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GRADIENT ESTIMATES FOR HARMONIC FUNCTIONS ON MANIFOLDS WITH LIPSCHITZ METRICS

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ABSTRACT. We introduce a distributional Ricci curvature on complete smooth manifolds with Lipschitz continuous metrics. Under an assumption on the volume growth of geodesics balls, we obtain a gradient estimate for weakly harmonic functions if the distributional Ricci curvature is bounded below.

1. **Introduction.** The study of harmonic functions has a long history. On Riemannian manifolds, Yau [12] proved a fundamental gradient estimate for harmonic functions in terms of the lower bound of Ricci curvature. As one of the many applications of Yau's gradient estimate, we have a generalization of Liouville's theorem to the effect that a positive harmonic function on a complete Riemannian manifold of nonnegative Ricci curvature is constant.

Recently there is an increasing interest in the study of harmonic functions (and harmonic mappings) on nonsmooth spaces. In this paper, we study the behavior of weakly harmonic functions on smooth manifolds with Lipschitz Riemannian metrics. We introduce the concept of distributional Ricci curvature and, under a mild volume growth condition, we derive a gradient estimate for positive (weakly) harmonic functions in terms of the lower bound of distributional Ricci curvature. Liouville's theorem also follows from our gradient estimate. The main result of our work can be stated as follows.

THEOREM 1.1. *Let M be a complete differentiable manifold with a Lipschitz Riemannian metric such that volumes of geodesic balls satisfies sub-quadratic exponential growth condition, i.e.,*

$$\mathrm{Vol}\big(B(R)\big) \leq e^{o(R^2)}.$$

Assume that the distributional Ricci curvature is bounded from below by $-(n-1)K$ for some nonnegative constant K. If u is positive weakly harmonic function on M, then

$$\left|\frac{\nabla u}{u}\right| \leq \sqrt{2n(n-1)K}$$

In particular, if the distributional Ricci curvature is nonnegative, then any positive weakly harmonic function is constant.

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Our argument is based on an early observation attributed to R. Schoen in Yau [13]. One feature of this argument is that it avoids using the maximum principle, thus allowing us to handle weakly harmonic functions on manifolds with Lipschitz metrics.

Let us make a few remarks on the Ricci curvature on a smooth manifold with a Lipschitz metric. For Riemannian metrics which are not C^2, it is natural from analytical point of view to understand the Ricci curvature in the sense of distribution. Our basic assumption is that the Ricci curvature is bounded from below by a constant in the sense of distribution. From a well-known result in distribution theory this assumption implies that the distributional Ricci curvature is in fact a measure. Such a definition of distributional Ricci curvature is enough for us to carry out the analytical part of our argument. However, we do not know whether a constant lower bound on the distributional Ricci curvature is sufficient for obtaining an exponential volume growth for geodesic balls. In the case of smooth metrics, such an implication is well known. Unlike Alexandrov's interpretation of sectional curvature through triangle comparison inequalities, to our knowledge so far there is no adequate interpretation for Ricci curvature in nonsmooth cases. In this respect, Lin [9] has recently studied the Liouville type theorems for L-harmonic functions, *i.e.* solutions of the elliptic equation

$$Lu = \frac{\partial}{\partial x^i}\left(A^{ij}(x)\frac{\partial}{\partial x^j}\right)u = 0,$$

where $A^{ij}(x)$ are bounded measurable functions and

$$\lambda_1 I \leq \left(A^{ij}(x)\right) \leq \lambda_2 I$$

for positive constants $\lambda_1 < \lambda_2$, under some assumptions on the asymptotic behavior of L and on the growth of volume of geodesic balls. We also point out that recent works of Colding [3] [4] provide an interesting integral version of the Toponogov comparison theorem for Ricci curvature.

The rest of the paper is arranged as follows. In Section 2 we define weakly harmonic functions and distributional Ricci curvatures. In Section 3 we follow R. Schoen's suggestion presented in Yau [13] and prove an integrated gradient estimate for positive harmonic functions on smooth Riemannian manifold with Ricci curvature bounded from below. In Section 4 we show how to prove the pointwise gradient estimate for weakly harmonic functions using the integrated gradient estimate and distributional Ricci curvature. Liouville's theorem for manifolds with nonnegative Ricci curvature follows immediately. We also prove a variation of Liouville's theorem for weakly harmonic functions satisfying certain growth conditions. Finally in Section 5 we give two examples where our theory can be applied.

2. **Weakly harmonic functions and Ricci curvature.** Let M be an n-dimensional smooth differentiable manifold and g a Riemannian metric on M. Throughout this paper we assume that the metric g is locally Lipschitz on M.

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Recall that in local coordinates $x = \{x^i\}$ the Riemannian volume $v_g = \sqrt{\det g}\, dx$ and the Laplace Beltrami operator

$$\Delta = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g}\, g^{ij} \partial_j),$$

where $g = \{g_{ij}\}$ in local coordinates, $g^{-1} = \{g^{ij}\}$. By definition, a function $u \in W^{1,2}_{\mathrm{loc}}(M)$ is (weakly) harmonic on M if it satisfies the equation $\Delta h = 0$ in the weak sense, *i.e.*, for any test function $\phi \in C^\infty_c(M)$ with support in a local coordinate neighborhood,

$$\int_M g^{ij} \partial_i \phi \partial_j u \, dv_g = 0.$$

According to standard theory of elliptic equations, for uniformly elliptic operators in divergence form with bounded measurable coefficients, generalized solutions are in fact Hölder continuous (Gilbarg and Trudinger [6], p. 205); if the coefficients are continuous, then the solutions are in the Sobolev space $W^{2,p}_{\mathrm{loc}}(M)$ for $p > 1$ (Gilbarg and Trudinger [6], p. 241). Furthermore, if the coefficients are Hölder continuous with exponent α, then solutions are $C^{1,\alpha}_{\mathrm{loc}}(M)$ (Gilbarg and Trudinger [6], p. 211). Hence, under our assumption on the metric g, a weakly harmonic function $u \in C^{1,\alpha}_{\mathrm{loc}}(M) \cap W^{2,p}_{\mathrm{loc}}(M)$, for $p > 1$.

We now explain the meaning of the Ricci curvature of (M, g) for a locally Lipschitz metric. Recall the local expression for the Riemannian curvature tensor:

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{jk} \Gamma^l_{im} - \Gamma^m_{ik} \Gamma^l_{km}, \tag{1}$$

where

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \{\partial_j g_{li} + \partial_i g_{lj} - \partial_l g_{ij}\} \tag{2}$$

are Christoffel symbols. First of all since we assume that the metric is Lipschitz, the Christoffel symbols Γ^l_{ij} are locally bounded measurable functions. It follows that each component R^l_{ijk} is a distribution of order 1 (for the definition of the order of a distribution, see Hörmander [8], 33–34.) In fact R^i_{ijk} can be written as a linear combination of bounded measurable functions and partial derivatives of bounded measurable functions. The Ricci curvature components are given by $R_{ij} = g_{lk} R^l_{ikj}$. Because g_{lk} are locally Lipschitz, we see that the Ricci curvature components are again distributions of order 1. Now if $X = a^i \partial_i$ is a smooth vector field with compact support, then $\mathrm{Ric}(X, X) = a^i a^j R_{ij}$ is a distribution of order 1 independent of the choice of local coordinates. We call $\mathrm{Ric}(X, X)$ the distributional Ricci curvature along the vector field X. Of course when the metric g is smooth the distributional Ricci curvature agrees with the usual Ricci curvature.

For a smooth function ϕ, we use $\mathrm{Ric}(X, X)(\phi)$ or $\int_M \mathrm{Ric}(X, X)\phi$ to denote the value of the distribution (as a linear functional) on the test function ϕ.

DEFINITION 2.1. Let M be a differentiable manifold with a locally Lipschitz Riemannian metric g. We say that the Ricci curvature of (M, g) is bounded from below by a

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constant $-(n-1)K$ if for any smooth vector field X with compact support, the (distributional) Ricci curvature $\mathrm{Ric}(X,X) \geq -(n-1)K|X|^2$ on M in the sense of distribution, *i.e.*, for any nonnegative smooth function ϕ with compact support,

$$\mathrm{Ric}(X,X)(\phi) \geq -(n-1)K \int_M \phi |X|^2 \, dv_g.$$

As a matter of fact, since $\mathrm{Ric}(X,X)$ is a linear combination of bounded measurable functions and derivatives of such functions, we can allow the test function $\phi \in W^{1,1}_{\mathrm{loc}}(M)$ in the above inequality.

Now suppose that the (distributional) Ricci curvature of (M,g) is bounded from below by K. Then the distribution $\mathrm{Ric}(X,X) - K|X|^2 v_g \geq 0$. Every positive distribution is a positive measure (see Hörmander [8], p. 38). It follows that $\mathrm{Ric}(X,X)$ is a measure on M. Now in local coordinates,

$$R_{ij} = \mathrm{Ric}(\partial_i, \partial_j) = \frac{1}{4}\{\mathrm{Ric}(\partial_i + \partial_j, \partial_i + \partial_j) - \mathrm{Ric}(\partial_i - \partial_j, \partial_i - \partial_j)\}$$

therefore each component R_{ij} is (locally) a measure.

3. **An inequality on integrated gradient.** We will prove gradient estimates for positive weakly harmonic functions through approximating the Lipschitz metric by a sequence of smooth metrics and approximating harmonic functions by a sequence of smooth (not necessarily harmonic) functions. For this purpose we need an integral form of gradient estimates. The advantage of this integral version is that one does not need the maximum principle nor need to worry about the cut locus. An argument with these features is especially useful in a nonsmooth setting. We would like to point out that the main inequality of this section (Theorem 3.1 below) holds for any smooth functions which are bounded from below by positive constants. In the next section we show how this inequality can be used together with an assumption of exponential volume growth to obtain a gradient estimate for positive weakly harmonic functions.

THEOREM 3.1. *Let (M,g) be a smooth Riemannian manifold whose Ricci curvature is bounded from below by $-(n-1)K$ and u a positive smooth function on M. Then for any smooth positive function ϕ with compact support and sufficiently large p we have*

$$\begin{aligned}\left(\int_M \phi^{2p} |\nabla \log u|^p\right)^{1/p} &\leq 2n(n-1)K\left\{\int_M \phi^{2p}\right\}^{1/p} \\ &\quad + 8\sqrt{np}\left\{\int \phi^{2p}\left|\frac{\triangle u}{u}\right|^p\right\}^{1/p} \\ &\quad + 4np\left\{\int_M |\nabla\phi|^{2p}\right\}^{1/p}.\end{aligned}$$

PROOF. Let $F = u^{-2}|\nabla u|^2$. Using Weitzenböck's formula and the inequality $|\mathrm{Hess}\, f|^2 \geq n^{-1}(\triangle f)^2$ we have (see also Yau [12])

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$$\begin{aligned}
\triangle F &= \triangle|\nabla \log u|^2 \\
&= 2|\operatorname{Hess}(\log u)|^2 + 2\operatorname{Ric}(\nabla \log u, \nabla \log u) + 2\nabla \log u \nabla \triangle \log u \\
&\geq \frac{2}{n}(\triangle \log u)^2 + 2\operatorname{Ric}(\nabla \log u, \nabla \log u) + 2\nabla \log u \nabla \triangle \log u \\
&= \frac{2}{n}(u^{-1}\triangle u - F)^2 + 2\operatorname{Ric}(\nabla \log u, \nabla \log u) + 2u^{-1}\nabla u \nabla (u^{-1}\triangle u - F) \\
&= \frac{2}{n}F^2 - 2u^{-1}\nabla u \nabla F + \frac{2}{n}(u^{-1}\triangle u)^2 + 2u^{-2}\nabla u \nabla \triangle u \\
&\quad - \left(\frac{4}{n} + 2\right) F u^{-1} \triangle u + 2\operatorname{Ric}(\nabla \log u, \nabla \log u).
\end{aligned}$$

We multiply the above inequality by $\phi^{2p}F^{p-2}$ and integrate with respect to the Riemannian volume. This gives the inequality

$$\begin{aligned}
\frac{2}{n}\int_M \phi^{2p}F^p &\leq \int_M \phi^{2p}F^{p-2}\triangle F + 2\int_M \phi^{2p}F^{p-2}u_\epsilon^{-1}\nabla u \nabla F \\
&\quad - 2\int_M \phi^{2p}F^{p-2}u^{-2}nu\nabla \triangle u - \frac{2}{n}\int_M \phi^{2p}F^{p-2}u^{-2}(\triangle u)^2 \\
&\quad + \left(2 + \frac{4}{n}\right)\int_M \phi^{2p}F^{p-1}u^{-2}\triangle u \\
&\quad - 2\int_M \phi^{2p}F^{p-2}u^{-2}\operatorname{Ric}(\nabla u, \nabla u) \\
&:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned} \tag{3}$$

We will leave I_4, I_5, and I_6 alone and estimate integrals I_1, I_2, and I_3. We use integration by parts in I_1 and obtain

$$\begin{aligned}
I_1 &= \int_M \phi^{2p}F^{p-2}\triangle F \\
&= -2p\int_M \phi^{2p-1}F^{p-2}\nabla\phi\nabla F - (p-2)\int_M \phi^{2p}F^{p-3}|\nabla F|^2.
\end{aligned}$$

Applying the inequality $2ab \leq \lambda^{-1}a^2 + \lambda b^2$ with $\lambda = 2$ to the first term on the right-hand side, we obtain

$$\begin{aligned}
2\int_M \phi^{2p-1}F^{p-2}\nabla\phi\nabla F &= 2\int_M \{\phi^p|\nabla F|F^{\frac{p-3}{2}}\}\{\phi^{p-1}|\nabla\phi|F^{\frac{p-1}{2}}\} \\
&\leq \frac{1}{2}\int_M \phi^{2p}F^{p-3}|\nabla F|^2 + 2\int_M \phi^{2p-2}F^{p-1}|\nabla\phi|^2.
\end{aligned}$$

Hence we have

$$I_1 \leq -\left(\frac{p}{2} - 2\right)\int_M \phi^{2p}F^{p-3}|\nabla F|^2 + 2p\int_M \phi^{2p-2}F^{p-1}|\nabla\phi|^2. \tag{4}$$

For I_2, we have

$$I_2 = 2\int_M \phi^{2p}F^{p-2}u^{-1}\nabla u \nabla F \tag{5}$$

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$$\begin{aligned}
&\le 2\int_M \phi^{2p}F^{p-2+\frac{1}{2}}|\nabla F| \\
&= 2\int_M \{\phi^p F^{\frac{p-3}{2}}|\nabla F|\}\{\phi^p F^{\frac{p}{2}}\} \\
&\le n\int_M \phi^{2p}F^{p-3}|\nabla F|^2 + \frac{1}{n}\int_M \phi^{2p}F^p.
\end{aligned}$$

Note that the last term is half of the left-hand side of (3). We now estimate I_3. Using integration by parts we have

$$\begin{aligned}
I_3 &= -2\int_M \phi^{2p}F^{p-2}u^{-2}\nabla u\nabla\triangle u \\
&= 2\int_M \phi^{2p}F^{p-2}u^{-2}(\triangle u)^2 - 4\int_M \phi^{2p}F^{p-1}u^{-1}\triangle u \\
&\quad + 4p\int_M \phi^{2p-1}F^{p-2}u^{-2}\nabla\phi\nabla u\triangle u \\
&\quad + 2(p-2)\int_M \phi^{2p}F^{p-3}u^{-2}\nabla F\nabla u\triangle u.
\end{aligned} \tag{6}$$

The third term on the right-hand side of (6) can be estimated as follows:

$$\begin{aligned}
2\int_M \phi^{2p-1}F^{p-2}u^{-2}\nabla\phi\nabla u\triangle u &\le 2\int_M \phi^{2p-1}F^{p-2+\frac{1}{2}}|\nabla\phi|u^{-1}\triangle u \\
&\le \int_M \phi^{2p-2}F^{p-1}|\nabla\phi|^2 + \int_M \phi^{2p}F^{p-2}u^{-2}(\triangle u)^2.
\end{aligned}$$

The fourth term on the right-hand side of (6) can be estimated as follows:

$$\begin{aligned}
2\int_M \phi^{2p}F^{p-3}u^{-2}\nabla u\nabla F\triangle u &\le 2\int_M \phi^{2p}F^{p-3+\frac{1}{2}}u^{-1}|\nabla F|\,|\triangle u| \\
&\le \frac{1}{4}\int_M \phi^{2p}F^{p-3}|\nabla F|^2 + 4\int_M \phi^{2p}F^{p-2}u^{-2}(\triangle u)^2.
\end{aligned} \tag{7}$$

Hence we have

$$\begin{aligned}
I_3 &\le 6(p-1)\int_M \phi^{2p}F^{p-2}u^{-2}(\triangle u)^2 + 4\int_M \phi^{2p}F^{p-1}u^{-1}|\triangle u| \\
&\quad + \frac{p-2}{4}\int_M \phi^{2p}F^{p-3}|\nabla F|^2 + 2p\int_M \phi^{2p-2}F^{p-1}|\nabla\phi|^2.
\end{aligned} \tag{8}$$

Now from (3), (4), (5), and (8) we have

$$\begin{aligned}
\frac{1}{n}\int \phi^{2p}F^p \le &-\Big(\frac{p}{4} - \frac{3}{2} - n\Big)\int_M \phi^{2p}F^{p-3}|\nabla F|^2 \\
&+ 4p\int_M \phi^{2p-2}F^{p-1}|\nabla\phi|^2 \\
&+ \Big(6 + \frac{4}{n}\Big)\int_M \phi^{2p}F^{p-1}u^{-1}|\triangle u| \\
&+ \Big(6p - 6 - \frac{2}{n}\Big)\int_M \phi^{2p}F^{p-2}u^{-2}(\triangle u)^2 \\
&- 2\int_M \phi^{2p}F^{p-2}u^{-2}\,\mathrm{Ric}(\nabla u, \nabla u).
\end{aligned} \tag{9}$$

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If we assume that $p \geq 4n + 6$, the first term on the right-hand side can be dropped. We will use Hölder's inequality on the remaining four terms. Let

$$G = \left(\int_M \phi^{2p} F^p \right)^{1/p}$$

for the sake of simplicity. Using the lower bound for the Ricci curvature,

$$-\int_M \phi^{2p} F^{p-2} u^{-2} \operatorname{Ric}(\nabla u, \nabla u) \leq (n-1)K \int_M \phi^{2p} F^{p-1} \leq (n-1)KG^{p-1} \left\{ \int_M \phi^{2p} \right\}^{1/p}. \tag{10}$$

We also have

$$\int_M \phi^{2p-2} F^{p-1} |\nabla \phi|^2 \leq G^{p-1} \left\{ \int_M |\nabla \phi|^{2p} \right\}^{1/p},$$

$$\int_M \phi^{2p} F^{p-1} u^{-1} |\triangle u| \leq G^{p-1} \left\{ \int_M \phi^{2p} \left| \frac{\triangle u}{u} \right|^p \right\}^{1/p},$$

$$\int_M \phi^{2p} F^{p-2} u^{-2} (\triangle u)^2 \leq G^{p-2} \left\{ \int_M \phi^{2p} \left| \frac{\triangle u}{u} \right|^p \right\}^{2/p}.$$

Using these estimates in (9) we have a quadratic inequality for G:

$$\frac{1}{n} G^2 \leq 4p \left\{ \int_M |\nabla \phi|^{2p} \right\}^{1/p} G + 10 \left\{ \int_M \phi^{2p} \left| \frac{\triangle u}{u} \right|^p \right\}^{1/p} G + 2(n-1)K \left\{ \int_M \phi^{2p} \right\}^{1/p} G + 6p \left\{ \int_M \phi^{2p} \left| \frac{\triangle u}{u} \right|^p \right\}^{2/p}.$$

This inequality has the form $n^{-1}G^2 \leq aG + b$, from which we have $G \leq \sqrt{nb} + na$. It follows that

$$G \leq 2n(n-1)K \left\{ \int_M \phi^{2p} \right\}^{1/p} + 8\sqrt{np} \left\{ \int \phi^{2p} \left| \frac{\triangle u}{u} \right|^p \right\}^{1/p} + 4np \left\{ \int_M |\nabla \phi|^{2p} \right\}^{1/p}.$$

This completes the proof of Theorem 4.1. ∎

4. **Gradient estimate for positive weakly harmonic functions.** In this section we study positive weakly harmonic functions on a complete differentiable manifold with a Lipschitz continuous Riemannian metric. We prove a gradient estimate for such functions under the conditions that the distributional Ricci curvature is bounded from below by a constant and that the volume of geodesic balls grow at most exponentially. As a consequence we obtain Liouville's theorem for such harmonic functions when the Ricci curvature is nonnegative.

PROPOSITION 4.1. *Let M be a complete differentiable manifold with a Lipschitz continuous Riemannian metric g. Suppose that the distributional Ricci curvature is bounded from below by $-(n-1)K$. If u is a positive weakly harmonic function on M and ϕ a smooth positive function with compact support, then*

$$\left(\int_M \phi^{2p} |\nabla \log u|^p \right)^{1/p} \leq 2n(n-1)K \left\{ \int_M \phi^{2p} \right\}^{1/p} + 4np \left\{ \int_M |\nabla \phi|^{2p} \right\}^{1/p}. \tag{11}$$

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PROOF. The inequality basically follows from Theorem 3.1, because we have $\Delta u = 0$. But since the metric is assumed Lipschitz, we need an approximation argument. Let (U_i, ϕ_i) be a paracompact atlas of M, where $\phi_i \colon U_i \longrightarrow \mathbb{R}^n$ are diffeomorphisms onto their images $\phi_i(U_i)$. Let ρ be a mollifier, *i.e.*, a nonnegative function in $C^\infty(\mathbb{R}^n)$ vanishing outside the unit ball $B_1(0)$ and satisfying

$$\int_{\mathbb{R}^n} \rho(y)\, dy = 1.$$

Let $\{\psi_i\}$ be a partition of unity subordinate to the covering $\{U_i\}$. Define the mollified function

$$u_\epsilon(x) = \sum_i \frac{1}{\epsilon^n} \int_{\mathbb{R}^n} \rho\left(\frac{\phi_i(x) - y}{\epsilon}\right)(\psi_i u)\big(\phi_i^{-1}(y)\big)\, dy.$$

The metric can also be mollified. More precisely, let $\{x_i^\alpha\}$ be the Cartesian coordinate functions on $\phi_i(U_i)$ and denote the metric matrix on U_i by $g_{i,\alpha\beta}$. Suppose that X, Y are two vector fields at $x \in M$, then the mollified metric g_ϵ is defined by

$$\begin{aligned} g_\epsilon(X, Y) = \sum_{i,j,a,b,c,d} & X(\phi_j^a)(x) Y(\phi_j^b)(x) \partial_a(\phi_i^c \circ \phi_j^{-1}) \partial_b(\phi_i^d \circ \phi_j^{-1}) \\ & \times \frac{1}{\epsilon^n} \int_{\mathbb{R}^n} \rho\left(\frac{\phi_i(x) - y}{\epsilon}\right) g(\partial_c, \partial_d)\big(\phi_i^{-1}(y)\big)\, dy. \end{aligned}$$

We set

$$F = |\nabla \log u|_g^2, \quad F_\epsilon = |\nabla_\epsilon \log u_\epsilon|_{g_\epsilon}^2.$$

Since mollified functions and metrics are smooth, the part of the proof of Theorem 3.1 holds up to (9). Thus (9) holds with g and u replaced by g_ϵ and u_ϵ. Let us call the resulting inequality $(9)_\epsilon$. As we pointed out in Section 1, we have $u \in W^{2,p}_{\mathrm{loc}}(M)$, since u is weakly harmonic. By basic properties of mollifiers (Gilbarg and Trudinger [6], 147–155), we have

$$\begin{aligned} u_\epsilon &\longrightarrow u \quad \text{in } W^{2,p}_{\mathrm{loc}}(M), \\ \nabla_\epsilon u_\epsilon &\longrightarrow \nabla u \quad \text{in } W^{1,p}_{\mathrm{loc}}(M), \end{aligned}$$

and

$$\Delta_\epsilon u_\epsilon \longrightarrow \Delta u \quad \text{in } L^p_{\mathrm{loc}}(M).$$

Thus in the term on the left-hand side and the first four terms on the right-hand side of $(9)_\epsilon$ we can pass to the limit as $\epsilon \longrightarrow 0$. Of course the third and fourth terms on the right-hand side converge to zero because u is weakly harmonic. We need to study the limit of the last term as $\epsilon \longrightarrow 0$. We will show that

$$\liminf_{\epsilon \longrightarrow 0} (-2) \int_M \phi^{2p} F_\epsilon^{p-2} u_\epsilon^{-2} \operatorname{Ric}_\epsilon(\nabla_\epsilon u_\epsilon, \nabla_\epsilon u_\epsilon) \le 2(n-1)K \int_M \phi^{2p} F^{p-1}.$$

In a smooth local coordinate system $x = \{x^i\}$, the expression under the limit can be written as

$$\int_M H_{\epsilon,ij} \operatorname{Ric}_\epsilon(\partial_i, \partial_j)\, dx, \tag{12}$$

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where

$$H_{\epsilon,ij} = -2\phi^{2p}F_\epsilon^{p-2}u_\epsilon^{-2}\sqrt{g_\epsilon}\nabla_{\epsilon,i}u_\epsilon\nabla_{\epsilon,j}u_\epsilon.$$

Again from basic properties of mollifiers and the fact that $u \in W^{2,p}_{\mathrm{loc}}(M)$, we have

$$H_{\epsilon,ij} \longrightarrow H_{ij} \quad \text{in } W^{1,p}_{\mathrm{loc}}(M) \tag{13}$$

where obviously

$$H_{ij} = -2\phi^{2p}F^{p-2}u^{-2}\sqrt{g}\nabla_i u \cdot \nabla_j u.$$

On the other hand, for the mollified Riemannian metric we have

$$g_{\epsilon,ij} \longrightarrow g_{ij} \quad \text{in } C^{1,0}_{\mathrm{loc}}(M). \tag{14}$$

From the formula for the Ricci curvature $\mathrm{Ric}_\epsilon(\partial_i,\partial_j)$ in terms of the metric matrix $\{g_{\epsilon,ij}\}$ (see (1) and (2)) we know that it involves at most second derivatives of the metric matrix $g_{\epsilon,ij}$. Therefore for the terms in (12) involving second derivatives of the metric matrix we can shift one derivative from $\mathrm{Ric}_{\epsilon,ij}$ to $H_{\epsilon,ij}$ by integration by parts. Having done this, we can pass to the limit as $\epsilon \longrightarrow 0$ by virtue of (13) and (14). The resulting limit is just the definition of the action of $\mathrm{Ric}(\partial_i,\partial_j)$ on H_{ij}. Note that $H_{ij} \in W^{1,p}_{\mathrm{loc}}(M)$ and that $\mathrm{Ric}(\partial_i,\partial_j)$ is a distribution of order 1. Therefore the action of $\mathrm{Ric}(\partial_i,\partial_j)$ on H_{ij} makes sense. Now we can write

$$\int_M H_{\epsilon,ij}\,\mathrm{Ric}_\epsilon(\partial_i,\partial_j)\,dx \longrightarrow \int_M H_{ij}\,\mathrm{Ric}(\partial_i,\partial_j)\,dx,$$

where the right-hand side is understood in the sense of distribution. By the assumption that the distributional Ricci curvature is bounded from below by $-(n-1)K$ (see Definition 2.1) and the definition of H_{ij} we have

$$\begin{aligned}\int_M H_{ij}\,\mathrm{Ric}(\partial_i,\partial_j)\,dx &\le 2(n-1)K\int_M \phi^{2p}F^{p-2}u^{-2}|\nabla u|^2 \\ &\le 2(n-1)K\int_M \phi^{2p}F^{p-1}.\end{aligned} \tag{15}$$

Up to this point we have shown that (9) in the proof of Theorem 3.1 holds with the last term there replaced by the last term of (15). We can now follow the proof of Theorem 3.1 to complete the proof of the present proposition. ∎

Now we are ready to prove our main result.

THEOREM 4.2. *Let M be a complete differentiable manifold with a Lipschitz Riemannian metric such that volumes of geodesic balls satisfies sub-quadratic exponential growth condition, i.e.,*

$$\mathrm{Vol}\big(B(R)\big) \le e^{o(R^2)}.$$

Assume that the distributional Ricci curvature is bounded from below by $-(n-1)K$ for some nonnegative constant K. If u is positive weakly harmonic function on M, then

$$\left|\frac{\nabla u}{u}\right| \le \sqrt{2n(n-1)K} \tag{16}$$

In particular, if the distributional Ricci curvature is nonnegative, then any positive weakly harmonic function is constant.

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PROOF. Let ϕ be a smooth nonnegative cut-off function on M such that

$$\begin{aligned}\phi(x) &= 1 \quad \text{for } x \in B(R)\\ \phi(x) &\in [0,1] \quad \text{for } x \in B(2R) \setminus B(R)\\ \phi(x) &= 0 \quad \text{for } x \in M \setminus B(2R)\\ |\nabla\phi(x)| &\le CR^{-1} \quad \text{for } x \in M,\end{aligned}$$

where as usual $B(R)$ and $B(2R)$ are the balls centered at a fixed point $x_0 \in M$ with radius R and $2R$ respectively. From (11) we have

$$\Big(\int_{B(R)} F^p\Big)^{1/p} \le 2n(n-1)K\,\mathrm{Vol}\big(B(2R)\big)^{1/p} + \frac{4Cnp}{R^2}\,\mathrm{Vol}\big(B(2R)\big)^{1/p}.$$

Fix a positive integer N. Set $p = N \log \mathrm{Vol}\big(B(2R)\big)$ and let $R \longrightarrow \infty$. Using the volume growth condition and the definition of F we have

$$\sup_M \Big|\frac{\nabla u}{u}\Big|^2 \le 2n(n-1)Ke^{1/N}.$$

Letting $N \longrightarrow \infty$ we obtain (16). ∎

REMARK 4.3. It is well know if the metric is smooth then $\mathrm{Ric} \ge -(n-1)K$ implies

$$\mathrm{Vol}\big(B(R)\big) \le C_n R^n e^{\sqrt{(n-1)K}R}$$

for a constant C_n. In this case the volume growth condition in the above theorem is implied by the condition on the lower bound of the Ricci curvature. In the nonsmooth case, we do not know whether a constant lower bound on the distributional Ricci curvature implies the above volume growth condition.

For harmonic functions which may not be bounded from one side but with some growth condition, the following results are well known. On a complete smooth Riemannian manifold with nonnegative Ricci curvature, Cheng [2] proved that any sublinear growth harmonic function must be a constant. For manifolds with nonnegative Ricci outside a compact set, Donnelly [5] observed that the space of bounded harmonic functions is finite dimensional and S. Y. Cheng proved that the same result holds for the space of positive harmonic functions. Later, Li and Tam [10] [11] studied systematically harmonic functions with various growth conditions. In the nonsmooth case, we have

THEOREM 3.3. *Let M be a complete differentiable manifold with a Lipschitz Riemannian metric. Assume that the distributional Ricci curvature is nonnegative. Let $r(x)$ be the distance from x to some fixed point $q \in M$.*

(1) Suppose that

$$\mathrm{Vol}\big(B(R)\big) \le e^{CR^{2(1-\alpha)}}$$

for some positive constant C and $\alpha \in [0,1)$. If a weakly harmonic function u satisfies

$$|u(x)| = o\big(r(x)^\alpha\big),$$

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then u is a constant.

(2) Suppose that

$$\mathrm{Vol}\big(B(R)\big) \le CR^k$$

for some $k > 0$. *If a weakly harmonic function u satisfies*

$$|u(x)| = o\left(\frac{r(x)}{\sqrt{\log r(x)}}\right),$$

then u is a constant.

PROOF. For any $R > 0$, the function $\sup_{B(2R)} u - u + 1$ is a positive weakly harmonic function on $B(2R)$. Let ϕ be the cut-off function with support in $B(2R)$ as defined before. By Theorem 3.1, we have for any fixed $p > 4n + 6$

$$\left(\int_M \phi^{2p} \left|\nabla \log\big(\sup_{B(2R)} u - u + 1\big)\right|^{2p}\right)^{1/p} \le 4Cnp\left(\int_M |\nabla\phi|^{2p}\right)^{1/p}.$$

This implies that

$$\left(\int_{B(R)} |\nabla u|^{2p}\right)^{1/p} \le 4CnpR^{-2}V\big(B(2R)\big)^{1/p}\big(\sup_{B(2R)} u - \inf_{B(2R)} u + 1\big)^2.$$

Now we let $p = R^{2(1-\alpha)}$ in case (1) and $p = \log R$ in case (2), then let R go to infinity. We see immediately in both cases

$$\sup_M |\nabla u|^2 = 0.$$

Hence u must be a constant. ∎

4. **Examples.** In this section we give two examples of smooth manifolds with Lipschitz Riemannian metrics for which the above theory applies.

EXAMPLE 5.1. We consider radially symmetric manifolds. Without loss of generality we consider the case $n = 2$. Let $M = \mathbb{R}^2$ with a radially symmetric Riemannian manifold given by

$$g = dr^2 + G(r)^2\, d\theta^2.$$

Fix a positive constant K. To make the matter simple, we assume that G is smooth in a neighborhood of 0, $G(0) = 0$, $G'(0) = 1$, and $-G''/G \ge -K$ in a neighborhood of 0 so that M has curvature $\ge -K$ in the ordinary sense. Now let ϕ be defined by $G = e^{\phi}$. We assume that ϕ is a concave function such that $|\phi'|^2 \le K$ for a fixed constant K. Thus ϕ is locally Lipschitz but in general not $C^{1,\alpha}$ for any $0 < \alpha < 1$ let alone C^2. Note that $\phi(r) \sim \log r$ when $r \longrightarrow 0$. Now it is clear that $(\mathbb{R}^2, g)$ has a Lipschitz (but not necessarily C^2) Riemannian metric. We now verify that that $\mathrm{Ric}_M \ge -K$. It is enough to verify this outside a neighborhood of 0. When the metric is smooth the curvature is given by $-G''/G$ in the local coordinates $\{r, \theta\}$. Hence, according to our definition, M has distributional (Ricci) curvature $\ge -K$ if

$$\int_{\mathbb{R}^2} G' f_r\, dr\, d\theta \ge -K \int_{\mathbb{R}^2} fG\, dr\, d\theta \tag{17}$$

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for all smooth, nonnegative functions $f = f(r,\theta)$ with compact support in $M \setminus \{0\} = \mathbb{R}^2 \setminus \{0\}$. Substituting $G = e^{\phi}$ and integrating by parts we have

$$\begin{aligned}\int_{\mathbb{R}^2} G' f_r \, dr\, d\theta &= \int_{\mathbb{R}^2} \phi' e^{\phi} f_r \, dr\, d\theta \\ &= -\int_{\mathbb{R}^2} f e^{\phi} \, d\phi' \, d\theta - \int_{\mathbb{R}^2} f |\phi'|^2 e^{\phi} \, dr\, d\theta.\end{aligned}$$

Since ϕ is concave, $d\phi'$ is a nonpositive measure. Therefore the first term is nonnegative. Using the hypothesis that $|\phi'|^2 \le K$ the second term is bounded from below by $-K\int_{\mathbb{R}^2} fG$. This shows (17).

From the assumption on ϕ, there is a constant C such that $\phi(r) \le Cr$. Hence

$$\mathrm{Vol}\big(B(R)\big) = \int_{B(R)} e^{\phi(r)} \, dr\, d\theta \le C_1 e^{CR}.$$

Therefore Theorem 1.1 applies.

If we take the ϕ to be $\phi(r) = \log r$ for $r \le r_0$ and $\phi(r) = \log r_0$ for $r \ge r_0$ for some $r_0 > 0$. Then we can take $K = 0$. This gives an example of Lipschitz metric with nonnegative (Ricci) curvature and every nonnegative weakly harmonic function must be constant. In fact the curvature is a nonnegative measure concentrated the circle $r = r_0$.

EXAMPLE 5.2. Let M be a connected smooth n-dimensional manifold. Let $\{g_i\}$ be a sequence of complete Riemannian metrics on M. Suppose that there exists a sequence of compact subdomains $(\Omega_k, g_i) \subseteq (M, g_i)$ which exhausts M, such that

1. $|\text{sectional curvature}|_{(\Omega_k, g_i)} \le K^2(k)$;
2. the injectivity radius of $(\Omega_k, g_i) > \delta(k)$;
3. the volume $\mathrm{Vol}_{g_i}(\Omega_k) \le V(k)$;

where $V(k)$ and $\delta(k)$ are positive numbers independent of i. According to the theorems of Green and Wu [7] and Anderson and Cheeger [1], there is for each fixed k a subsequence of (Ω_k, g_i) converges to (Ω_k, g_∞), and moreover $g_\infty \in C^{1,\alpha(k)}$ where $0 < \alpha(k) < 1$ and the convergence can be taken to be in the $C^{1,\alpha(k)}$ norm. By taking the diagonal process (picking a subsequence from the previous subsequence when k increases), we have a subsequence of $\{g_i\}$ which converges to g_∞ in $C^{1,\alpha(\Omega)}$-norm for every compact $\Omega \subset M$. Note that there may not be a positive α such that g_∞ is $C^{1,\alpha}$ globally on M. Nevertheless, g_∞ is C^1 on M. In particular, the distributional Ricci curvature for the limit metric g_∞ is well-defined. In general the limit metric is not C^2, so the usual definition of Ricci curvature does not apply. Now we assume in addition that the Ricci curvature of g_i is nonnegative for each i. It is easy to see that the distributional Ricci curvature of g_∞ is also nonnegative. Moreover, for a fixed $\epsilon > 0$, if i is sufficiently large,

$$\mathrm{Vol}_{g_\infty}\big(B_{x_0}(R)\big) \le (1+\epsilon)\,\mathrm{Vol}_{g_i}\big(B_{x_0}(R)\big) \le (1+\epsilon) C_n R^n,$$

where we have used the standard volume comparison theorem to obtain the the second inequality. Hence $\mathrm{Vol}_{g_\infty}\big(B_{x_0}(R)\big) \le C_n R^n$. Therefore Theorem 1.1 applies to the limit Riemannian manifold (M, g_∞).

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REFERENCES

1. M. Anderson and J. Cheeger, *C^{α}-compactness for manifolds with Ricci curvature and injectivity radius bounded below*. J. Differential Geom. **35**(1992), 265–281.
2. S. Y. Cheng, *Eigenvalue comparison theorems and its geometric applications*. Math. Z. **143**(1975), 289–297.
3. T. H. Colding, *Ricci curvature and volume convergence*. (1995), preprint.
4. ———, *Large manifolds with positive Ricci curvature*. (1995), preprint.
5. H. Donnelly, *Bounded harmonic functions and positive Ricci curvature*. Math. Z. **191**(1986), 559–565.
6. D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, Berlin, 1983.
7. R. Green and H. Wu, *Lipschitz convergence of Riemannian manifolds*. Pacific J. Math. **131**(1988), 119–141.
8. L. Hörmander, *The Analysis of Linear Partial Differential Operators I*. Springer-Verlag, Berlin, 1983.
9. F. H. Lin, *Asymptotic conic elliptic operators and Liouville type theorems.* (1995), preprint.
10. P. Li and L. F. Tam, *Harmonic functions and the structure of complete manifolds*. J. Differential Geom. **35**(1992), 359–383.
11. ———, *Positive harmonic functions on complete manifolds with non-negative curvature outside a compact set*. Ann. of Math. (2) **125**(1987), 171–207.
12. S.-T. Yau, *Harmonic functions on complete Riemannian manifolds*. Comm. Pure Appl. Math. **28**(1975), 201–228.
13. ———, *Survey on partial differential equations in differential geometry*. Ann. of Math. Study **102**(1982), 3–73.

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