EVALUATION OF AN E-FUNCTION WHEN TWO OF THE UPPER PARAMETERS DIFFER BY AN INTEGER

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1. Introductory. If $p \geqq q+1$, [**1**, p. 353]

$$E(p\,;\alpha_r:q\,;\rho_s:z)=\sum_{r=1}^{p}\left[\prod_{t=1}^{p}{}'\Gamma(\alpha_t-\alpha_r)\right]\left[\prod_{s=1}^{p}\Gamma(\rho_s-\alpha_r)\right]^{-1}\Gamma(\alpha_r)$$

$$\times z^{\alpha_r}F\left\{\begin{matrix}\alpha_r,\ \alpha_r-\rho_1+1,\ \ldots,\ \alpha_r-\rho_q+1:(-1)^{p-q}z\\ \alpha_r-\alpha_1+1,\ \ldots *\ldots,\ -\alpha_r\alpha_p+1\end{matrix}\right\}$$

$$=\sum_{r=1}^{p}z^{\alpha_r}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_r+n)\prod_{t=1}^{p}{}'\Gamma(\alpha_t-\alpha_r-n)}{n!\prod_{s=1}^{p}\Gamma(\rho_s-\alpha_r-n)}(-z)^n, \tag{1}$$

where, if $p=q+1$, $|z|<1$. The dash in the product sign indicates that the factor for which $t=r$ is omitted, while the asterisk indicates that the parameter $\alpha_r-\alpha_r+1$ is omitted.

Now, if two or more of the α's are equal or differ by integral values, some of the series on the right cease to exist. For instance, if $\alpha_1=\alpha+l$, $\alpha_2=\alpha$, where l is a positive integer, the first two series are non-existent. In § 3 it will be shown that they can be replaced by the expression

$$(-1)^l z^{\alpha+l}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+l+n)\prod_{t=3}^{p}\Gamma(\alpha_t-\alpha-l-n)}{n!(l+n)!\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-l-n)}\Delta_n z^n$$

$$+z^{\alpha}\sum_{n=0}^{l-1}\frac{\Gamma(a+n)(l-n-1)!\prod_{t=3}^{p}\Gamma(\alpha_t-\alpha-n)}{n!\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-n)}(-z)^n, \tag{2}$$

where

$$\Delta_n=\psi(l+n)+\psi(n)-\psi(\alpha+l+n-1)-\log z+\sum_{t=3}^{p}\psi(\alpha_t-\alpha-l-n-1)-\sum_{s=1}^{q}\psi(\rho_s-\alpha-l-n-1).$$

Here [**1**, p. 141]

$$\psi(z)=\frac{d}{dz}\log\Gamma(z+1), \tag{3}$$

so that

$$\frac{d}{dz}\Gamma(z+1)=\Gamma(z+1)\psi(z). \tag{4}$$

Formulae required in the proof are given in § 2; and, in § 4 certain integrals are evaluated with the aid of (1) and (2).

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2. Formulae required in the proof. If n is a positive integer,

$$\psi(z+n) = \psi(z) + \sum_{r=1}^{n} \frac{1}{z+r}; \tag{5}$$

$$\psi(0) = -\gamma, \tag{6}$$

where γ is Euler's constant;

$$\psi(n) = \phi(n) - \gamma, \tag{7}$$

where

$$\phi(n) = 1 + \tfrac{1}{2} + \ldots + \frac{1}{n}, \quad \phi(0) = 0; \tag{8}$$

$$\psi(z) = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z}\right) - \gamma; \tag{9}$$

$$\psi(\tfrac{1}{2}+n) = 2\phi(2n+1) - \phi(n) - 2\log 2 - \gamma. \tag{10}$$

Note. The approximate value of γ is 0·5772156649 … .
From the formula

$$\Gamma(z+1)\Gamma(-z) = -\pi \operatorname{cosec} \pi z \tag{11}$$

it follows that

$$\psi(-z-1) = \psi(z) + \pi \cot \pi z; \tag{12}$$

and from the formula [**1**, p. 154]

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{mz-\frac{1}{2}} \Gamma(z)\Gamma\left(z+\frac{1}{m}\right) \ldots \Gamma\left(z+\frac{m-1}{m}\right), \tag{13}$$

where m is a positive integer, that

$$m\psi(mz-1) = m \log m + \sum_{t=0}^{m-1} \psi\left(z+\frac{t}{m}-1\right). \tag{14}$$

From (13), on replacing z by $z-t/m$, where $t = 1, 2, 3, \ldots, m-1$, it follows that

$$\Gamma\left(z-\frac{t}{m}\right)\prod_{s=1}^{m-1}{}'\,\Gamma\left(z+\frac{s-t}{m}\right) = (2\pi)^{\frac{1}{2}m-\frac{1}{2}} m^{-mz+\frac{1}{2}+t} \frac{\Gamma(1-z)}{\Gamma(1-mz+t)} \frac{\sin \pi(z+n)}{\sin \pi(mz+mn)} (-1)^{mn+n+t},$$

where the dash on the product sign indicates that the factor for which $s=t$ is omitted. Here let $z \to -n$, where n is a positive integer, and so obtain

$$\Gamma\left(-\frac{t}{m}-n\right)\prod_{s=1}^{m-1}{}'\,\Gamma\left(\frac{s-t}{m}-n\right) = (2\pi)^{\frac{1}{2}m-\frac{1}{2}} m^{mn-\frac{1}{2}+t} \frac{n!}{(mn+t)!} (-1)^{mn+n+t}. \tag{15}$$

Similarly it can be deduced from (13) that, if n is a positive integer,

$$\prod_{t=1}^{m-1} \Gamma\left(\frac{t}{m}-n\right) = (-1)^{mn+n} (2\pi)^{\frac{1}{2}m-\frac{1}{2}} m^{mn-\frac{1}{2}} n!/(mn)!. \tag{16}$$

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Again, from (14) and (12),

$$\sum_{t=1}^{m-1}\psi\left(\frac{t}{m}+z-1\right) = m\psi(-mz)-\psi(-z)-m\log m-m\pi\cot(\pi mz)+\pi\cot(\pi z).$$

But, when $z\to -n$,

$$\pi\cot(\pi z)-m\pi\cot(\pi mz)\to 0.$$

Hence, if n is a positive integer,

$$\sum_{t=1}^{m-1}\psi\left(\frac{t}{m}-n-1\right) = m\phi(mn)-\phi(n)-(m-1)\gamma-m\log m. \tag{17}$$

The following integral [1, p. 406] will also be required.
If m is a positive integer and $R(k)>0$,

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(p\,;\alpha_r:q\,;\rho_s:z/\lambda^m)\,d\lambda = m^{k-\frac{1}{2}}(2\pi)^{\frac{1}{2}-\frac{1}{2}m}E(p+m\,;\alpha_r:q\,;\rho_s:z/m^m), \tag{18}$$

where

$$\alpha_{p+1+v} = (k+v)/m \quad (v=0,1,2,\ldots,m-1).$$

3. Proof of the formula. If $\alpha_1=\alpha+l$, $\alpha_2=\alpha+\varepsilon$, where l is zero or a positive integer and ε is small, the sum of the first two series on the right of (1) can be written

$$z^{\alpha+l}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+l+n)\Gamma(-l-n+\varepsilon)\prod_{t=3}^{p}\Gamma(a_t-\alpha-l-n)}{n!\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-l-n)}(-z)^n$$

$$+z^{\alpha+\varepsilon}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n+\varepsilon)\Gamma(l-n-\varepsilon)\prod_{t=3}^{p}\Gamma(\alpha_t-\alpha-n-\varepsilon)}{n!\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-n-\varepsilon)}(-z)^n$$

$$=(-1)^l z^{\alpha+l}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+l+n)\prod_{t=3}^{p}\Gamma(\alpha_t-\alpha-l-n)}{n!\,\Gamma(1+l+n-\varepsilon)\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-l-n)}\frac{\pi}{\sin\pi\varepsilon}z^n$$

$$=(-1)^l z^{\alpha+\varepsilon}\sum_{n=l}^{\infty}\frac{\Gamma(\alpha+n+\varepsilon)\prod_{t=3}^{p}\Gamma(\alpha_t-\alpha-n-\varepsilon)}{n!\,\Gamma(1-l+n+\varepsilon)\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-n-\varepsilon)}\frac{\pi}{\sin\pi\varepsilon}z^n$$

$$+z^{\alpha+\varepsilon}\sum_{n=0}^{l-1}\frac{\Gamma(\alpha+n+\varepsilon)\Gamma(l-n-\varepsilon)\prod_{t=3}^{p}\Gamma(\alpha_t-\alpha-n-\varepsilon)}{n!\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-n-\varepsilon)}(-z)^n.$$

The limit when $\varepsilon \to 0$ of the first two terms is obtained by removing the factor $\pi/\sin \pi\varepsilon$, then differentiating with respect to ε, and finally making $\varepsilon \to 0$. On replacing n by $l+n$ in the second series formula (2) is obtained.

4. Evaluation of certain integrals. Formula (2) can be employed to evaluate certain integrals.

For example, if $|\operatorname{amp} z| < \pi$,

$$\int_0^\infty \frac{e^{-t}\,dt}{z+t} = z^{-1}\int_0^\infty e^{-t}E(1::z/t)\,dt = z^{-1}E(1,1::z),$$

by (18). From (2), with $l=0$, $\alpha=1$, $p=2$, $q=0$, this becomes

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}[\psi(n)-\log z].$$

Hence, if $|\operatorname{amp} z| < \pi$,

$$\int_0^\infty \frac{e^{-t}\,dt}{z+t} = \sum_{n=0}^{\infty}\frac{\phi(n)}{n!}z^n-(\gamma+\log z)e^z. \tag{19}$$

Again, from (18), (1) and (2), if $|\operatorname{amp} z| < \frac{1}{2}\pi$,

$$\int_0^\infty \frac{e^{-t}\,dt}{z^2+t^2} = z^{-2}\int_0^\infty e^{-t}E(1::z^2/t^2)\,dt = \pi^{-\frac{1}{2}}z^{-2}E(1,1,\tfrac{1}{2}::\tfrac{1}{4}z^2)$$

$$=\frac{1}{4\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\Gamma(-\frac{1}{2}-n)}{n!}\begin{bmatrix}\psi(n)-2\log(\frac{1}{2}z)\\ +\psi(-n-\frac{3}{2})\end{bmatrix}\left(\frac{z^2}{4}\right)^n+\tfrac{1}{2}\pi z^{-1}F(\,;\tfrac{1}{2};-\tfrac{1}{4}z^2).$$

Here apply (12) and (10), and so get

$$\int_0^\infty \frac{e^{-t}\,dt}{z^2+t^2} = \tfrac{1}{2}\pi z^{-1}\cos z-\sum_{n=0}^{\infty}\frac{\phi(2n+1)}{(2n+1)!}(-z^2)^n+(\gamma+\log z)\sin z/z, \tag{20}$$

where $|\operatorname{amp} z| < \frac{1}{2}\pi$.

Note. For large values of $|z|$ the asymptotic expansions of the E-functions can be employed in evaluating the integrals in (19) and (20).

More generally, if $R(k)>0$, $|\operatorname{amp} z| < \pi$, and if l and m are positive integers,

$$\int_0^\infty \frac{e^{-t}t^{k-1}}{(z+t^m)^l}dt = \frac{z^{-l}}{\Gamma(l)}\int_0^\infty e^{-t}t^{k-1}E(l::z/t^m)\,dt;$$

and therefore, from (18),

$$\int_0^\infty \frac{e^{-t}t^{k-1}}{(z+t^m)^l}dt = \frac{m^{k-\frac{1}{2}}z^{-l}}{(2\pi)^{\frac{1}{2}m-\frac{1}{2}}\Gamma(l)}E\left(l,\frac{k}{m},\frac{k+1}{m},\ldots,\frac{k+m-1}{m}::z/m^m\right). \tag{21}$$

In particular, if $l=1$, $k=m$,

$$\int_0^\infty \frac{e^{-t}t^{m-1}}{z+t^m}\,dt = m^{m-\frac{1}{2}}(2\pi)^{\frac{1}{2}-\frac{1}{2}m}z^{-1}E\left(1,1,1+\frac{1}{m},\ldots,1+\frac{m-1}{m}::z/m^m\right).$$

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Now, from (1) and (2), the E-function is equal to

$$z\sum_{n=0}^{\infty}\left\{\prod_{t=1}^{m-1}\Gamma\left(\frac{t}{m}-n\right)\right\}\frac{z^n}{n!}\left[\psi(n)-\log z+\sum_{t=1}^{m-1}\psi\left(\frac{t}{m}-n-1\right)\right]$$

$$+z\sum_{t=1}^{m-1}z^{t/m}\sum_{n=0}^{\infty}\Gamma\left(1+\frac{t}{m}+n\right)\left\{\Gamma\left(-\frac{t}{m}-n\right)\right\}^2\left\{\prod_{s=1}^{m-1}{}'\,\Gamma\left(\frac{s-t}{m}-n\right)\right\}\frac{(-z)^n}{n!}.$$

Here apply (16), (17) and (15), and so get

$$\int_0^\infty \frac{e^{-t}t^{m-1}}{z+t^m}\,dt = m^{m-1}\sum_{n=0}^{\infty}\frac{\{(-1)^{m+1}m^m z\}^n}{(mn)!}\left[\begin{array}{c}\phi(mn)-m\gamma\\ -m\log m-\log z\end{array}\right]$$

$$-\pi m^{m-1}\sum_{t=1}^{m-1}\frac{(-mz^{1/m})^t}{\sin(\pi t/m)}\sum_{n=0}^{\infty}\frac{\{(-1)^{m+1}m^m z\}^n}{(mn+t)!}, \qquad (22)$$

where m is a positive integer and $|\operatorname{amp} z| < \pi$.

Again, in (21), put $l=1$, $k=1$ and get

$$\int_0^\infty \frac{e^{-t}\,dt}{z+t^m} = (2\pi)^{\frac{1}{2}-\frac{1}{2}m}m^{\frac{1}{2}}z^{-1}E\left(1, 1, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m} :: z/m^m\right)$$

$$=(2\pi)^{\frac{1}{2}-\frac{1}{2}m}m^{\frac{1}{2}-m}\sum_{n=0}^{\infty}\frac{\prod_{t=1}^{m-1}\Gamma\left(\frac{t}{m}-1-n\right)}{n!}\left(\frac{z}{m^m}\right)^n\left[\begin{array}{c}\psi(n)-\log(z/m^m)\\ +\sum_{t=1}^{m-1}\psi\left(\frac{t}{m}-2-n\right)\end{array}\right]$$

$$+(2\pi)^{\frac{1}{2}-\frac{1}{2}m}m^{\frac{1}{2}}z^{-1}\sum_{t=1}^{m-1}\left(\frac{z}{m^m}\right)^{t/m}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{t}{m}+n\right)\left\{\Gamma\left(1-\frac{t}{m}-n\right)\right\}^2\left\{\prod_{s=1}^{m-1}{}'\,\Gamma\left(\frac{s-t}{m}-n\right)\right\}}{n!}\left(\frac{-}{m^m}\right)^n.$$

Here apply formulae (16) and (17) with $n+1$ in place of n, and formula (15); then, if $|\operatorname{amp} z| < \pi$,

$$\int_0^\infty \frac{e^{-t}\,dt}{z+t^m} = \frac{(-1)^{m+1}}{m}\sum_{n=0}^{\infty}\frac{\{(-1)^{m+1}z\}^n}{(mn+m-1)!}\left[\begin{array}{c}m\phi(mn+m-1)\\ -m\gamma-\log z\end{array}\right]$$

$$-\frac{\pi}{mz}\sum_{t=1}^{m-1}\frac{(-z^{1/m})^t}{\sin(\pi t/m)}\sum_{n=0}^{\infty}\frac{\{(-1)^{m+1}z\}^n}{(mn+t-1)!}. \qquad (23)$$

Note. Formulae (19) and (20) are particular cases of (22) and (23).

REFERENCE

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