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PURE STATES ON FREE GROUP C*-ALGEBRAS

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Abstract. We prove that all the pure states of the reduced C*-algebra of a free group on an uncountable set of generators are *-automorphism equivalent and extract some consequences of this fact.

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1. Preliminaries. For any set R with two or more elements, let F_R denote the free group on R with generators $\{u_r : r \in R\}$ and let $C_r^*(F_R)$ denote the reduced group C*-algebra. We shall not distinguish between the elements of F_R and the corresponding unitary operators in $C_r^*(F_R)$. In what follows, r_0 will be a fixed element of R, u will denote the generator u_{r_0} and F_u will denote the subgroup of F_R generated by u. We view $C_r^*(F_S)$ as the C*-subalgebra of $C_r^*(F_R)$ generated by the unitaries $\{u_s : s \in S\}$ and $C_r^*(F_u)$ as the C*-subalgebra generated by u. Let P_u (resp. P_S) denote the unique trace preserving conditional expectation from $C_r^*(F_R)$ onto $C_r^*(F_u)$ (respectively $C_r^*(F_S)$). Recall that $C_r^*(F_u)$ is *-isomorphic to the *-algebra of continuous complex valued functions on the unit circle, with u going into the function $\theta(z) = z$. Let f_0 denote the (unique!) pure state of $C_r^*(F_u)$ that satisfies $f_0(u) = 1$ and let $f = f_0 \circ P_u$.

2. Results. If R is uncountable, then $C_r^*(F_R)$ is inseparable. For $\text{Card}(R) = \aleph_1$, the algebra $C_r^*(F_R)$ is discussed in [**11**, Corollary 6.7], where it is shown that $C_r^*(F_R)$ is inseparable, but that every abelian subalgebra is separable. Powers [**12**] showed that for $\text{Card}(R) = 2$, $C_r^*(F_R)$ is simple and has unique trace. Powers' method extends to general R. For general free products of groups, simplicity and uniqueness of trace follow by results of Avitzour [**7**]. In [**1**] and [**3**] the methods of [**4**] were used to extend the simplicity and uniqueness of trace results to a host of other group of C*-algebras where free sets lurked in the underlying groups. In [**6**] Archbold also obtained related results.

LEMMA 2.1. *If $S \subset R$, Card $(S) > 1$ and α is a *-automorphism of $C_r^*(F_S)$, then α has an extension to a *-automorphism of $C_r^*(F_R)$.*

Proof. Check that if α' is defined on the *-algebra A generated by $C_r^*(F_S)$ and the generators in $R\backslash S$ by applying α to elements of $C_r^*(F_S)$ and leaving the other generators alone, then α' is a *-automorphism of A. Every element of $C_r^*(F_R)$ is representable in the form of an element of $l^2(F_R)$, and the trace of such an element is simply the coefficient of the identity. Since the trace is unique on $C_r^*(F_S)$ by [**1**, Proposition 1] (see also [**7**, 3.1]), α preserves the trace. Thus it is easy to verify that α' preserves the trace on A. Again by density of A in $l^2(F_R)$, for any $a \in A$ and any $\epsilon > 0$ there exists $b \in A$ such that $\|b\|_2 = 1$ and $\|ab\|_2 > \|a\| - \epsilon$. So $\|\alpha'(b)\|_2 = \|b\|_2 = 1$ by invariance of the trace, and hence $\|\alpha'(a)\| \geq \|\alpha'(a)\alpha'(b)\|_2 = \|\alpha'(ab)\|_2 = \|ab\|_2 > \|a\| - \epsilon$. Since a similar inequality holds for α^{-1}, we see that α' extend by continuity to an automorphism of $C_r^*(F_R)$. □

LEMMA 2.2. *The state f is the unique state extension of f_0 to $C_r^*(F_R)$, and f is a pure state of $C_r^*(F_R)$. Moreover, $f|_{C_r^*(F_S)}$ is pure for any subset S of R that contains r_0.*

Proof. Let g be a state of $C_r^*(F_R)$ such that $g(u) = 1$. The Cauchy–Schwarz inequality applies to show that $g((1-u)a) = g(a(1-u)) = 0$ for any $a \in C_r^*(F_R)$. By induction, $g(u^n) = g(u^{-n}) = 1$ for every natural number n. Fix $s \in F_R \setminus F_u$. By the Cauchy–Schwarz inequality again, as above, $g(u^n s u^{-n}) = g(s)$ for every natural number n. Taking ξ to be the canonical trace vector in $l^2(F_R)$, $l^2(F_R) = H_0 \oplus H_1$, where H_0 is the closed linear span of all vectors of form $w\xi$ with w a reduced word in F_R with a non-zero power of u on the left, and H_1 is the closed linear span of those $w\xi$ with w not ending in a non-zero power of u on the left. Then, $u^n H_1 \subset H_0$ for any non-zero integer n and $sH_0 \subset H_1$. By [**8**, Lemma 2.2] (see also [**7**, Lemma 3.0])

$$|g(s)| = \lim_{k\to\infty} \left|(1/k)\sum_{n=1}^{k} g(u^n s u^{-n})\right| \leq \lim_{k\to\infty} \left\|(1/k)\sum_{n=1}^{k} u^n s u^{-n}\right\| \leq \lim_{k\to\infty} \frac{2}{\sqrt{k}} = 0.$$

By linearity and continuity of g, this implies that $g = g|_{C_r^*(F_u)} \circ P_u$ and hence that $g = f$. An easy convexity argument shows that f is a pure state.

The conclusion of the last sentence of the Lemma follows immediately from the conclusion of the first sentence. □

PROPOSITION 2.3. *Let $\{G_r\}_{r\in R}$ be a set of nontrivial countable groups and for non-empty $S \subset R$, let G_S be the free product $(*_{r\in S} G_r)$. Given a nonempty countable subset S_0 of R, if g is a pure state on $C_r^*(G_R)$ there is a countable subset S of R containing S_0 such that $g|_{C_r^*(G_S)}$ is a pure state of $C_r^*(G_S)$. Moreover, $C_r^*(G_S)$ is separable and also simple if $|G_s| > 2$ for some $s \in S$.*

Proof. Assume without loss of generality that R is uncountable. For any non-empty countable $S \subset R$, $C_r^*(G_S)$ is separable, and by [**7**, 3.1] simple if $|G_s| > 2$ for some $s \in S$. If (π_g, H_g, ξ_g) is the representation of $C_r^*(G_R)$ corresponding to g by the Gelfand-Naimark-Segal construction, sequences of sets

$$S_1 \subset S_2 \subset \cdots \subset R,$$

with each S_i countably infinite, closed separable linear subspaces

$$\mathbb{C}\xi_g = H_1 \subset H_2 \subset \cdots \subset H_g$$

and, for each $i \geq 2$, a countable dense subset X_i of the unit sphere of H_i such that

$$X_2 \subset X_3 \subset \cdots$$

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are constructed inductively so that

$$\pi_g(C_r^*(F_{S_i}))H_i \subseteq H_{i+1}$$

for $i \geq 1$. Let S_1 be a non-empty countable subset of R containing S_0. For the inductive step, given S_i and H_i, let H_{i+1} be the closed linear span of $\pi_g(C_r^*(G_{S_i}))H_i$, which is separable, and let X_{i+1} be a countable dense subset of the unit sphere of H_{i+1} containing X_i. By Kadison's transitivity theorem there is a countable set $\mathcal{U}_{i+1}$ of unitaries in $C_r^*(G_R)$ such that for $\xi, \eta \in X_{i+1}$, $\pi_g(u)\xi = \eta$ for some $u \in \mathcal{U}_{i+1}$. Since each such u is a norm-limit of a sequence of finite linear combinations of elements of G_R, there is a countable subset S'_{i+1} of R such that $\mathcal{U}_{i+1} \subset C_r^*(G_{S'_{i+1}})$. Let $S_{i+1} = S'_{i+1} \cup S_i$. Now let

$$S = \bigcup_{i=1}^{\infty} S_i, \quad X = \bigcup_{i=2}^{\infty} X_i, \quad H = \overline{\bigcup_{i=1}^{\infty} H_i}.$$

Then S and X are countable, H is separable, $\pi_g(C_r^*(G_S))H \subseteq H$ and X is dense in the unit sphere of H. If $\xi, \eta \in X$, then $\pi_g(v)\xi = \eta$ for some unitary $v \in C_r^*(G_S)$. It follows that for any $\varepsilon > 0$ and unit vectors $\xi, \eta \in H$, $\|\pi_g(w)\xi - \eta\| < \varepsilon$ for some unitary $w \in C_r^*(G_S)$, which implies that $\pi_g(C_r^*(G_S))|_H$ acts irreducibly on H. Since $g|_{C_r^*(G_S)}$ is the state of $C_r^*(G_S)$ corresponding to ξ_g, $g|_{C_r^*(G_S)}$ is pure. □

THEOREM 2.4. *Any two pure states of $C_r^*(F_R)$ are *-automorphism equivalent.*

Proof. If R is countable, the conclusion is immediate from [**10**]. Assume that R is uncountable. Let g be a pure state of $C_r^*(F_R)$. We shall show that g is *-automorphism equivalent to f. By Proposition 2.3 there is a countably infinite subset $S \subset R$ such that $r_0 \in S$ and $g|_{C_r^*(F_S)}$ is pure. We have already noted that $C_r^*(F_S)$ is simple, and it is obviously separable, so by [**10**] choose a *-automorphism γ_0 of $C_r^*(F_S)$ such that $g|_{C_r^*(F_S)} = \gamma_0^*(f|_{C_r^*(F_S)})$. By Lemma 2.1, extend γ_0 to a *-automorphism γ of $C_r^*(F_R)$. We must show that $\gamma^*(f) = g$. Lemma 2.2 shows that $f|_{C_r^*(F_S)}$ has unique state extension to $C_r^*(F_R)$. Since γ is a *-automorphism extending γ_0, the same uniqueness of state extension must follow for $\gamma^*(f|_{C_r^*(F_S)}) = g|_{C_r^*(F_S)}$. Thus $\gamma^*(f) = g$. □

The next result is in contrast to Corollary 0.9 of [**5**].

THEOREM 2.5. *If g is a pure state on $C_r^*(F_R)$, then its hereditary kernel,*

$$\{a \in C_r^*(F_R) : g(a^*a + aa^*) = 0\},$$

contains a sequential abelian approximate unit, and hence a strictly positive element.

Proof. By Theorem 2.4 it suffices to prove this for f. Choose an excising sequence $\{a_n\}$ for f_0 in $C_r^*(F_u)$, as defined in [**2**]. Let $p = \lim a_n$ in $C_r^*(F_R)^{**}$. By Lemma 2.2, p is a minimal projection there. By [**2**, Prop. 2.2], $\{a_n\}$ will excise f and $\{1 - a_n\}$ will be an approximate unit for $\{a \in C_r^*(F_R) : f(a^*a + aa^*) = 0\}$, so $\sum_i^{\infty} 2^{-n}(1 - a_n)$ is strictly positive there. □

THEOREM 2.6. *Let $r_0 \in S \subset R$.*
1. Any pure state of $C_r^*(F_S)$ has a unique extension to a pure state of $C_r^*(F_R)$.
2. The projection P_S is the unique conditional expectation of $C_r^*(F_R)$ onto $C_r^*(F_S)$.

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Proof. 1. By Theorem 2.4, any pure state of $C_r^*(F_S)$ is *-automorphism equivalent to $f|_{C_r^*(F_S)}$, and thus has the unique extension property since Lemma 2.2 shows that $f|_{C_r^*(F_S)}$ has that property.

2. If there were another conditional expectation $Q : C_r^*(F_R) \to C_r^*(F_S)$ distinct from P_S, then the duals Q^* and P_S^* would have to be different on some element of $C_r^*(F_S)^*$, hence on some state of $C_r^*(F_S)$, hence on some pure state of $C_r^*(F_S)$ by the Krein Milman Theorem [**9**, p. 32]. This is impossible by part 1 of this theorem. □

3. Concluding remarks. 1. A very similar proof to that of Proposition 2.3 shows the related result that if B is a separable C*-subalgebra of an inseparable C*-algebra A, then if g is a pure state of A, there is a separable C*-subalgebra C of A such that $B \subseteq C$ and $g|_C$ is pure. An analogous induction argument shows moreover that if A is simple, then a simple C with these properties can be found.

2. The proof of Theorem 2.4 and the preceding lemmas generalize in an obvious way to the general free product groups $G_R = *_{r\in R} G_r$ considered in Proposition 2.3, provided that one of the constituent groups G_{r_0} is abelian with an element of infinite order. Thus any two pure states of $C_r^*(G_R)$ are *-automorphism equivalent. The corresponding generalizations of Theorems 2.5 and 2.6 to these free product groups then follow, with G_{r_0} taking the place of F_u.

REFERENCES

1. C. A. Akemann, Operator algebras associated with Fuchsian groups, *Houston J. Math.* **7**(3) (1981), 295–301.

2. C. A. Akemann, J. Anderson and G. K. Pedersen, Excising states of C*-algebras, *Canad. J. Math.* **38**(5) (1986), 223–230.

3. C. A. Akemann and T.-Y. Lee, Computing norms in group C*-algebras, *Indiana U. Math. J.* **29**(4) (1980), 505–511.

4. C. A. Akemann and P. A. Ostrand, Simple C*-algebras associated with free groups, *Amer. J. Math.* **98**(4) (1976), 1015–1047.

5. C. A. Akemann and N. Weaver, Classically normal pure states, *Positivity*.

6. R. Archbold, A mean ergodic theorem associated with the free group on two generators, *J. Lond. Math. Soc.* **13**(2) (1976), 339–345.

7. D. Avitsour, Free products of C*-algebras, *Trans. A.M.S.* **271** (1982), 423–435.

8. M.-D. Choi, A simple C*-algebra generated by two finite-order unitaries, *Canad. J. Math.* **21** (1979), 867–880.

9. R. V. Kadison and J. Ringrose, *Fundamentals of the theory of operator algebras, vol. 1* (Academic Press, London, 1983).

10. A. Kishimoto, N. Ozawa and S. Sakai, Homogeneity of the pure state space of a separable C*-algebra, *Canad. Math. Bull.* **46** (2003), 365–372.

11. S. Popa, Orthogonal pairs of *-subalgebras in finite von Neumann algebras, *J. Operator Theory* **9** (183), 253–268.

12. R. Powers, Simplicity of the C*-algebra associated with the free group on two generators, *Duke Math. J.* **42** (1975), 151–156.