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$H_2(T, G, \partial)$ AND CENTRAL EXTENSIONS FOR CROSSED MODULES

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We prove in this paper that if (T, G, ∂) is a perfect and aspherical ($\mathrm{Ker}\,\partial = 1$) crossed module, then it admits a universal central extension, whose kernel is the invariant $H_2(T, G, \partial)$, that we introduced in [9].

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Introduction

In previous work [4] the first author and Doncel-Juárez started on a detailed study of crossed modules as algebraic objects in their own right. It is well known that they model all homotopy 2-types and study of their homological algebra from that viewpoint has been started by Ellis [5] and others. In [4] it was proved that (T, G, ∂) is a q-perfect crossed module if and only if it admits a universal q-central extension by (T, G, ∂). We got as corollaries the results obtained in [1] and [10]. Later in [9], we introduced for a crossed module (T, G, ∂) two crossed module valued invariants $H_1(T, G, \partial)$ and $H_2(T, G, \partial)$. These are connected by a five-term exact sequence, associated to an extension of crossed modules.

This paper is divided into three sections. In the first section, we introduce some concepts in the theory of crossed modules, with special mention of commutator, centre and tensor product of G-crossed modules introduced in [10], [11] and [3]. Section 2 is a summary, without proof, of some results of our paper [9], that are necessary to apply in Section 3. The main result in this section is the theorem that says "If (T, G, ∂) is a perfect crossed module $(H_1(T, G, \partial) = (1, 1, 1))$ and aspherical $(\mathrm{Ker}\,\partial = 1)$, then $H_2(T, G, \partial)$ is the kernel of the universal central extension by (T, G, ∂)". We get as corollaries the analogous results for groups [12] and an isomorphism of crossed modules.

1. Generalities on crossed modules

Recall that a *crossed module* (T, G, ∂) is a group homomorphism $\partial : T \longrightarrow G$ together with an action of G on T satisfying:

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(i) ∂ is a precrossed module, i.e., $\partial({}^g t) = g \partial t g^{-1}$, for all $g \in G$, $t \in T$.

(ii) The subgroup of Peiffer is trivial, i.e., : ${}^{\partial t}s = tst^{-1}$, for all $t, s \in T$.

Examples. (1) If X is a path connected topological space and Y is a path connected subspace, $Y \subset X$, then $\partial : \pi_2(X, Y) \longrightarrow \pi_1(Y)$ is a crossed module. This was the motivating example for Whitehead [**13**].

(2) (N, G, i), where N is a normal subgroup of a group G, i is the inclusion and G acts on N by conjugation. In this way, every group G can be seen as a crossed module in the two obvious ways: $(1, G, i)$ or (G, G, id).

(3) $(A, G, 0)$ where A is a G-module and the boundary operator is the zero map.

A *morphism of crossed modules* $(\alpha, \phi) : (T, G, \partial) \longrightarrow (M, P, \mu)$ is a pair of group morphism $\alpha : T \longrightarrow M$ and $\phi : G \longrightarrow P$, such that $\mu\alpha = \phi\partial$ and α is a G-group morphism, via $\phi(\alpha({}^g t) = {}^{\phi(g)} \alpha(t)$, for all $g \in G$, $t \in T$).

A crossed module (S, H, σ) is a *crossed submodule* of a crossed module (T, G, ∂) if:

(i) S is a subgroup of T and H is a subgroup of G.

(ii) $\sigma = \partial|_S$

(iii) The action of H on S is induced by that of G on T.

A crossed submodule (S, H, σ) of a crossed module (T, G, ∂) is a *normal crossed submodule* if:

(i) H is a normal subgroup of G.

(ii) ${}^g s \in S$, for all $g \in G$, $s \in S$.

(iii) ${}^h t \cdot t^{-1} \in S$, for all $h \in H$, $t \in T$.

Taking objects and morphisms as defined above we obtain the category $\mathcal{CM}$ of crossed modules. $\mathcal{CM}$ has pullbacks, zero object, kernels and cokernels [**2, 8**].

A sequence of crossed module morphisms

$$(M, P, \mu) \xrightarrow{(\alpha,\phi)} (T, G, \partial) \xrightarrow{(\beta,\psi)} (U, Q, \omega)$$

is called *exact* if the crossed submodules of (T, G, ∂), Im (α, ϕ) and Ker (β, ψ), coincide.

If (S, H, ∂) and (R, K, ∂) are two normal crossed submodules of a crossed module (T, G, ∂), then the *commutator crossed submodule* [**10**] of (S, H, ∂) and (R, K, ∂), denoted by $[(S, H, \partial), (R, K, \partial)]$, is $([K, S][H, R], [H, K], \partial)$. In particular, the commutator crossed submodule of (T, G, ∂), denoted by $(T, G, \partial)' = [(T, G, \partial), (T, G, \partial)]$, is $([G, T], G', \partial)$, where $[G, T] = \langle\{{}^g t t^{-1} / t \in T, g \in G\}\rangle$ is the displacement subgroup of T relative to G, and $G' = [G, G]$ is the commutator subgroup of G.

Examples. (1) If N is a normal subgroup of G then $(N, G, i)' = ([G, N], G', i)$. Regarding a group G as a crossed module in the two usual ways, $N = 1$ or $N = G$, then $(G, G, Id)' = (G', G', Id)$ or $(1, G, i)' = (1, G', i)$.

(2) If A is a G-module then $(A, G, 0)' = (A \cdot IG, G', 0)$, where IG is the augmentation ideal of G, [**6**].

The *centre* of (T, G, ∂) is $Z(T, G, \partial) = (T^G, Z(G) \cap st_G(T), \partial)$ where $T^G = \{t \in T / {}^g t = t$ for all $g \in G\}$ and $st_G(T)$ is the stabilizer in G of T, i.e., $st_G(T) = \{g \in G / {}^g t = t$ for all $t \in T\}$ [**10**]. The crossed module (T, G, ∂) is *abelian* if it coincides with its centre. (T, G, ∂) is abelian if and only if G is abelian and the action of G on T is trivial, which implies that T is also abelian.

Suppose that we are given two crossed modules (M, G, μ) and (T, G, ∂). Each of the groups M and T acts on the other, and also acts on itself, via the action of G. The (*non-abelian*) *tensor product* [**3**] $M \otimes T$ is the group generated by the symbols $m \otimes t$ with $m \in M$, $t \in T$ subject to the relations

$$mm' \otimes t = ({}^m m' \otimes {}^m t)(m \otimes t),$$
$$m \otimes tt' = (m \otimes t)({}^t m \otimes {}^t t') \qquad \text{for} \quad m, m' \in M \quad \text{and} \quad t, t' \in T.$$

The commutator map $\kappa : G \otimes G \longrightarrow G$, $g \otimes g' \longrightarrow [g, g']$, $g, g' \in G$ is a homomorphism with image the commutator subgroup G' of G and central kernel. If G is perfect, the commutator map gives the universal central extension by G [**3**].

The tensor square $G \otimes G$ acts on $G \otimes T$ componentwise via the commutator map, i.e.,

$$^{g \otimes g'}(h \otimes t) = {}^{[g,g']}h \otimes {}^{[g,g']}t \qquad \text{with} \quad g, g', h \in G, t \in T$$

and so the map $1 \otimes \partial : G \otimes T \longrightarrow G \otimes G$, $g \otimes t \longrightarrow g \otimes \partial t$, is a crossed module.

Let $U : \mathcal{G}r \longrightarrow \mathcal{S}et$ be the forgetful functor. We denote by $(\mathcal{S}et \downarrow U)$ the comma category determined by the functors $Id : \mathcal{S}et \longrightarrow \mathcal{S}et$ and $U : \mathcal{G}r \longrightarrow \mathcal{S}et$. There exists an adjunction

$$(\mathcal{S}et \downarrow U) \underset{\mathcal{U}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{CM}$$

defined by $\mathcal{F}(f : S \longrightarrow UG) = (C, G, \partial)$, where $C = F_r(G \times S)/P$, $F_r(G \times S)$ is the free group over $G \times S$, P is the Peiffer subgroup of $F_r(G \times S)$ and the action of G over the generators is given by ${}^{g'}(g, s) = (g'g, s)$, $g, g' \in G$, $s \in S$. It is said that $\mathcal{F}(f : S \longrightarrow UG)$ is the *free crossed module* over $f : S \longrightarrow UG$. The construction of this object, without adjunctions, appears for the first time in [**13**].

We are interested in free presentations of a crossed module (T, G, ∂):

$$(V, R, \mu) \rightarrowtail (Y, F, \mu) \twoheadrightarrow (T, G, \partial)$$

where the codomain F of the free crossed module (Y, F, μ) is a free group. This corresponds to another notion of freedom on crossed modules, given by the adjunction:

$$(\mathcal{S}et \downarrow \mathcal{S}et) \underset{\mathcal{U}_{\mathcal{S}et}}{\overset{\mathcal{F}_{\mathcal{S}et}}{\rightleftarrows}} \mathcal{CM}$$

composite of two adjunctions: $(\mathcal{S}et \downarrow \mathcal{S}et) \rightleftarrows (\mathcal{S}et \downarrow U) \rightleftarrows \mathcal{CM}$

where the first one associates to $A \longrightarrow B$ the map $A \longrightarrow F_r(B)$, with $F_r(B)$ the free group over B. We will call $\mathcal{F}_{\mathcal{S}et}(f : A \longrightarrow B)$ *$\mathcal{S}et$-free crossed module* over $f : A \longrightarrow B$.

Let $\mathcal{E}$ be the class of epimorphisms in $\mathcal{CM}$ $(\alpha, \phi): (M, P, \mu) \longrightarrow (T, G, \partial)$ such that α, ϕ and the morphism Ker $\mu \longrightarrow$ Ker ∂ are all surjective. $\mathcal{S}et$-free crossed modules are $\mathcal{E}$-projective and every crossed module (T, G, ∂) is the quotient of $\mathcal{S}et$-free crossed module $(\alpha, \phi): (M, P, \mu) \longrightarrow (T, G, \partial)$ with $(\alpha, \phi) \in \mathcal{E}$ **[9]**.

2. Crossed modules and homology

Let $\mathcal{ACM}$ denote the category of abelian crossed modules, and consider the abelianization functor $\mathcal{A}b : \mathcal{CM} \longrightarrow \mathcal{ACM}$, that to each crossed module (T, G, ∂) associates its abelianization, i.e., $(T, G, \partial)/(T, G, \partial)' = (T/[G, T], G/(G, G), \bar{\partial})$, and to each morphism the induced one. The functor $\mathcal{A}b$ is left adjoint to the inclusion functor $\mathcal{U} : \mathcal{ACM} \longrightarrow \mathcal{CM}$. This follows from the universal property **[10]** of the commutator crossed submodule.

We define the *first homology crossed module* of a crossed module (T, G, ∂) by

$$H_1(T, G, \partial) = (T, G, \partial)/(T, G, \partial)' = (T/[G, T], G/[G, G], \bar{\partial}).$$

Examples. (1) If N is normal subgroup of G, then $H_1(N, G, i) = (N/[G, N], H_1(G), \bar{i})$.

(2) Seeing a group G as a crossed module in the two usual ways, we obtain the first group of integral homology $H_1(1, G, i) = (1, H_1(G), i)$, or $H_1(G, G, Id) = (H_1(G), H_1(G), Id)$.

(3) If A is a G-module, then $H_1(A, G, 0) = (H_0(G, A), H_1(G), 0)$.

A crossed module (T, G, ∂) is called *perfect* if it coincides with its commutator crossed submodule $(H_1(T, G, \partial) = (1, 1, 1))$. This is equivalent to saying that G is a perfect group **[12]** and $T = [T, G]$.

Examples. (1) Note that $(G, G, 1)$ or $(1, G, i)$ are perfect crossed modules if and only if G is a perfect group.

(2) $(A/B, G, 0)$ is a perfect crossed module where A is a G-module, G is a perfect group and $B = \langle\{{}^g a - 2a / a \in A, g \in G\}\rangle$.

Given an $\mathcal{E}$-projective presentation

$$(V, R, \mu) \rightarrowtail (Y, F, \mu) \twoheadrightarrow (T, G, \partial)$$

of the crossed module (T, G, ∂) we define the *second homology crossed module* of (T, G, ∂) by

$$\begin{aligned} H_2(T, G, \partial) &= ((V, R, \mu) \cap [(Y, F, \mu), (Y, F, \mu)])/[(Y, F, \mu), (V, R, \mu)] \\ &= (V \cap [F, Y]/[R, Y][F, V], R \cap [F, F]/[F, R], \mu *). \end{aligned}$$

$H_2(T, G, \partial)$ is independent of the chosen $\mathcal{E}$-projective presentation **[9]**. If (Y, F, μ) is a $\mathcal{S}e$-free crossed module, then $H_2(Y, F, \mu) = (1, 1, 1)$.

Examples. (1) If we consider a group G as a crossed module in the two usual ways, then we obtain the classic formula of Hopf [7], $H_2(G, G, id) = (H_2(G), H_2(G), id)$, or $H_2(1, G, i) = (1, H_2(G), i)$.

(2) If A is an abelian group, then $H_2(A, 0, 0) = (0, 0, 0)$ since $[F, Y]/[F, Y][F, V] = 0$, where $(V, F, \mu) \rightarrowtail (Y, F, \mu) \twoheadrightarrow (A, 0, 0)$ is an $\mathcal{E}$-projective presentation. Considering $A = Z \times Z$ $(H_2(A) = Z)$ we remark that, in general, $H_2(T, G, \partial) \neq (H_2(T), H_2(G), H_2(\partial))$.

3. Central extensions for crossed modules

Suppose that (R, K, ∂) is a normal crossed submodule of (T, G, ∂) and that (S, H, ∂') is a crossed module such that $(T/R, G/K, \bar{\partial}) \cong (S, H, \partial')$, then we call (T, G, ∂) an *extension* of (R, K, ∂) by (S, H, ∂'). If there exists a surjective morphism $\psi: (X_1, X_2, \xi) \longrightarrow (T, G, \partial)$, then trivially (X_1, X_2, ξ) is an extension of the crossed module Ker ψ by (T, G, ∂).

An extension $((X_1, X_2, \xi), \psi)$ by (T, G, ∂) is a *central extension* if Ker ψ is contained in $Z(X_1, X_2, \xi)$. It is easy to see that the extension $((X_1, X_2, \xi), \psi)$ by (T, G, ∂) is central if and only if (T, G, ∂) acts trivially on Ker ψ (the morphism of (T, G, ∂) to the actor **[11]** of Ker ψ is trivial).

We will use the following lemmas **[10]**:

Lemma 1. *If (Y_1, Y_2, η) is perfect, there exists at most one morphism from (Y_1, Y_2, η) to (X_1, X_2, ξ) over (T, G, ∂).*

Lemma 2. *If* $((X_1, X_2, \xi), \psi)$ *is a central extension by a perfect crossed module* (P_1, P_2, ρ) *then the commutator crossed submodule* $(X_1, X_2, \xi)'$ *of* (X_1, X_2, ξ) *is perfect and maps onto* (P_1, P_2, ρ).

A central extension $((U_1, U_2, \nu), \phi)$ by (T, G, ∂) is called a *universal central extension* if for every central extension $(X_1, X_2, \xi), \psi$ by (T, G, ∂) there exists one and only one morphism $h\colon (U_1, U_2, \nu) \longrightarrow (X_1, X_2, \xi)$, making the following diagram commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \operatorname{Ker}\phi & \longrightarrow & (U_1, U_2, \nu) & \xrightarrow{\phi} & (T, G, \partial) & \longrightarrow & 1 \\ & & h\big\downarrow & & h\big\downarrow & & \big\| & & \\ 1 & \longrightarrow & \operatorname{Ker}\psi & \longrightarrow & (X_1, X_2, \xi) & \xrightarrow{\psi} & (T, G, \partial) & \longrightarrow & 1. \end{array}$$

By definition, if such a universal central extension exists then it is unique up to isomorphism over (T, G, ∂).

Proposition 1. *Let* $(P, N, \partial) \rightarrowtail (T, G, \partial) \twoheadrightarrow (U, Q, \omega)$ *be a central extension of crossed modules where* $(T, G, \partial) \longrightarrow (U, Q, \omega)$ *belongs to* $\mathcal{E}$. *Then we have the following exact (and natural) sequence:*

$$H_2(T, G, \partial) \longrightarrow H_2(U, Q, \omega) \longrightarrow (P, N, \partial) \longrightarrow H_1(T, G, \partial) \longrightarrow H_1(U, Q, \omega) \longrightarrow (1, 1, 1).$$

Proof. Since $(P, N, \partial) \subseteq Z(T, G, \partial) = (T^G, Z(G) \cap st_G(T), \partial)$ we have $[G, P] = 1$ $(P \subseteq T^G)$, $[G, N] = 1$ $(N \subseteq Z(G))$ and $[N, T] = 1$ $(N \subseteq st_G(T))$. The result is consequence of [**9**, 4.1 Theorem].

Now it is possible to define various classes of central extensions (commutator, stem, stem cover) for crossed modules in the same way that is done for groups [**12**].

Theorem. *If* (T, G, ∂) *is a perfect* $(H_1(T, G, \partial) = (1, 1, 1))$ *and aspherical* $(\operatorname{Ker}\partial = 1)$ *crossed module, then*

$$H_2(T, G, \partial) \rightarrowtail ([F, Y]/[R, Y][F, V], [F, F]/[F, R], \mu*) \twoheadrightarrow (T, G, \partial)$$

is the universal central extension by (T, G, ∂), *where* $(V, R, \mu) \rightarrowtail (Y, F, \mu) \twoheadrightarrow (T, G, \partial)$ *is an* $\mathcal{E}$-*projective presentation of* (T, G, ∂).

Proof. Let $(V, R, \mu) \rightarrowtail (Y, F, \mu) \twoheadrightarrow (T, G, \partial)$ be a presentation $\mathcal{S}et$-free of (T, G, ∂). Since $[(V, R, \mu), (Y, F, \mu)] = ([R, Y][F, V], [F, R], \mu)$ is a normal crossed submodule of (V, R, μ) and (T, G, ∂) is aspherical we have the following extension of crossed modules

$$(V/[R, Y][F, V], R/[F, R], \mu*) \rightarrowtail (Y/[R, Y][F, V], F/[F, R], \mu*) \xrightarrow{\alpha} \!\!\!\!\rightarrow (T, G, \partial) \tag{1}$$

where the epimorphism $\alpha \in \mathcal{E}$.

Further if $\bar{v} = v[R, Y][F, V] \in V/[R, Y][F, V]$ and $\bar{f} = f[F, R] \in F/[F, R]$, then $^{\bar{f}}\bar{v} = \overline{^{f}v} = \bar{v}$ because $^{f}vv^{-1} \in [F, V]$. In the same way if $\bar{r} = r[F, R] \in R/[F, R]$ and $\bar{y} = y[R, Y][F, V] \in Y/[R, Y][F, V]$ then $^{\bar{r}}\bar{y} = \overline{^{r}y} = \bar{y}$, because $^{r}yy^{-1} \in [R, Y]$. So

$$(V/[R, Y][F, V], R/[F, R], \mu*) \subseteq Z(Y/[R, Y][F, V], F/[F, R], \mu*)$$
$$= (Y/[R, Y][F, V]^{F/[F,R]}, Z(F/[F, R]) \cap st_{F/[F,R]}(Y/[R, Y][F, V], \mu*)$$

and therefore the extension (1) is central.

The epimorphism α in the central extension (1) maps the commutator crossed submodule of $(Y/[R, Y][F, V], F/[F, R], \mu*) = ([F, Y]/[R, Y][F, V], [F, F]/[F, R], \mu_*)$ that is perfect (Lemma 2) onto (T, G, ∂).

So we have the central extension of crossed modules

$$H_2(T, G, \partial) \rightarrowtail ([F, Y]/[R, Y][F, V], [F, F]/[F, R], \mu*) \xrightarrow{\beta} \!\!\!\!\rightarrow (T, G, \partial) \tag{2}$$

where $\beta \in \mathcal{E}$ as (T, G, ∂) is aspherical ($\text{Ker}\, \partial = 1$). To show that the central extension (2) is the universal, it is enough to prove by Lemma 1 that exists a morphism from (2) to an arbitrary central extension by (T, G, ∂).

Let $\text{Ker}\, \gamma \rightarrowtail (L, M, \lambda) \xrightarrow{\gamma} \!\!\!\!\rightarrow (T, G, \partial)$ be an arbitrary central extension by the aspherical crossed module (T, G, ∂). Since $\gamma \in \mathcal{E}$ and (Y, F, μ) is $\mathcal{E}$-projective, there exists a morphism $\phi = (\phi_1, \phi_2): (Y, F, \mu) \longrightarrow (L, M, \lambda)$ such that the diagram

$$\begin{array}{ccc} (Y, F, \mu) & & \\ \phi \downarrow & \searrow & \\ (L, M, \lambda) & \xrightarrow[\gamma]{} & (T, G, \partial) \end{array}$$

is commutative.

Since $\text{Ker}\ \gamma \subseteq Z(L, M, \lambda) = (L^M, Z(M) \cap st_M(L), \lambda)$, we have $\phi_1([R, Y][F, V]) = 1$, $\phi_2([F, R]) = 1$, and so we get a morphism $h: (Y/[R, Y][F, V], F/[F, R], \mu*) \longrightarrow (L, M, \lambda)$ such that $hi^c = \phi$, i^c being the cokernel of $i: ([R, Y][F, V], [F, R], \mu) \longrightarrow (Y, F, \mu)$. The restriction h' of h to the commutator crossed submodule verifies $\gamma h' = \beta$ and yields the wanted morphism.

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$$\begin{array}{ccccc}
 & & ([F,Y]/[R,Y][F,V],[F,F]/[F,R],\mu *) & \stackrel{\beta}{\longrightarrow} & (T,G,\partial) \\
 & & \updownarrow h' & & \| \\
(Y,F,\mu) & \stackrel{i^c}{\longrightarrow} & (Y/[R,Y][F,V],F/[F,R],\mu *) & \stackrel{\alpha}{\longrightarrow} & (T,G,\partial) \\
 & \searrow^{\phi} & h \downarrow & & \| \\
 & & (L,M,\lambda) & \stackrel{\gamma}{\longrightarrow} & (T,G,\partial)
\end{array}$$

Corollary 1. *If G is a perfect group, then* $H_2(G) \rightarrowtail [F,F]/[R,R] \twoheadrightarrow G$ *is the universal central extension by G,* $R \rightarrowtail F \twoheadrightarrow G$ *being a free presentation of G.*

Proof. Consider a group G as a crossed module in the two usual ways, (G, G, id) or $(1, G, i)$.

Corollary 2. *If* (T, G, ∂) *is an aspherical perfect crossed module, we have an isomorphism of crossed modules* $(G \otimes T, G \otimes G, 1 \otimes \partial) \cong ([F,Y]/[R,Y][F,V],[F,F]/[F,R],\mu *)$, $(V,R,\mu) \rightarrowtail (Y,F,\mu) \twoheadrightarrow (T,G,\partial)$ *being an* $\mathcal{E}$*-projective presentation of* (T,G,∂).

Proof. If (T, G, ∂) is a perfect crossed module, then

$$1 \longrightarrow \operatorname{Ker} c \longrightarrow (G \otimes T, G \otimes G, 1 \otimes \partial) \stackrel{c}{\longrightarrow} (T,G,\partial) \longrightarrow 1$$

is the universal central extension by (T, G, ∂) [10, 2.68 Theorem]. The result is a consequence of the above Theorem and the uniqueness of the morphism from the universal central extension to another central extension by (T, G, ∂).

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