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On Lebesgue points of entropy solutions to the eikonal equation

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We consider entropy solutions to the eikonal equation $|\nabla u| = 1$ in two-space dimensions. These solutions are motivated by a class of variational problems and fail in general to have bounded variation. Nevertheless, they share several of their fine properties with BV functions: we show in particular that the set of non-Lebesgue points has at least one co-dimension.

Keywords: Aviles–Giga; eikonal equation; Lagrangian representation

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1. Introduction

We consider an open set $\Omega \subset \mathbb{R}^2$ and $m\colon \Omega \to \mathbb{R}^2$ a solution of the eikonal equation

$$|m| = 1 \text{ a.e., and } \nabla \cdot m = 0 \text{ in } \Omega. \tag{1.1}$$

We are interested in particular in solutions that arise as limits as $\varepsilon \to 0$ of vector fields m_ε with equi-bounded energy $\sup_{\varepsilon>0} F_\varepsilon(m_\varepsilon, \Omega) < \infty$, where

$$\begin{aligned} &F_\varepsilon(m;\Omega) = \frac{\varepsilon}{2}\int_\Omega |\nabla m|^2 + \frac{1}{2\varepsilon}\int_\Omega (1-|m|^2)^2, \\ &m\colon \Omega \to \mathbb{R}^2, \quad \nabla \cdot m = 0 \end{aligned} \tag{1.2}$$

are the functionals introduced by Aviles and Giga [5]. We refer to the introduction of [11] for a description of several physical applications.

The notion of entropy, borrowed from the field of conservation laws, plays a fundamental role in the study of the singular limit as $\varepsilon \to 0$ of these functionals. We say that a compactly supported function $\Phi \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ is an *entropy* for (1.1) if for every open set U and every smooth $m : U \to \mathbb{R}^2$ solving $\nabla \cdot m = 0$ and $|m| = 1$ it holds $\nabla \cdot \Phi(m) = 0$. It is shown in [3, 8] that functions with equi-bounded energy as $\varepsilon \to 0$ are pre-compact in $L^2(\Omega)$ and any limit is an entropy solution of (1.1): namely, for every entropy $\Phi \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ the distribution $\nabla \cdot \Phi(m)$ is a finite Radon measure. Remarkably, the same class of entropy solutions to (1.1) contains the asymptotic domain of other families of functionals: see [**2**, **17**] for two micromagnetics models.

It is shown in [**9**] that m is an entropy solution if and only if the following kinetic equation (introduced in [**10**]) is satisfied:

$$\mathrm{e}^{is} \cdot \nabla_x \mathbf{1}_{m(x)\cdot \mathrm{e}^{is}>0} = \partial_s \sigma, \quad \sigma \in \mathcal{M}_{\mathrm{loc}}(\Omega \times \mathbb{R}/2\pi\mathbb{Z}). \tag{1.3}$$

We denote by $\nu \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ the entropy dissipation measure given by

$$\nu(A) = |\sigma|(A \times \mathbb{R}/2\pi\mathbb{Z}), \quad A \subset \Omega. \tag{1.4}$$

It is known [**7**] that $\mathcal{H}^1$-a.e. point $x \in \Omega$ at which $\nu(B_r(x))/r \to 0$ as $r \to 0^+$ is a vanishing mean oscillation (VMO) point of m, that is,

$$⨍_{B_r(x)} \left| m - ⨍_{B_r(x)} m \right| \longrightarrow 0 \quad \text{as } r \to 0^+.$$

It is conjectured in [**7**, conjecture 1(b′)] that $\mathcal{H}^1$-a.e. such point is in fact a Lebesgue point. Our main result states that this conjecture is true under the additional assumption that $\nu(B_r(x))/r$ decays algebraically to 0.

Theorem 1.1. *Let $m\colon \Omega \to \mathbb{R}^2$ be an entropy solution (1.3) of the eikonal equation (1.1). Then $\mathcal{H}^1$-a.e. $x \in \Omega$ such that $\lim_{r\to 0^+} \nu(B_r(x))/r^{1+a} = 0$ for some $a > 0$ is a Lebesgue point of m. In particular, the set of non-Lebesgue points of m has Hausdorff dimension at most 1.*

Remark 1.2. After this work was submitted, we became aware that the bound on the Hausdorff dimension can also be obtained as a consequence of classical capacity estimates [**1**, theorem 6.21] and of the regularity $m \in B^{1/3}_{3,\infty}$ [**9**] (which implies $m \in W^{s,3}$ for any $s < 1/3$). Note however that the information provided by theorem 1.1 is stronger, in that it directly relates oscillations at a point x to the local energy dissipation $\nu(B_r(x))$. Also note that, as will be clear from the proof, the assumption of algebraic energy decay $\nu(B_r)/r = \mathcal{O}(r^a)$ can be relaxed to $\nu(B_r)/r = \mathcal{O}(|\ln r|^{-14})$. Via a covering argument this implies that non-Lebesgue points are finite for the Hausdorff measure defined by the function $r \mapsto r|\ln r|^{-14}$ (see e.g. [**1**, § 5.1]), a fact which does not follow directly from capacity estimates.

Analogues of theorem 1.1 have been obtained previously in [**13**] for Burgers' equation, and in [**16**] for general scalar conservations laws. To prove theorem 1.1 we follow the scheme laid out in [**16**], where it is shown that oscillations of averages $⨍_{B_r(x)} u$ of the solution u are controlled by the entropy dissipation. This, together with the VMO property, implies the Lebesgue point property. However, a key feature for the argument of [**16**] is that the solution u takes values in the ordered set $\mathbb{R}$. Here, our solution m takes values in $\mathbb{S}^1$, and adapting the argument of [**16**] is not enough to conclude (see proposition 1.4). Our proof of theorem 1.1 relies instead

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on the following dichotomy: either the oscillations of $\fint_{B_r(x)} m$ are controlled by the entropy dissipation ν, or m takes very different values in large subsets of $B_R(x)$ – this second alternative is ruled out by the VMO property. That dichotomy is made quantitative in the next statement.

PROPOSITION 1.3. *Assume $B_1 \subset \Omega$. Let $r \in (0, 1/2)$ and*

$$h = h(r) = \max_{x_1, x_2 \in \overline{B}_{2r}} \left| \fint_{B_r(x_1)} m - \fint_{B_r(x_2)} m \right|. \tag{1.5}$$

There exist absolute constants $c, \delta > 0$ such that, if

$$R = \frac{32r}{\delta h^2} \leqslant 1,$$

then either

$$\nu(B_R) \geqslant c\, h^{11} r, \tag{1.6}$$

or there exist $s_0 \in \mathbb{R}$ such that

$$\left| B_R \cap \left\{ m \cdot e^{is} \geqslant -\frac{1}{2} \right\} \right| \geqslant c\, R^2 \quad \textit{for } \operatorname{dist}(s, \{s_0, s_0 + \pi\}) \leqslant \pi/4. \tag{1.7}$$

Here and in the rest of the article, we denote by $|A|$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}^d$. Theorem 1.1 is a rather direct consequence of proposition 1.3, as we explain now.

Proof of theorem 1.1. Let $x \in \Omega$ be a VMO point of m such that $\nu(B_r(x))/r^{1+a} \to 0$ for some $a > 0$. Translating and rescaling we assume without loss of generality that $x = 0$ and $B_1 \subset \Omega$. We claim that $h(r) = \mathcal{O}(r^b)$ for $b = a/(13 + 2a) > 0$. This, together with the fact that 0 is a VMO point of m, implies that 0 is a Lebesgue point (see [**16**, lemma 4.6]). (Note, in connection with remark 1.2, that $h(r) = \mathcal{O}(|\ln r|^{-1-\varepsilon})$ for some $\varepsilon > 0$ would imply the same conclusion.) To prove that $h(r) = \mathcal{O}(r^b)$ we argue by contradiction and assume that $h(r)/r^b \to \infty$ along a sequence $r \to 0^+$. Then, along the same sequence,

$$R = \frac{32r}{\delta h^2} = \frac{32}{\delta} r^{1-2b} \left(\frac{r^b}{h} \right)^2 \to 0 \quad \text{because } b < \frac{1}{2},$$
$$\text{and } \frac{R^{1+a}}{h^{11} r} = \frac{32^{1+a}}{\delta^{1+a}} \left(\frac{r^b}{h} \right)^{13+2a} \to 0.$$

Therefore, applying proposition 1.3 along the sequence $R \to 0$, condition (1.6) cannot be satisfied because $\nu(B_R)/R^{1+a} \to 0$, so we have (1.7). This contradicts the VMO property: for all small enough R, the projection $z_R \in \mathbb{S}^1$ of $\fint_{B_R} m$ onto $\mathbb{S}^1$

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satisfies

$$|B_R \cap \{|m - z_R| \geqslant \pi/12\}| \leqslant \frac{c}{2} R^2. \tag{1.8}$$

But one can choose $s \in \mathbb{R}$ such that $\mathrm{dist}(s, \{s_0, s_0 + \pi\}) \leqslant \pi/4$ and

$$z \cdot \mathrm{e}^{is} \geqslant -\frac{1}{2} \quad \Longrightarrow \quad |z - z_R| \geqslant \pi/12,$$

for any $z \in \mathbb{S}^1$ (if $z_R = \mathrm{e}^{is_R}$, any $s \in [s_R + 3\pi/4, s_R + 5\pi/4]$ has that property). According to (1.7) this implies $|B_R \cap \{|m - z_R| \geqslant \pi/12\}| \geqslant cR^2$, in contradiction with (1.8). Hence, we have proved that x is a Lebesgue point. The estimate on the Hausdorff dimension of non-Lebesgue points follows via a covering argument (see e.g. [**4**, theorem 2.56]). □

The proof of proposition 1.3 has two main ingredients. The first ingredient consists in adapting the arguments of [**16**] to prove a dichotomy similar to proposition 1.3, but where the second option (1.7) is replaced by a statement which is not strong enough to conclude.

Proposition 1.4. *Let $r \in (0, 1/2)$ and h be as in proposition 1.3. There exist absolute constants $c, \delta > 0$ such that, if $R = 32r/(\delta h^2) \leqslant 1$, then we have either $\nu(B_R) \geqslant ch^{11} r$, or*

$$\left| B_{R/2} \cap \left\{ m \cdot m_0 \geqslant \frac{1}{2} \right\} \right| \geqslant chr^2 \quad \text{and} \quad \left| B_{R/2} \cap \left\{ m \cdot m_0 \leqslant -\frac{1}{2} \right\} \right| \geqslant chr^2, \tag{1.9}$$

for some $m_0 \in \mathbb{S}^1$.

The main idea behind the argument in [**16**] is that a large value of h implies the existence of a configuration which would be impossible in the absence of entropy dissipation. In the presence of dissipation, such configuration provides a lower bound on the dissipation, and there is no dichotomy. Here instead, not all configurations created by large values of h can be ruled out in the absence of dissipation: in particular the vortex solution $m(x) = x^\perp/|x|$ has zero dissipation but the values of $h(r)$ around the origin are not vanishing. This is reflected in the second alternative (1.9) of the dichotomy.

The second ingredient in our proof of proposition 1.3 consists in using the methods developed in [**6**, **12**, **14**, **15**] in order to pass from (1.9) to (1.7).

Proposition 1.5. *Let $m_0 = \mathrm{e}^{is_0} \in \mathbb{S}^1$, and $R > 0$ such that $B_R \subset \Omega$. Then we have either*

$$\nu(B_R) \geqslant \frac{c}{R} \min(|X_+|, |X_-|), \quad X_\pm = B_{R/2} \cap \{\pm m \cdot m_0 \geqslant 1/2\}, \tag{1.10}$$

or $\nu(B_R) \geqslant cR$, or (1.7), for some absolute constant $c > 0$.

Proposition 1.3 follows readily from propositions 1.4 and 1.5. Thanks to proposition 1.4, we know indeed that either $\nu(B_R) \geqslant ch^{11} r$, in which case we are done, or estimate (1.9) is valid. But according to proposition 1.5, if (1.9) is satisfied, then we have either $\nu(B_R) \geqslant chr^2/R \geqslant ch^{11}r$, or $\nu(B_R) \geqslant cR \geqslant ch^{11}r$, or (1.7). In all cases, proposition 1.3 is verified.

The proofs of propositions 1.4 and 1.5 are presented in § 2 and § 3.

Notations.

We denote by $|A|$ the Lebesgue measure of a set $A \subset \mathbb{R}^d$. We use the symbol $\gtrsim$ to signify inequality up to an absolute multiplicative constant.

2. Proof of proposition 1.4

Let x_1, x_2 attain the maximum in definition (1.5) of h, and define, for $j = 1, 2$, $\rho_j(s)$ as the proportion of points $x \in B_r(x_j)$ at which $m(x)$ lies in the semi-circle of direction e^{is}, that is, for every $s \in \mathbb{R}/2\pi\mathbb{Z}$, we set

$$\rho_j(s) = \frac{1}{|B_r|}\left|B_r(x_j) \cap \left\{m \cdot \mathrm{e}^{is} > 0\right\}\right| = \frac{1}{|B_r|}\int_{B_r(x_j)} \mathbf{1}_{E_m}(x, s)\,\mathrm{d}x,$$

where

$$E_m = \left\{(x, s) \in \Omega \times \mathbb{R}/2\pi\mathbb{Z}\colon m(x) \cdot \mathrm{e}^{is} > 0\right\}. \tag{2.1}$$

Note that $|\rho_j| \leqslant 1$ and, since for every $x \in \Omega$ it holds $|D_s\mathbf{1}_{E_m}(x, \cdot)|(\mathbb{R}/2\pi\mathbb{Z}) = 2$, then $\rho_j \in BV(\mathbb{R}/2\pi\mathbb{Z})$ with $|D\rho_j|(\mathbb{R}/2\pi\mathbb{Z}) \leqslant 2$. Moreover, by Fubini theorem, these functions satisfy the identities

$$\int_{\mathbb{R}/2\pi\mathbb{Z}} \mathrm{e}^{is}\rho_j(s)\,\mathrm{d}s = \int_{\mathbb{R}/2\pi\mathbb{Z}} \fint_{B_r(x_j)} \mathbf{1}_{E_m}(x, s)\mathrm{e}^{is}\,\mathrm{d}x\,\mathrm{d}s = 2\fint_{B_r(x_j)} m(x)\,\mathrm{d}x.$$

For $s \in \mathbb{R}$ and $\rho > 0$ we denote by $I_\rho(s)$ the segment

$$I_\rho(s) = [s - \rho, s + \rho].$$

For a small enough absolute constant $\delta \in (0, 1)$, the subset $S \subset \mathbb{R}/2\pi\mathbb{Z}$ given by

$$S = \left\{s \in \mathbb{R}/2\pi\mathbb{Z}\colon (|D\rho_1| + |D\rho_2|)(I_{\delta h^2}(s)) \geqslant \frac{h}{4\pi}\right\},$$

satisfies $|S| \leqslant h/2$ (as follows e.g. from a Besicovitch covering argument). Thus, we have

$$\begin{aligned} h &= \frac{1}{2}\left|\int_{\mathbb{R}/2\pi\mathbb{Z}} \mathrm{e}^{is}\rho_1(s)\,\mathrm{d}s - \int_{\mathbb{R}/2\pi\mathbb{Z}} \mathrm{e}^{is}\rho_2(s)\,\mathrm{d}s\right| \leqslant \frac{1}{2}\int_{\mathbb{R}/2\pi\mathbb{Z}} |\rho_1(s) - \rho_2(s)|\,\mathrm{d}s \\ &\leqslant \frac{1}{2}\int_{(\mathbb{R}/2\pi\mathbb{Z})\setminus S} |\rho_1(s) - \rho_2(s)|\,\mathrm{d}s + \frac{h}{2}. \end{aligned}$$

We may therefore find $s \in \mathbb{R}/2\pi\mathbb{Z}$ such that $s \notin S$ and $|\rho_1(s) - \rho_2(s)| \geqslant h/2\pi$. We assume without loss of generality that $\rho_1(s) - \rho_2(s) \geqslant h/2\pi$, and by definition of S

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we deduce

$$\inf_{I_{\delta h^2}(s)} \rho_1 - \sup_{I_{\delta h^2}(s)} \rho_2 \geqslant \frac{h}{4\pi}.$$

In particular, setting $s_0 = s - \pi/2 - 3\delta h^2/4$, we have

$$\inf_{I_{\delta h^2/4}(s_0+\pi/2)} \rho_1 - \sup_{I_{\delta h^2/4}(s_0+\pi/2+\delta h^2)} \rho_2 \geqslant \frac{h}{4\pi},$$

$$\inf_{I_{\delta h^2/4}(s_0+\pi/2+\delta h^2)} \rho_1 - \sup_{I_{\delta h^2/4}(s_0+\pi/2)} \rho_2 \geqslant \frac{h}{4\pi}.$$

As $\rho_j(s+\pi) = 1 - \rho_j(s)$ for a.e. $s \in \mathbb{R}/2\pi\mathbb{Z}$, this implies

$$\operatorname*{ess\,inf}_{I_{\delta h^2/4}(s_0+\pi/2)} \rho_1 + \operatorname*{ess\,inf}_{I_{\delta h^2/4}(s_0-\pi/2+\delta h^2)} \rho_2 \geqslant 1 + \frac{h}{4\pi}, \tag{2.2}$$

$$\operatorname*{ess\,inf}_{I_{\delta h^2/4}(s_0+\pi/2+\delta h^2)} \rho_1 + \operatorname*{ess\,inf}_{I_{\delta h^2/4}(s_0-\pi/2)} \rho_2 \geqslant 1 + \frac{h}{4\pi}. \tag{2.3}$$

The relevance of (2.2)–(2.3) comes from the following geometric observation. Given two directions $s_1 \in I_{\delta h^2/4}(s_0+\pi/2)$ and $s_2 \in I_{\delta h^2/4}(s_0 - \pi/2 + \delta h^2)$ and two points $y_1 \in B_r(x_1) \cap \{m \cdot \mathrm{e}^{is_1} > 0\}$, $y_2 \in B_r(x_2) \cap \{m \cdot \mathrm{e}^{is_2} > 0\}$, we have $|s_1 - s_2| \geqslant \delta h^2$, and the two lines $y_j + \mathbb{R}\mathrm{e}^{is_j}$ intersect at a point $z \in B_{8r/(\delta h^2)}$. In the absence of dissipation, one would have $m(z) \cdot \mathrm{e}^{is_j} > 0$ for $j = 1, 2$, and therefore $m(z) \cdot \mathrm{e}^{is_0} \geqslant \cos(2\delta h^2) \geqslant 1/2$. The last lower bound is valid provided $\delta \leqslant \pi/24$, since $|h| \leqslant 2$. The same argument with $s_1 \in I_{\delta h^2/4}(s_0 + \pi/2 + \delta h^2)$ and $s_2 \in I_{\delta h^2/4}(s_0 - \pi/2)$ implies instead $m(z) \cdot \mathrm{e}^{is_0} \leqslant -1/2$.

Thanks to the techniques in [**16**], in the presence of dissipation this can be made quantitative. The main idea is that (1.3) provides an estimate on the difference between the 'epigraph' E_m defined in (2.1) and its free transport $\mathrm{FT}(E_m, t)$, where the free transport operator $\mathrm{FT}(\cdot, t)$ is defined for $t \in \mathbb{R}$ by

$$\mathrm{FT}(E, t) = \left\{(x, s) \in \Omega \times \mathbb{R}/2\pi\mathbb{Z} \colon (x - t\mathrm{e}^{is}, s) \in E\right\}.$$

Lemma 2.1. *Let $t \in \mathbb{R}$ and $\rho > 0$ such that $B_{\rho+|t|} \subset \Omega$. For all $\phi \in C^1_c(B_\rho \times \mathbb{R}/2\pi\mathbb{Z})$ we have*

$$\int_{\Omega\times\mathbb{R}/2\pi\mathbb{Z}} \phi(x,s)\left(\mathbf{1}_{\mathrm{FT}(E_m,t)} - \mathbf{1}_{E_m}\right) \mathrm{d}x\, \mathrm{d}s \leqslant \left(|t|\, \|\partial_s\phi\|_\infty + t^2 \|\nabla_x \phi\|_\infty\right) \nu(B_{\rho+|t|}).$$

Proof of lemma 2.1. Define $\chi, \chi^{\mathrm{FT}} \colon [-|t|, |t|] \times \Omega \times \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ by

$$\chi(\tau, x, s) = \mathbf{1}_{(x,s)\in E_m}, \quad \chi^{\mathrm{FT}}(\tau, x, s) = \mathbf{1}_{(x,s)\in \mathrm{FT}(E_m,\tau)} = \chi(x - \tau\mathrm{e}^{is}, s),$$

so we have, in the sense of distributions,

$$\partial_\tau \chi + \mathrm{e}^{is} \cdot \nabla_x \chi = \partial_s \sigma(x,s), \quad \partial_\tau \chi^{\mathrm{FT}} + \mathrm{e}^{is} \cdot \nabla_x \chi^{\mathrm{FT}} = 0.$$

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Setting $\hat\chi = \chi^{\mathrm{FT}} - \chi$, and $\psi(\tau,x,s) = \phi(x + \mathrm{e}^{is}(t-\tau), s)$ which satisfies $\partial_\tau\psi + \mathrm{e}^{is}\cdot\nabla_x\psi = 0$, we deduce

$$\partial_\tau\left[\psi\hat\chi\right] + \mathrm{e}^{is}\cdot\nabla_x\left[\psi\hat\chi\right] = -\psi\partial_s\sigma.$$

Integrating with respect to (x,s) (this is formal but makes sense distributionally) we deduce

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\Omega\times\mathbb{R}/2\pi\mathbb{Z}} \psi\hat\chi\,\mathrm{d}x\,\mathrm{d}s = \int_{\Omega\times\mathbb{R}/2\pi\mathbb{Z}} \partial_s\psi\,\mathrm{d}\sigma(x,s).$$

Integrating this from 0 to t and recalling $\nu(A) = |\sigma|(A\times\mathbb{R}/2\pi\mathbb{Z})$ for $A\subset\Omega$, we obtain

$$\begin{aligned}\int_{\Omega\times\mathbb{R}/2\pi\mathbb{Z}} \phi(x,s)\left(\mathbf{1}_{\mathrm{FT}(E_m,t)} - \mathbf{1}_{E_m}\right)\,\mathrm{d}x\,\mathrm{d}s &= \int_0^t\int_{\Omega\times\mathbb{R}/2\pi\mathbb{Z}} \partial_s\psi\,\mathrm{d}\sigma\,\mathrm{d}\tau\\ &\leqslant |t|\,\|\partial_s\psi\|_\infty|\nu|(B_{\rho+|t|}\times\mathbb{R}/2\pi\mathbb{Z}).\end{aligned}$$

Noting that $\|\partial_s\psi\|_\infty \leqslant \|\partial_s\phi\|_\infty + |t|\,\|\nabla_x\phi\|_\infty$ completes the proof. □

Equipped with lemma 2.1 we continue the proof of proposition 1.4. First, we make use of (2.2). We define $\hat z\in\mathbb{R}^2$ as the intersection of the lines $x_1 + \mathbb{R}\,\mathrm{e}^{i(s_0+\pi/2)}$ and $x_2 + \mathbb{R}\,\mathrm{e}^{i(s_0-\pi/2+\delta h^2)}$, that is,

$$x_1 + t_1\,\mathrm{e}^{i(s_0+\pi/2)} = x_2 + t_2\,\mathrm{e}^{i(s_0-\pi/2+\delta h^2)} = \hat z,$$

for some $t_1, t_2\in\mathbb{R}$. Since $|x_1 - x_2|\leqslant 4r$, we have

$$|t_1|, |t_2| \leqslant \frac{4r}{\sin(\delta h^2)} \leqslant \frac{8r}{\delta h^2}, \tag{2.4}$$

and therefore $B_r(\hat z)\subset B_{R/2}$. We will use lemma 2.1 to compare E_m with $\mathrm{FT}(E_m,t_1)$ and $\mathrm{FT}(E_m,t_2)$ on $B_r(\hat z)$. We define

$$\begin{aligned}&C_1 = B_r(\hat z)\times I_c(s_0+\pi/2), \quad C_2 = B_r(\hat z)\times I_c(s_0-\pi/2+\delta h^2),\\ &A_1 = E_m\cap C_1, \quad A_2 = E_m\cap C_2\end{aligned}$$

with $c = \delta h^3/128\pi \leqslant \delta h^2/4$, and their free transport counterparts

$$A_1^{\mathrm{FT}} = \mathrm{FT}(E_m,t_1)\cap C_1, \quad A_2^{\mathrm{FT}} = \mathrm{FT}(E_m,t_2)\cap C_2.$$

We estimate

$$\begin{aligned}|A_1^{\mathrm{FT}}| &= |E_m\cap\mathrm{FT}(\cdot,t_1)^{-1}(C_1)|\\ &\geqslant |E_m\cap(B_r(x_1)\times I_c(s_0+\pi/2))|\\ &\quad - |\mathrm{FT}(\cdot,t_1)^{-1}(C_1)\backslash(B_r(x_1)\times I_c(s_0+\pi/2))|\\ &= \int_{s_0+\pi/2-c}^{s_0+\pi/2+c}\rho_1(s)\,\mathrm{d}s - |\mathrm{FT}(\cdot,t_1)^{-1}(C_1)\backslash(B_r(x_1)\times I_c(s_0+\pi/2))|.\end{aligned}$$

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Moreover,

$$
\begin{aligned}
|\mathrm{FT}(\cdot,t_1)^{-1}(C_1)\backslash(B_r(x_1)\times I_c(s_0+\pi/2))| &= \int_{s_0+\pi/2-c}^{s_0+\pi/2+c} |B_r(\hat{z}-t_1\mathrm{e}^{is})\backslash B_r(x_1)|\,\mathrm{d}s \\
&\leqslant 2r\int_{s_0+\pi/2-c}^{s_0+\pi/2+c} |\hat{z}-t_1\mathrm{e}^{is}-x_1|\,\mathrm{d}s \\
&\leqslant 2r\int_{s_0+\pi/2-c}^{s_0+\pi/2+c} |t_1||\mathrm{e}^{i(s_0+\pi/2)}-\mathrm{e}^{is}|\,\mathrm{d}s \\
&\leqslant 32\frac{c^2r^2}{\delta h^2},
\end{aligned}
$$

where in the last inequality we used (2.4). Therefore, we have

$$
|A_1^{\mathrm{FT}}| \geqslant \int_{s_0+\pi/2-c}^{s_0+\pi/2+c} \rho_1(s)\,\mathrm{d}s - 32\frac{c^2r^2}{\delta h^2}, \tag{2.5}
$$

and similarly

$$
|A_2^{\mathrm{FT}}| \geqslant \int_{s_0-\pi/2+\delta h^2-c}^{s_0-\pi/2+\delta h^2+c} \rho_2(s)\,\mathrm{d}s - 32\frac{c^2r^2}{\delta h^2}. \tag{2.6}
$$

From (2.2) we know that

$$
\rho_1\left(s_0+\frac{\pi}{2}+s\right)+\rho_2\left(s_0-\frac{\pi}{2}+\delta h^2+s\right) \geqslant 1+\frac{h}{4\pi} \quad \text{for all } |s|\leqslant\frac{\delta}{4}h^2.
$$

Integrating this inequality in $s\in[-c,c]$, it follows from (2.5) and (2.6) that

$$
|A_1^{\mathrm{FT}}|+|A_2^{\mathrm{FT}}| \geqslant 2c|B_r|\left(1+\frac{h}{4\pi}\right)-64\frac{c^2r^2}{\delta h^2} \geqslant 2c|B_r|\left(1+\frac{h}{8\pi}\right), \tag{2.7}
$$

by the choice $c=\delta h^3/128\pi$. Next, we consider two cases, depending on whether A_1 and A_2 satisfy a similar inequality.

Case 1. Assume first that

$$
|A_1|+|A_2| \geqslant 2c|B_r|\left(1+\frac{h}{16\pi}\right),
$$

then

$$
|\pi_x(A_1)|+|\pi_x(A_2)| \geqslant |B_r|\left(1+\frac{h}{16\pi}\right).
$$

Moreover, since $\pi_x(A_1)\cup\pi_x(A_2)\subset B_r(\hat{z})$, it follows that $A:=\pi_x(A_1)\cap\pi_x(A_2)$ satisfies $|A|\geqslant h|B_r|/16$. By construction, we have

$$
\begin{aligned}
A = \Big\{ x\in B_r(\hat{z})\colon \exists s_1\in I_c\left(s_0+\frac{\pi}{2}\right),\ s_2\in I_c\left(s_0-\frac{\pi}{2}+\delta h^2\right), \\
m(x)\cdot\mathrm{e}^{is_1}>0 \text{ and } m(x)\cdot\mathrm{e}^{is_2}>0 \Big\} \\
\subset B_r(\hat{z})\cap\left\{m\cdot\mathrm{e}^{is_0}\geqslant\cos(2\delta h^2)\right\},
\end{aligned}
$$

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so this implies

$$\left|B_{R/2}\cap\left\{m\cdot m_0\geqslant\frac{1}{2}\right\}\right|\gtrsim hr^2. \tag{2.8}$$

Case 2. Assume now that

$$|A_1|+|A_2|<2c|B_r|\left(1+\frac{h}{16\pi}\right).$$

Then using (2.7) we obtain

$$|A_1^{\mathrm{FT}}|-|A_1|+|A_2^{\mathrm{FT}}|-|A_2|>2c|B_r|\frac{h}{16\pi},$$

so either $|A_1^{\mathrm{FT}}|-|A_1|$ or $|A_2^{\mathrm{FT}}|-|A_2|$ is larger than half the right-hand side. We consider without loss of generality only the first case:

$$|A_1^{\mathrm{FT}}|-|A_1|>|B_r|\frac{ch}{16\pi}.$$

This implies a lower bound on the entropy dissipation $\nu(B_R)$ thanks to lemma 2.1. Specifically, we apply lemma 2.1 to $t=t_1$ and $\phi\in C_c^\infty(B_{2r(\hat z)}\times I_{2c}(s_0+\pi/2))$ such that

$$\mathbf{1}_{x\in B_r(\hat z)}\mathbf{1}_{s\in I_c(s_0+\pi/2)}\leqslant\phi(x,s)\leqslant\mathbf{1}_{x\in B_{(1+\varepsilon)r}(\hat z)}\mathbf{1}_{s\in I_{(1+\varepsilon)c}(s_0+\pi/2)},$$

and $|\partial_s\phi|\leqslant 2/(\varepsilon c)$, $|\nabla_x\phi|\leqslant 2/(\varepsilon r)$. We choose $\varepsilon=h/192\pi$ to ensure

$$\left|\left(B_{(1+\varepsilon)r}(\hat z)\times I_{(1+\varepsilon)c}\right)\setminus\left(B_r(\hat z)\times I_c\right)\right|\leqslant\frac{ch}{32\pi}|B_r|.$$

Since $|t_1|\leqslant 8r/(\delta h^2)$ and $B_{2r+|t_1|}(\hat z)\subset B_R$, we deduce $\nu(B_R)\gtrsim\delta^3h^{11}r\gtrsim h^{11}r$.

Similarly, using (2.3) we have two cases: either

$$\left|B_{R/2}\cap\left\{m\cdot m_0\leqslant-\frac{1}{2}\right\}\right|\gtrsim hr^2, \tag{2.9}$$

or $\nu(B_R)\gtrsim h^{11}r$. So gathering all cases, we see that either both (2.8) and (2.9) are satisfied, or $\nu(B_R)\gtrsim h^{11}r$, which is exactly the dichotomy of proposition 1.4.

3. Proof of proposition 1.5

To prove proposition 1.5, we briefly recall from [**14**] the notion of *Lagrangian representation* of an entropy solution m of the eikonal equation. In [**14**, **15**], the second author shows the existence of a finite non-negative Radon measure ω on the set of curves:

$$\begin{aligned}\Gamma=\big\{(\gamma,t_\gamma^-,t_\gamma^+)\colon 0\leqslant t_\gamma^-\leqslant t_\gamma^+\leqslant 1,\\ \gamma=(\gamma_x,\gamma_s)\in\mathrm{BV}((t_\gamma^-,t_\gamma^+);\Omega\times\mathbb{R}/2\pi\mathbb{Z}),\\ \gamma_x\text{is Lipschitz}\big\},\end{aligned}$$

with the following three properties:

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- for every $t \in (0,1)$, the pushforward of ω, restricted to the section $\Gamma(t) = \{(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma \colon t_\gamma^- < t < t_\gamma^+\}$, by the evaluation map $e_t \colon \gamma \mapsto \gamma(t)$ (a right-continuous representative of γ_s is always considered), is uniform on the 'epigraph' $E_m = \{m(x) \cdot \mathrm{e}^{is} > 0\}$, that is,

$$(e_t)_\sharp \left[\omega \lfloor \Gamma(t)\right] = \mathbf{1}_{m(x)\cdot \mathrm{e}^{is}>0} \,\mathrm{d}x\,\mathrm{d}s; \tag{3.1}$$

- the measure ω is concentrated on curves $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma$ solving the characteristic equation:

$$\dot{\gamma}_x(t) = \mathrm{e}^{i\gamma_s(t)} \quad \text{for a.e. } t \in (t_\gamma^-, t_\gamma^+); \tag{3.2}$$

- the entropy dissipation measure (1.4) disintegrates along the Lagrangian curves as

$$\nu(A) = \int_\Gamma \mu_\gamma(\gamma_x^{-1}(A))\,\mathrm{d}\omega(\gamma) \quad \text{for all measurable } A \subset \Omega, \tag{3.3}$$

  where $\mu_\gamma = |D_t \gamma_s|$, with the convention that a jump of γ_s from s^- to s^+ at time $t_0 \in (t_\gamma^-, t_\gamma^+)$ contributes $\mathrm{dist}_{\mathbb{R}/2\pi\mathbb{Z}}(s^-, s^+)\delta_{t=t_0}$ to the jump part of μ_γ (see [**14**, proposition 2.5]).

Moreover, the Lagrangian property (3.1) implies that ω is concentrated on curves γ such that $\gamma_x(t)$ is a Lebesgue point of m with $m(\gamma_x(t)) \cdot \mathrm{e}^{i\gamma_s(t^+)} > 0$, for a.e. $t \in (0,1)$ [**14**, lemma 2.7]. We denote by $\Gamma_g \subset \Gamma$ the full-measure subset of Lagrangian curves which satisfy that property together with the characteristic equation (3.2).

The proof of proposition 1.5 is based on two main tools. The first, lemma 3.1, is a dichotomy stating that either Lagrangian curves passing through a given set create a lot of dissipation, or one can find an almost-straight Lagrangian curve passing through that set. The second ([**12**, lemma 5.2], a slightly more precise version of [**14**, lemma 3.1], itself adapted from [**15**, lemma 22]) is another dichotomy: given an almost-straight Lagrangian curve, either the density of points at which m lies in the semi-circle indicated by the s-component of that curve is high, or a lot of dissipation must be created. The succession of these two dichotomies is reflected in the three alternatives in the conclusion of proposition 1.5. We first state and prove the first tool, and then proceed to the proof of proposition 1.5.

LEMMA 3.1. *For any $R > 0$ such that $B_R \subset \Omega$, any measurable set $A \subset B_R \times \mathbb{R}/2\pi\mathbb{Z}$, and any $\eta \in (0,1)$, we have either*

$$\nu(B_R) \gtrsim \frac{\eta}{R} \left|\{(x,s) \in A \colon m(x) \cdot \mathrm{e}^{is} > 0\}\right|, \tag{3.4}$$

or there exists a curve $\gamma \in \Gamma_g$ and a connected component J of $\gamma_x^{-1}(B_R)$ such that

$$J \cap \gamma^{-1}(A) \neq \emptyset \quad \textit{and} \quad \mu_\gamma(J) \leqslant \eta.$$

Proof of lemma 3.1. Assume that the second alternative of lemma 3.1 is not verified: for every curve $\gamma \in \Gamma_g$ and every connected component J of $\gamma_x^{-1}(B_R)$

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intersecting $\gamma^{-1}(A)$, we have $\mu_\gamma(J) > \eta$. Then we claim that

$$\mu_\gamma(\gamma_x^{-1}(B_R)) \gtrsim \frac{\eta}{R} T(\gamma), \quad T(\gamma) = \left|\{t \in (t_\gamma^-, t_\gamma^+) \colon \gamma(t) \in A\}\right|, \tag{3.5}$$

for all $\gamma \in \Gamma_g$. To prove (3.5), denote by $J_k = (t_k^-, t_k^+)$ the connected components of $\gamma_x^{-1}(B_R)$ which intersect $\gamma^{-1}(A)$. We show next that $\mu_\gamma(J_k) \gtrsim \eta|J_k|/R$ for all k. On the one hand, if $|J_k| \leqslant 4R$ then $\mu_\gamma(J_k) \gtrsim \eta|J_k|/R$ because $\mu_\gamma(J_k) > \eta$ by assumption. On the other, from the characteristic equation (3.2) and the definition of $\mu_\gamma = |D_t\gamma_s|$, we have the inequality:

$$|\gamma_x(t_2) - \gamma_x(t_1) - \mathrm{e}^{i\gamma_s(t_1)}(t_2 - t_1)| \leqslant \mu_\gamma([t_1, t_2])|t_2 - t_1|,$$

and we deduce that in any interval $J \subset (0, 1)$ such that $\gamma_x(J) \subset B_R$ and $|J| \geqslant 4R$, we must have $\mu_\gamma(J) \geqslant 1/2$. Therefore, if $|J_k| \geqslant 4R$, cutting J_k in disjoint subintervals of length between $4R$ and $8R$, we obtain that $\mu_\gamma(J_k) \gtrsim |J_k|/R \geqslant \eta|J_k|/R$. So we have

$$\mu_\gamma(\gamma_x^{-1}(B_R)) \geqslant \sum_k \mu_\gamma(J_k) \gtrsim \frac{\eta}{R} \sum_k |J_k|,$$

which implies (3.5) since $\gamma^{-1}(A) \subset \bigcup_k J_k$. From (3.5) and the fact that $\omega(\Gamma\backslash\Gamma_g) = 0$ we infer

$$\nu(B_R) = \int_\Gamma \mu_\gamma(\gamma_x^{-1}(B_R))\, \mathrm{d}\omega(\gamma) \gtrsim \frac{\eta}{R} \int_\Gamma T(\gamma)\, \mathrm{d}\omega(\gamma),$$

where the first equality comes from disintegration (3.3). Making use of the Lagrangian property (3.1) to rewrite the last expression, we see that it is precisely equal to the right-hand side of (3.4), which concludes the proof of lemma 3.1. □

Proof of proposition 1.5. We recall that $m_0 = \mathrm{e}^{is_0}$ and the sets $X_\pm$ are defined by

$$X_\pm = B_{R/2} \cap \{\pm m \cdot \mathrm{e}^{is_0} \geqslant 1/2\}.$$

For any $\hat{s} \in [s_0 - \pi/4, s_0 + \pi/4]$, we apply lemma 3.1 to $A(\hat{s}) = B_{R/2} \times I_\eta(\hat{s})$, where $I_\eta(\hat{s}) = [\hat{s} - \eta, \hat{s} + \eta]$. If $\eta \in (0, \pi/12)$ then we have $m(x) \cdot \mathrm{e}^{is} > 0$ for all $(x, s) \in X_+ \times I_\eta(\hat{s})$, and therefore,

$$\left|\{(x, s) \in A(\hat{s}) \colon m(x) \cdot \mathrm{e}^{is} > 0\}\right| \geqslant \eta|X_+|.$$

So, we have either $\nu(B_R) \gtrsim \eta^2|X_+|/R$, or there exists a curve $\gamma \in \Gamma_g$ and a connected component J of $\gamma_x^{-1}(B_R)$ intersecting $A(\hat{s})$ such that $\mu_\gamma(J) < \eta$. In that second case, applying [**12**, lemma 5.2] we deduce that either $\nu(B_R) \gtrsim \eta^3 R$ or $|B_R \cap \{m \cdot \mathrm{e}^{i\hat{s}} \geqslant -2\eta\}| \gtrsim \eta R^2$. We fix $\eta = 1/4$ and summarize the preceding

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discussion: for all $\hat{s} \in [s_0 - \pi/4, s_0 + \pi/4]$, we have

$$\nu(B_R) \gtrsim \frac{|X_+|}{R}, \quad \text{or } \nu(B_R) \gtrsim R, \quad \text{or } \left|B_R \cap \{m \cdot \mathrm{e}^{i\hat{s}} \geqslant -1/2\}\right| \gtrsim R^2.$$

Similarly, for all $\hat{s} \in [s_0 + 3\pi/4, s_0 + 5\pi/4]$, we have

$$\nu(B_R) \gtrsim \frac{|X_-|}{R}, \quad \text{or } \nu(B_R) \gtrsim R, \quad \text{or } \left|B_R \cap \{m \cdot \mathrm{e}^{i\hat{s}} \geqslant -1/2\}\right| \gtrsim R^2.$$

We conclude that we have either (1.10), or $\nu(B_R) \gtrsim R$, or

$$\left|B_R \cap \{m \cdot \mathrm{e}^{is} \geqslant -1/2\}\right| \gtrsim R^2,$$

for all $s \in [s_0 - \pi/4, s_0 + \pi/4] \cup [s_0 + 3\pi/4, s_0 + 5\pi/4]$. This corresponds exactly to the three alternatives in the statement of proposition 1.5. □

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