2I_2 + B_2) + (2I_3 + B_3).$$

As we have already mentioned the case $N = 3$ in Macdonald's formula (3) was earlier obtained by Reeve. Reeve's formula

$$12V(P) = 2I_2 + B_2 - 2(2I_1 + B_1) + 2\chi(P) - \chi(\partial P)$$

in the case of lattice polyhedra that are 3-dimensional manifolds coincides with our formula (7) since for such polyhedra we have $2\chi(P) - \chi(\partial P) = 0$, see [**13**, Corollary 1.6].

Summarising the observations made in this note we have the following theorem.

Theorem. *If P is a lattice polyhedron in $\mathbb{R}^N$ that is an N-dimensional manifold, then*

$$V(P) = \begin{cases} \dfrac{1}{N!}\left[\displaystyle\sum_{j=1}^{N/2} (-1)^{(N/2)-j} \binom{N}{(N/2)-j} (2I_j + B_j) + (-1)^{N/2}\binom{N}{N/2}\chi(P)\right] & \text{if } N \text{ is even,} \\ \dfrac{1}{(N+1)!} \displaystyle\sum_{j=1}^{(N+1)/2} (-1)^{(N+1)/2-j} \binom{N+1}{(N+1)/2-j}\, j\,(2I_j + B_j) & \text{if } N \text{ is odd.} \end{cases}$$

For more lattice polyhedra volume formulae the reader is referred to [**7, 8, 9, 10, 11**]. More information concerning lattice polyhedra can be found in [**5, 6**].

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