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HYPERBOLIC 3-MANIFOLDS AND CLUSTER ALGEBRAS

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Abstract. We advocate the use of cluster algebras and their y-variables in the study of hyperbolic 3-manifolds. We study hyperbolic structures on the mapping tori of pseudo-Anosov mapping classes of punctured surfaces, and show that cluster y-variables naturally give the solutions of the edge-gluing conditions of ideal tetrahedra. We also comment on the completeness of hyperbolic structures.

§0. Introduction

0.1 Cluster algebras

Cluster algebras were introduced by Fomin and Zelevinsky [FZ02] around 2000. Since then, many authors have uncovered beautiful connections between the theory of cluster algebras and a wide range of mathematics such as

- dual canonical bases and their relations with preprojective algebras and quiver varieties [BFZ05], [Lec10], [Nak11a], [Kim12]
- total positivity [Fom10]
- (higher) Teichmüller theory and its quantization [FG06, FG07, FG09], [Tes07, Tes]
- 2-dimensional hyperbolic geometry [GSV03], [FST08]
- cluster categories [Kel10], [Ami09], [Pla11]
- discrete integrable systems [Ked08], [KNS11]
- Donaldson–Thomas theory [KS], [Nag13]
- supersymmetric gauge theories [GMN], [CNV], [EF12].

The goal of this paper is to add yet another item to this list: *the theory of hyperbolic* 3*-manifolds*. This paper is a companion to [TY14], which

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discusses the application of cluster algebras to the physics of 3d $\mathcal{N}=2$ supersymmetric gauge theories.

0.2 Hyperbolic 3-manifolds

A hyperbolic 3-manifold (with cusps) has a decomposition into ideal tetrahedra. This makes it possible for us to compute invariants of the 3-manifold, such as the hyperbolic volume and the Chern–Simons invariant [NZ85, Neu92].

An ideal tetrahedron is parametrized by a complex number called a *shape parameter*. Given a topological decomposition of the 3-manifold into ideal tetrahedra, we need to find shape parameters which satisfy *edge-gluing equations* (§4.2) in order to obtain a hyperbolic structure on the 3-manifold. Moreover, the *cusp equations* (§4.4) should hold for the complete hyperbolic structure. In general, it is a rather nontrivial problem to systematically find solutions of these equations.

In this paper, we study mapping tori M_φ of mapping classes φ of a surface Σ with punctures. We mainly discuss the case that the mapping torus admits a hyperbolic structure.

The main results of this paper are summarized as follows:

- Solving the periodicity equation in Theorem 4.4 for cluster transformations, we get a solution of the edge-gluing equations of the mapping torus M_φ with an ideal triangulation induced by the cluster transformations.
- Shape parameters of tetrahedra are given by the cluster y-variables, where the initial values of the y-variables are taken to be the solution of the periodicity equation.
- The cusp condition is written as a simple condition on a product of the initial values of the y-variables.

REMARK 0.1. The complete hyperbolic structure gives a nonzero solution of the periodicity equation, thanks to the result of [KT15, Corollary 2.6].

REMARK 0.2. This paper has grown up from our attempts to formulate the results of [KN11] and [TY11, TY13] in mathematically rigorously.

In [TY11], the authors conjectured an equivalence of the partition function of a $3d$ $\mathcal{N}=2$ gauge theory on a duality wall and that of the $\mathrm{SL}(2,\mathbb{R})$ Chern–Simons theory on a mapping torus. This is a $3d/3d$ counterpart of the $4d/2d$ correspondence, known as the AGT relation [AGT10].

In [TY13], the authors demonstrated that a limit of the 3d $\mathrm{N} = 2$ partition function reproduces the hyperbolic volume of the mapping torus in the case of the once-punctured torus by using quantum cluster transformations. The key observation in [TY13] was that the shape parameters satisfying edge-gluing equations (as previously analyzed in [Gué06]) appear at the saddle point.

In [KN11], it was shown that classical dilogarithm identities [Nak11c] naturally emerge from quantum dilogarithm identities [Kel11], [Nag11] by the saddle-point method.

It will be interesting to learn from physics about the "quantum" aspects of hyperbolic geometry of 3-manifolds.

REMARK 0.3. There is a known relation between cluster transformations and integrable systems [FZ03, Kel13, Nak11b, Ked08]. With this, our theorem, which connects cluster transformations to 3-manifolds, gives a natural explanation for mysterious and interesting relations between 3-manifolds and conformal field theories/integrable systems, originally found in [GT96, NRT93, DS94]. We illustrate this point by an example in the final subsection (the corresponding 3-manifold is not hyperbolic).

The differences between the two setups, (a) integrable systems and (b) hyperbolic 3-manifolds, can be stated in several different languages:

- We have periodicity conditions on the cluster y-variables both in (a) and in (b). However, in (a), periodicity is imposed as identities of rational functions on y_i's, whereas in (b) we solve the periodicity equations to determine values of y_i, which in turn determines the hyperbolic structure of the mapping tori.
- In (a), the product of the quantum dilogarithms associated to the sequence of mutations is equal to 1 (quantum dilogarithm identity [Kel11]). In (b), the product gives a nontrivial action of the mapping class in the quantum Teichmüller theory.
- In terms of surface triangulations and flips, after a sequence of flips, in (a) we get the original triangulation (up to a permutation of vertices), while in (b) we get the original triangulation pulled back by the mapping class.
- A mutation provides a derived equivalence of 3-dimensional Calabi–Yau categories associated to quivers with potential [KY11]. In (a), the composition of the derived equivalences is an identity functor, while in (b) it gives the action of the mapping class on the derived category (see [Nag]).

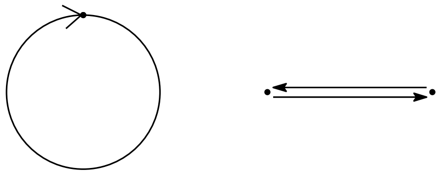

Figure 1.
A loop (left) and an oriented 2-cycle (right) of a quiver.

- The derived equivalence induced by mutation corresponds to a wall crossing in the space of stability conditions, and a sequence of mutations gives a new chamber. In (a), the new chamber coincides with the original one, while in (b) the chamber is obtained from the original one by the action of the mapping class on the space of stability conditions. In other words, the former is the wall crossing associated with a contractible cycle, whereas the latter corresponds to a noncontractible cycle with nontrivial monodromies (cf. [ADJM12]).

§1. Cluster algebras

1.1 Quiver mutation

In this paper, we always assume that a quiver has

- the vertex set $I = \{1, \dots, n\}$, and
- no loops and oriented 2-cycles (see Figure 1).

For vertices i and $j \in I$, we define

$$Q(i,j) = \sharp\{\text{arrows from } i \text{ to } j\}, \qquad \overline{Q}(i,j) = Q(i,j) - Q(j,i).$$

Note that the quiver Q is uniquely determined by the skew-symmetric matrix $\overline{Q}(i,j)$ (or equivalently $Q(i,j)$) under the assumption above[1].

For the vertex k, we define a new quiver $\mu_k Q$ (mutation of Q at vertex k) by an antisymmetric matrix

$$\overline{\mu_k Q}(i,j) = \begin{cases} -\overline{Q}(i,j), & i = k \text{ or } j = k, \\ \overline{Q}(i,j) + Q(i,k)Q(k,j) - Q(j,k)Q(k,i), & i, j \neq k. \end{cases}$$

1.2 Cluster variables

Given a sequence $\mathbf{k} = (k_1, \dots, k_l)$ of vertices and "time" parameters $t = 0, \dots, l$, we define

$$Q_0 := Q, \qquad Q_t := \mu_{k_{t-1}} \cdots \mu_{k_1} Q \quad (t > 0).$$

[1] In this paper, we restrict ourselves to cluster algebras associated with skew-symmetric matrices.

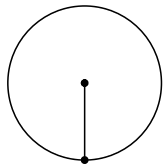

Figure 2.
We do not allow self-folded triangles as in this figure.

For initial values $x_i(0) = x_i$ and $y_i(0) = y_i$, we define the cluster x-variables $x_i(t)$ and the cluster y-variables (coefficients) $y_i(t)$ $(i \in I)$ by

$$x_i(t+1) = \frac{\prod_j x_j(t)^{Q_t(i,j)} + \prod_j x_j(t)^{Q_t(j,i)}}{x_i(t)}, \tag{1}$$

and

$$y_i(t+1) = \begin{cases} y_k(t)^{-1}, & i = k, \\ y_i(t) y_k(t)^{Q_t(k,i)} \big(1 + y_k(t)\big)^{\overline{Q_t}(i,k)}, & i \neq k. \end{cases} \tag{2}$$

§2. Triangulated surfaces and quivers

Let Σ be a closed connected oriented surface and M be a finite set of points on Σ, called *punctures*. We assume that M is nonempty and (Σ, M) is not a sphere with less than four punctures.

We choose an ideal triangulation τ of Σ, *i.e.*, we decompose Σ into triangles whose vertices are located at the punctures. We will not allow self-folded arcs (see Figure 2) in this paper.

2.1 Quiver associated to a triangulation

For a triangulation τ without self-folded arcs we will define a quiver Q_τ whose vertex set I is the set of arcs in τ.

For a triangle Δ and arcs i and j, we define a skew-symmetric integer matrix $\overline{Q}^\Delta$ by

$$\overline{Q}^\Delta(i,j) := \begin{cases} 1 & \Delta \text{ has sides } i \text{ and } j, \text{with } i \text{ following } j \\ & \text{in the clockwise order,} \\ -1 & \text{the same holds, but in the counter-clockwise order,} \\ 0 & \text{otherwise.} \end{cases}$$

We define

$$\overline{Q}_\tau := \sum_{\Delta \in \tau} \overline{Q}^\Delta,$$

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where the sum is taken over all triangles in τ. Let Q_τ denote the quiver associated to the matrix $\overline{Q}_\tau$.

For an arc i in the triangulation τ, we can flip the edge i to get a new triangulation $f_i(\tau)$. This operation is compatible with a mutation at vertex i:

$$Q_{f_i(\tau)} = \mu_i(Q_\tau).$$

2.2 Mapping class group action

For a triangulation τ, we define

$$T = T(\tau) := \mathbb{C}(y_e)_{e\in\tau_1}, \qquad T^\vee = T^\vee(\tau) := \mathbb{C}(x_e)_{e\in\tau_1}.$$

For a puncture $m \in M$, take a sufficiently small circle around m and let $e_1, \ldots, e_n$ be the sequence of arcs which intersect with the circle, where $e_1, \ldots, e_n$ may have multiplicity. We define

$$y_m := \prod_{i=1}^{n} y_{e_i} \tag{3}$$

and

$$\underline{T} = \underline{T}(\tau) := \mathbb{C}[y_e, y_e^{-1}]_{e\in\tau_1} / (y_m)_{m\in M}.$$

Let us fix a mapping class φ. Then the two triangulations τ and $\varphi(\tau)$ are related by a sequence of flips, together with appropriate changes of labels. More formally, there exists a sequence

$$\mathbf{k} = (k_1, \ldots, k_l) \in (\tau_1)^l$$

such that the two triangulations τ and $\varphi(\tau)$ are related by the sequence of flips associated to $\mathbf{k}$ (see [FST08, Proposition 3.8]). Note that a flip provides a canonical bijection of the edges of the triangulations. We can represent the composition of the bijections by a permutation $\sigma \in \mathfrak{S}_I$. We define the automorphisms

$$\mathrm{CT}_\varphi \colon T(\tau) = T(\varphi(\tau)) \xrightarrow{\sim} T(\tau), \qquad \mathrm{CT}^\vee_\varphi \colon T^\vee(\tau) = T^\vee(\varphi(\tau)) \xrightarrow{\sim} T^\vee(\tau)$$

by

$$\mathrm{CT}_\varphi(y_e) = y_{\sigma(e)}(l), \qquad \mathrm{CT}^\vee_\varphi(x_e) = x_{\sigma(e)}(l).$$

Thanks to the result [FST08, Theorem 3.10] and the pentagon relation of cluster transformations, CT_φ and $\mathrm{CT}^\vee_\varphi$ are independent of the choices of the sequences of flips and provides a well-defined action of the mapping class group on $T(\tau)$.

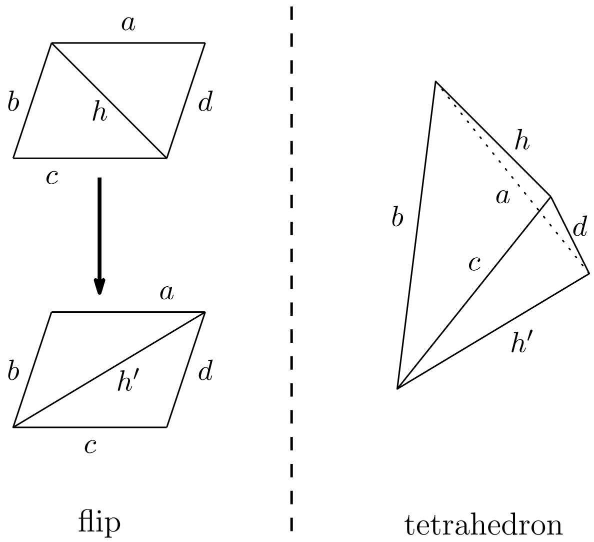

Figure 3.
A flip in a 2d triangulation can be traded for a 3d tetrahedron.

§3. Pseudo-Anosov mapping tori

Let τ, φ, $\mathbf{k}$ and σ be as in §2.2. We assume that no triangles are self-folded. Let $h = h(t)$ be the edge flipped at t and h' be the edge after the flip. Let a, b, c and d be the edges of the quadrilateral in the triangulations whose diagonals are h and h'. We associate a topological tetrahedron $\Delta = \Delta(t)$ whose edges are labeled by a, b, c, d, h and h' (see Figure 3).

For any pseudo-Anosov mapping class φ, this provides a topological tetrahedron decomposition of the mapping torus [Ago11]. A mapping class φ is pseudo-Anosov if and only if the mapping torus has a hyperbolic structure.

§4. Equations for hyperbolic structure

4.1 Shape parameters

For an ideal tetrahedron in $\mathbb{H}^3$ with vertices 0, 1, z and ∞ (Figure 4), we associate the *shape parameter* z with the edge connecting 0 and ∞. For an ideal tetrahedron, a pair of mutually nonintersecting edges has a common shape parameter, and the shape parameters for the three pairs of mutually nonintersecting edges are given by (Figure 5)

$$z, \qquad 1 - z^{-1}, \qquad \frac{1}{1-z}. \tag{4}$$

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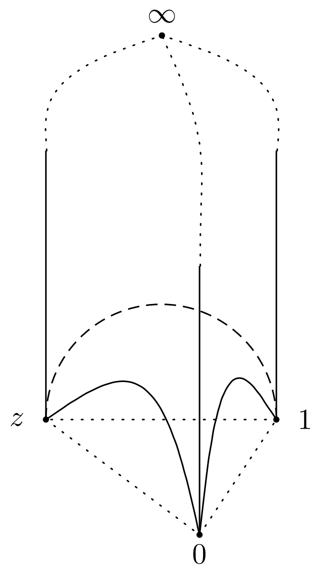

Figure 4.
An ideal tetrahedron with shape parameter z.

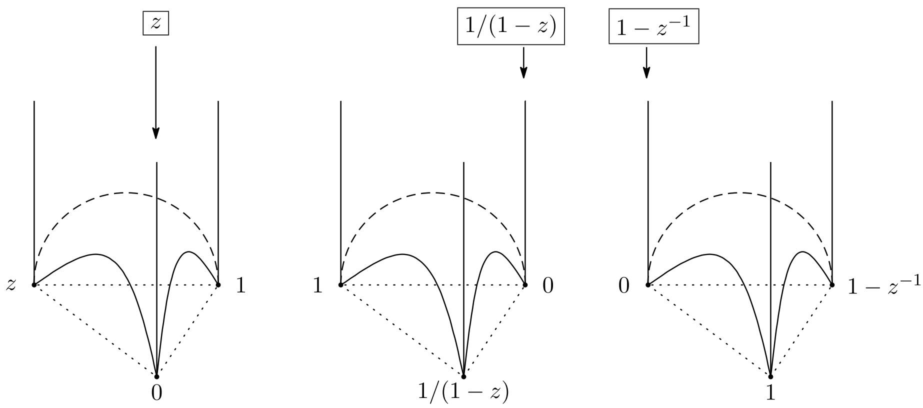

Figure 5.
The three shape parameters of an ideal tetrahedron.

We take a sequence of flips and associated topological decomposition of the mapping torus as in §3. For $t \in \mathbb{Z}$, let $\Delta(t)$ denote the tth tetrahedron, where $\Delta(t)$ and $\Delta(t+l)$ are identified for any t. Let $Z(t)$ denote the shape parameter of $\Delta(t)$ at the edge $h(t)$, the edge flipped at time t. Note that

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the sequence $(Z(t))$ satisfies shape parameter periodicity

$$Z(t+l) = Z(t). \tag{5}$$

For a tetrahedron Δ, let Δ_1 be the set of edges of Δ. We define

$$\overline{E} := \coprod_{t\in\mathbb{Z}} \Delta(t)_1.$$

Let $\widetilde{E}$ denote the set of all edges in the tetrahedron decomposition of $\Sigma \times \mathbb{R}$ and $\pi\colon \overline{E} \to \widetilde{E}$ be the canonical surjection.

Given parameters $(Z(t))_{t\in\mathbb{Z}}$, we can define associated parameter $Z_e = Z_e(t)$ for any $t \in \mathbb{Z}$ and $e \in \Delta(t)_1$ as the shape parameter of $\Delta(t)$ on the edge e, which is determined as in (4).

4.2 Edge-gluing conditions

Suppose that the shape parameters $(Z(t))_{t\in\mathbb{Z}}$ give an ideal tetrahedron decomposition[2]. This holds if and only if the following three conditions are satisfied.

First, we need the shape parameter periodicity condition as already discussed in (5). Second, we need

$$\operatorname{Im} Z(t) > 0 \quad \text{for any } t \quad \text{(positivity condition)},$$

so that the tetrahedron is positively oriented. Third, for each edge $g \in \widetilde{E}$, the product of all the shape parameters associated to the elements in $\pi^{-1}(g)$ must be 1 ([Thu79], see Figure 6):

$$\prod_{\bar{g}\in\pi^{-1}(g)} Z_{\bar{g}} = 1 \quad \text{(edge-gluing equation)}.$$

4.3 y-variables and gluing conditions

PROPOSITION 4.1. *Let $e(t) \in \tau(t)_1$ be the edge which we flip at t and $e'(t+1) \in \tau(t+1)_1$ be the edge which appears after the flip. The edge-gluing equation is satisfied for the shape parameters*

$$Z(t) := -y_{e(t)}(t) \left(= -y_{e'(t+1)}(t+1)^{-1}\right). \tag{6}$$

[2] Here we do not require completeness. See §4.4 for complete hyperbolic structures.

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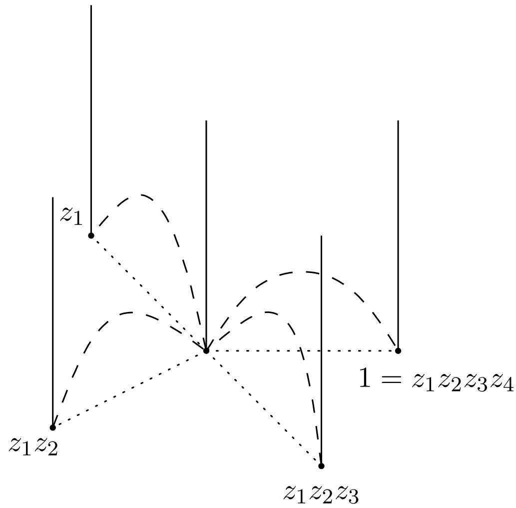

Figure 6.
The edge-gluing equation around an edge.

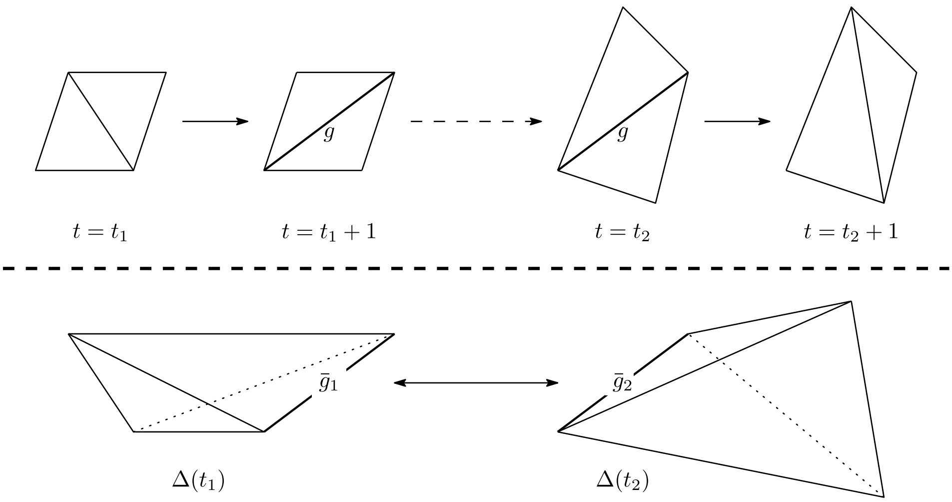

Figure 7.
An edge g in a tetrahedron decomposition appears at time t_1 and disappears at time t_2.

Proof. Let $g \in \widetilde{E}$ be an edge which appears at the t_1th flip at $\bar{g}'$ and disappears at the t_2th flip $\bar{g}''$ (Figure 7). Let $\bar{g}_1$ (resp. $\bar{g}_2$) be the unique element in $\Delta(t_1)_1 \cap \pi^{-1}(g)$ (resp. in $\Delta(t_2)_1 \cap \pi^{-1}(g)$). The gluing equation

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associated with g is

$$1 = \prod_{t=t_1}^{t_2} \prod_{\bar{g} \in \Delta(t)_1 \cap \pi^{-1}(g)} Z_{\bar{g}}$$
$$= Z(t_1) \times \left(\prod_{t=t_1+1}^{t_2-1} \prod_{\bar{g} \in \Delta(t)_1 \cap \pi^{-1}(g)} Z_{\bar{g}} \right) \times Z(t_2).$$

For this equation, we will show

$$Z(t_1) \times \left(\prod_{t=t_1+1}^{T} \prod_{\bar{g} \in \Delta(t)_1 \cap \pi^{-1}(g)} Z_{\bar{g}} \right) = -y_{g_0}(T+1)^{-1}, \tag{7}$$

where g_0 is the edge corresponding to g which $\tau(t)$ ($t = t_1 + 1, \ldots, t_2$) have in common. We show the equation above by induction with respect to T. The claim for $T = t_1$ trivially follows from the definition (6). Let us assume the above statement for $T \to T - 1$. To show the statement for T, we need to show

$$\prod_{\bar{g} \in \Delta(t)_1 \cap \pi^{-1}(g)} Z_{\bar{g}} = y_{g_0}(t)/y_{g_0}(t+1).$$

We will show this by classifying the positional relation of g_0 and $e(t)$.

- g_0 and $e(t)$ have no triangle in common: both side of the equation above is 1.
- g_0 and $e(t)$ have a single triangle in common:
  - $\overline{Q_t}(e(t), g_0) = 1$ (see Figure 8):
  $$(\text{LHS}) \overset{\text{equation (4)}}{=} 1 - Z(t)^{-1} \overset{\text{equation (2)}}{=} (\text{RHS}),$$
  - $\overline{Q_t}(e(t), g_0) = -1$:
  $$(\text{LHS}) \overset{\text{equation (4)}}{=} (1 - Z(t))^{-1} \overset{\text{equation (2)}}{=} (\text{RHS}),$$
- g_0 and $e(t)$ have two triangles in common:
  - $\overline{Q_t}(e(t), g_0) = \pm 2$ (see Figure 9):
  $$(\text{LHS}) \overset{\text{equation (4)}}{=} (1 - Z(t)^{\mp})^{\pm 2} \overset{\text{equation (2)}}{=} (\text{RHS}),$$
  - $\overline{Q_t}(e(t), g_0) = 0$: this cannot happen because we prohibit self-folded edges in this paper (see Figure 10).

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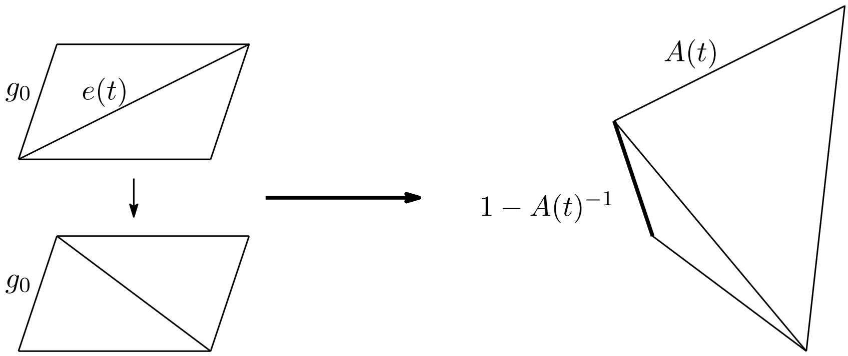

Figure 8.
The case with $\overline{Q_t}(e(t), g_0) = 1$.

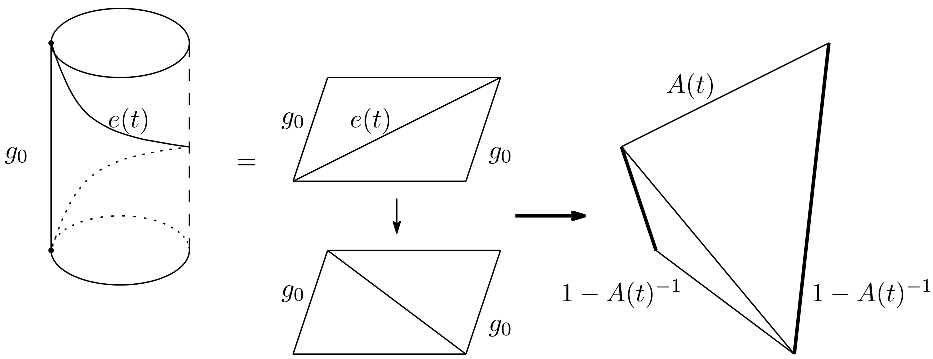

Figure 9.
The case with $\overline{Q_t}(e(t), g_0) = 2$.

4.4 Complete hyperbolic structures

Patching ideal tetrahedra with corners removed, we get a hyperbolic 3-manifold with boundaries, each of which is isomorphic to a torus. Note that such a boundary torus has two directions: the direction of "time" parameter t (*time direction*) and the direction of the original surface (*surface direction*)[3].

The intersection of a removed corner and a boundary torus gives a triangle on the torus with a shape parameter for each angle.

[3]We avoid to use the terms "longitude" and "meridian" to avoid a confusion.

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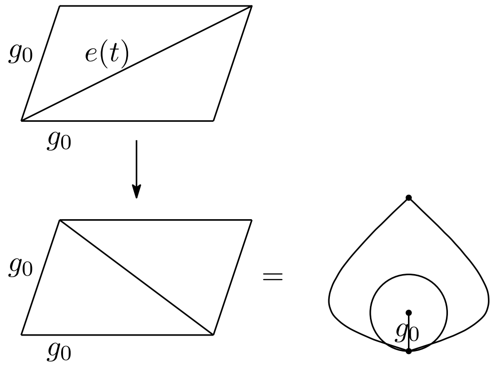

Figure 10.
The case with $\overline{Q_t}(e(t), g_0) = 0$.

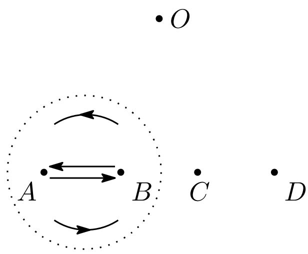

Figure 11.
A Dehn half-twist σ_1 along a circle containing A and B.

EXAMPLE 4.2. We take a five-punctured sphere. Let A, B, C, D, O be the punctures and σ_1 (resp. σ_2 or σ_3) be the Dehn half-twist along a circle containing A and B (resp. B and C, or C and D) in the anticlock direction (see Figure 11). Note that σ_1, σ_2 and σ_3 generate the braid group B_4. We take

- $\sigma_1\sigma_2\sigma_3^{-1}$ as a mapping class;
- the triangulation as in Figure 13;
- 8, 9, 5, 7, 1, 8 as a sequence of edges which we flip[4].

[4]Flipping at 8, 6 (resp. 6, 9, 5, 7 or 1, 8) corresponds to the half-twist σ_1 (resp. σ_2 or σ_3^{-1}). Canceling the doubled 6, we get the sequence above.

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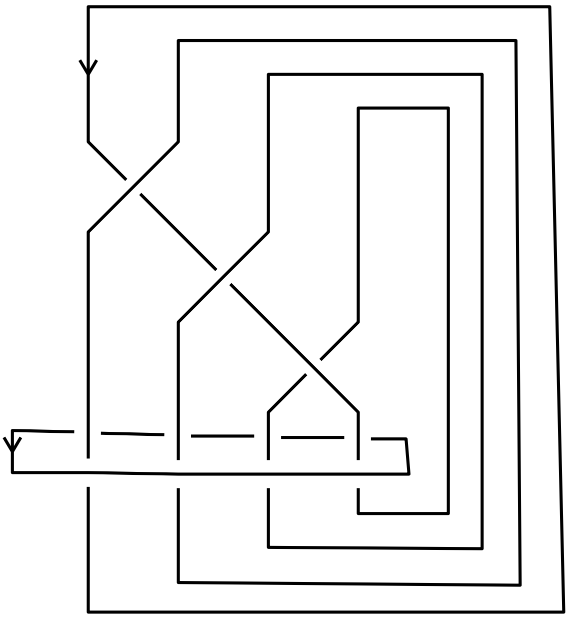

Figure 12.
The link corresponding to $\sigma_1\sigma_2\sigma_3^{-1}$.

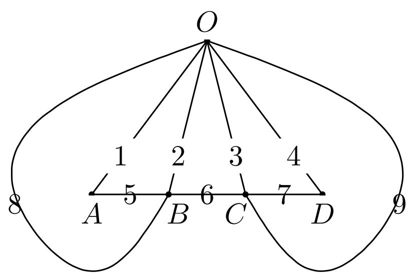

Figure 13.
A triangulation of the five-punctured sphere.

The mapping torus is the complement of the two-component link in $S^2 \times S^1$ (Figure 12), and hence we have two boundary components. We show the triangulation of the universal cover of one of the components in Figure 14.

Fix a puncture $m \in M$ of the surface and a time parameter t_0. Let F_i ($i \in \mathbb{Z}/n\mathbb{Z}$) be the triangle in $\tau(t_0)$ which is adjacent to e_{i-1}, m and e_i, where $(e_1, \ldots, e_n)$ is the sequence of arcs around m as before.

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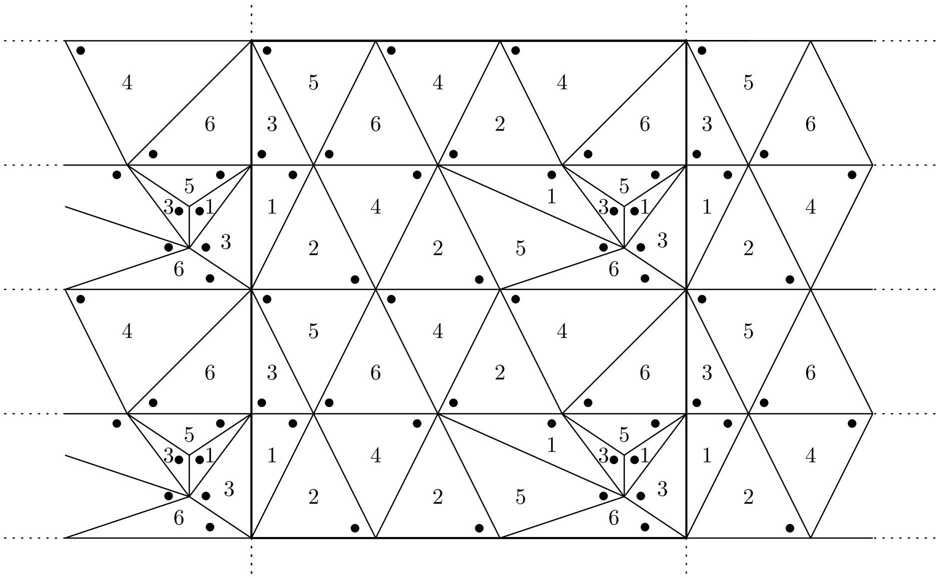

Figure 14.
The triangulation of the boundary torus. A triangle with number t represents the tth tetrahedron $\Delta(t)$, whose modulus $Z(t)$ corresponds to a dihedral angle represented by a black dot.

On the boundary torus, e_i represents a vertex and F_i represents an edge connecting e_{i-1} and e_i. The union of F_i's provides a (piecewise linear) closed curve on the boundary torus[5]. We call this a *vertical line* (see Figure 15).

The *holonomy* of along a cycle in the surface direction is given as follows. A vertical line divides the boundary torus into two parts. We fix one of them. For a vertex e_i on the vertical line, we take all angles in the universal cover which have e_i as the vertex and which are on the given side of the vertical line. We denote by H_i the product of the shape parameters associated to these angles. Then we have

$$\text{(the holonomy in the surface direction)} = \prod_i (-H_i). \tag{8}$$

A hyperbolic structure given by a sequence of shape parameters is complete if and only if the following condition holds:

[5] As a cycle, this represents the homology generator in the surface direction.

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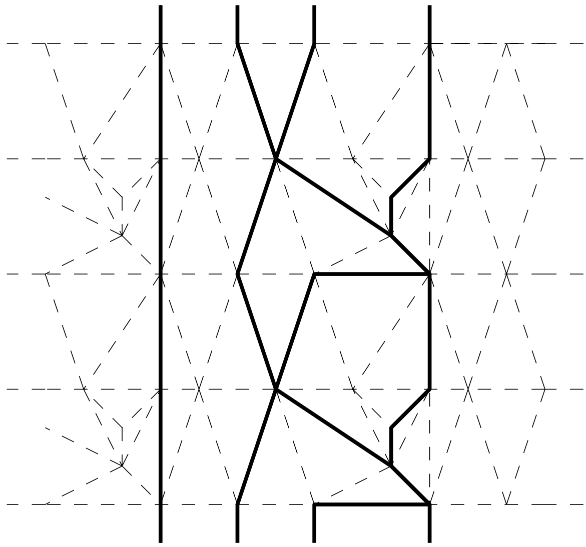

Figure 15.
Vertical lines, drawn in the triangulation of Figure 14.

the holonomy along the surface direction of each boundary is trivial (*cusp condition*, [NZ85]).

In Figure 7, we study the set of all tetrahedra which are adjacent to an edge. In this setting, the vertical line divides the set of these tetrahedra into two groups: tetrahedra which appear before/after $t = t_0$. Hence we have

$$H_i = Z(t_1) \times \left(\prod_{t=t_1+1}^{t_0-1} \prod_{\bar{e}_i \in \Delta(t)_1 \cap \pi^{-1}(e_i)} Z_{\bar{e}_i} \right).$$

By (7), the right hand side equals $-y_{e_i}(t_0)$. Therefore the holonomy (8) is equivalent with $y_m(t_0) = \prod_{i=1}^{n} y_{e_i}(t_0)$ (recall (3)). We can show this product is independent on the choice of t_0, either by induction or by using the edge-gluing conditions (vertical lines at different choices of t_0 are homologous in the triangulation of the boundary torus).

In summary, we get the following description of the holonomy:

PROPOSITION 4.3. *For a sequence of shape parameters determined by the result of Proposition 4.1, the holonomy around a puncture m in the surface direction is equal to y_m.*

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4.5 Main theorem

Let us summarize our results in the form of a theorem:

THEOREM 4.4. *Let $(y_e)_{e\in\tau_1}$ be nonzero complex numbers such that $y_m = 1$ for any puncture $m \in M$. Assume that $y_h(t)\Big|_{y_e(0)=y_e}$ is well defined for any h and t and that the periodicity equation is satisfied*

$$y_{\sigma(h)} = y_h(l)\Big|_{y_e(0)=y_e}.$$

Let us define the shape parameters $Z(t)$ by

$$Z(t) := -y_{e(t)}(t)\Big|_{y_e(0)=y_e},$$

where $e(t)$ is the edge flipped at time t, and suppose that $Z(t) \neq 0, 1$ for any t. Then $(Z(t))$ satisfies the edge-gluing equations in §4.2 and the cusp condition in §4.4.

This theorem gives a systematic method to identify for hyperbolic structures on mapping tori, formulated in the language of cluster algebras. For a genuine hyperbolic structure we also need to verify $\mathrm{Im}(Z(t)) > 0$; see the examples in the next section.

§5. Examples

In the last section, we demonstrate Theorem 4.4 in the case of a once-punctured torus and of a five-punctured sphere. The examples are chosen for the sake of simplicity, and the same methods apply to more general mapping classes of more general punctured surfaces (recall Remark 0.1). We also discuss an example of the six-punctured disc, to show that our formulation covers the nonhyperbolic cases not covered in Theorem 4.4.

5.1 Once-punctured torus and LR

Let us start with a once-punctured torus. We take a sequence of two flips as in Figure 16. This is the mapping class studied in [TY13, §3.1]. Then the shape parameter periodicity conditions are

$$\begin{aligned}
y_1 &= y_2^{-1}\left(1 + y_1^{-1}(1 + y_2^{-1})^2\right)^{-2},\\
y_2 &= y_3(1 + y_2)^2\left(1 + y_1(1 + y_2^{-1})^{-2}\right)^2,\\
y_3 &= y_1^{-1}(1 + y_2^{-1})^2,
\end{aligned}$$

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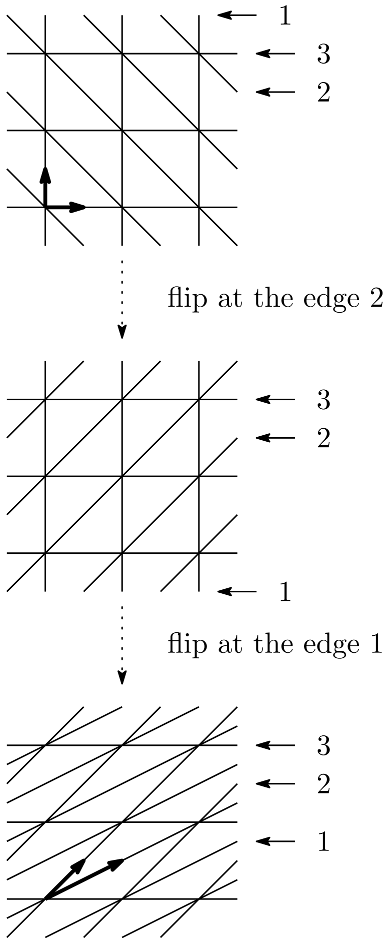

$$y_1(0) = y_1,$$

$$y_2(0) = y_2,$$

$$y_3(0) = y_3.$$

$$y_1(1) = y_1(1 + y_2^{-1})^{-2},$$

$$y_2(1) = y_2^{-1},$$

$$y_3(1) = y_3(1 + y_2)^2.$$

$$y_1(2) = y_1^{-1}(1 + y_2^{-1})^2,$$

$$y_2(2) = y_2^{-1}\left(1 + y_1^{-1}(1 + y_2^{-1})^2\right)^{-2},$$

$$y_3(2) = y_3(1 + y_2)^2\left(1 + y_1(1 + y_2^{-1})^{-2}\right)^2.$$

Figure 16.
Example: once-punctured torus and LR.

and the cusp condition is

$$y_1 y_2 y_3 = 1.$$

Solving these equations, we get a solution

$$y_1 = 1, \qquad y_2 = \frac{-1 - \sqrt{-3}}{2}, \qquad y_3 = \frac{-1 + \sqrt{-3}}{2}.$$

By Theorem 4.4, shape parameters

$$Z(0) = -y_2(0) = -y_2 = \frac{1 + \sqrt{-3}}{2},$$

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$$Z(1) = -y_1(1) = -y_1(1 + y_2^{-1})^{-2} = \frac{1 + \sqrt{-3}}{2},$$

satisfy edge-gluing conditions. Moreover, the imaginary parts of $Z(0)$ and $Z(1)$ are positive, and we obtain a complete hyperbolic structure on the mapping torus. The parameters coincide with the ones in [TY13, §3.1].

5.2 Five-punctured sphere and $\sigma_1\sigma_2\sigma_3^{-1}$

Let us take the example of a five-punctured sphere in Example 4.2.
The cusp conditions are

$$y_1y_2y_3y_4y_8y_9 = y_1y_5 = y_2y_5y_8y_6 = y_3y_6y_9y_7 = y_4y_7 = 1,$$

and the shape parameter periodicity conditions are

$$\begin{aligned}
y_6 &= \frac{1}{y_1y_5y_8^2y_9}\Big((1+y_5+y_5y_8)\,((1+y_9+y_7y_9)(1+y_9+y_8y_9) \\
&\quad + y_5(1+(1+y_7)(1+y_8)y_9)(1+(1+y_8+y_1y_8)y_9))\Big),\\
y_1 &= y_2(1+y_5+y_5y_8),\\
y_2 &= \frac{y_3y_7y_9(1+y_9+y_8y_9+y_5(1+y_8)(1+(1+y_8+y_1y_8)y_9))}{(1+y_9+y_7y_9)(1+y_9+y_8y_9)+y_5(1+(1+y_7)(1+y_8)y_9)(1+(1+y_8+y_1y_8)y_9)},\\
y_9 &= \frac{y_4(1+(1+y_7)(1+y_8)y_9)(1+y_9+y_8y_9+y_5(1+y_8)(1+(1+y_8+y_1y_8)y_9))}{(1+y_5+y_5y_8)(1+y_9+y_8y_9)},\\
y_8 &= \frac{y_1y_8y_9}{1+y_9+y_8y_9+y_5(1+y_8)(1+(1+y_8+y_1y_8)y_9)},\\
y_5 &= \frac{y_5y_6y_8}{1+y_5+y_5y_8},\\
y_4 &= \frac{(1+y_9+y_7y_9)(1+y_9+y_8y_9)+y_5(1+(1+y_7)(1+y_8)y_9)(1+(1+y_8+y_1y_8)y_9)}{y_7y_8y_9},\\
y_3 &= \frac{y_8(1+y_9+y_8y_9)}{(1+(1+y_7)(1+y_8)y_9)(1+y_9+y_8y_9+y_5(1+y_8)(1+(1+y_8+y_1y_8)y_9))},\\
y_7 &= \frac{y_1y_5y_7y_8y_9}{(1+y_9+y_7y_9)(1+y_9+y_8y_9)+y_5(1+(1+y_7)(1+y_8)y_9)(1+(1+y_8+y_1y_8)y_9)}.
\end{aligned}$$

Solving the shape parameter periodicity conditions with cusp conditions, we get 14 solutions. We take one of the solutions

$$\begin{aligned}
y_1 &= 1.781241 - 0.294452 \times \sqrt{-1},\\
y_2 &= 1,\\
y_3 &= -0.304877 + 0.754529 \times \sqrt{-1},\\
y_4 &= 0.460355 + 1.139318 \times \sqrt{-1},
\end{aligned}$$

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$$
\begin{aligned}
y_5 &= 0.546473 + 0.0903361 \times \sqrt{-1},\\
y_6 &= 1.155478 + 1.893847 \times \sqrt{-1},\\
y_7 &= 0.304877 - 0.754529 \times \sqrt{-1},\\
y_8 &= 0.304877 - 0.754529 \times \sqrt{-1},\\
y_9 &= -0.14865 - 0.664193 \times \sqrt{-1}.
\end{aligned}
$$

Following the algorithm in Theorem 4.4, we get the following six parameters

$$
\begin{gathered}
0.754529 \times \sqrt{-1} - 0.304877,\\
0.754529 \times \sqrt{-1} + 0.695123,\\
0.294452 \times \sqrt{-1} - 0.781241,\\
0.754529 \times \sqrt{-1} + 0.695122,\\
0.475124 \times \sqrt{-1} + 0.311704,\\
1.139320 \times \sqrt{-1} + 0.460354,
\end{gathered}
$$

whose imaginary parts are positive, which provide a complete hyperbolic structure. The volume of the mapping torus computed from the parameters above is[6]

$$4.851170.$$

This coincides with the value computed by SnapPea/SnapPy [CDW].

5.3 Nonhyperbolic example

Our formalism discussed in this paper applies to in general nonhyperbolic 3-manifolds which are themselves not covered in Theorem 4.4. To illustrate this point, let us consider a disk with six points, and we consider the $1/6$ rotation as a mapping class. The mapping class is realized as a sequence of three flips as in Figure 17.

[6]The hyperbolic volume of an ideal tetrahedron with modulus z is given by the Bloch–Wigner function

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|, \tag{9}$$

where $\mathrm{Li}_2(z) = -\int_0^z (\log(1-t)/t)\,dt$ is the Euler classical dilogarithm function. When a 3-manifold is triangulated by ideal tetrahedra, the hyperbolic volume of the 3-manifold is the sum of the hyperbolic volumes of the tetrahedra.

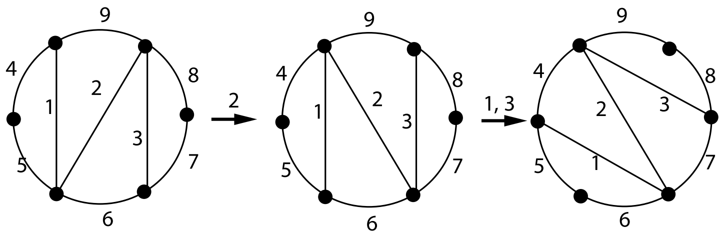

Figure 17.
A nonhyperbolic example, associated with the 1/6 rotation of the six-punctured disc.

The periodicity conditions are

$$y_1 = y_1(3), \qquad y_2 = y_2(3), \qquad y_3 = y_3(3), \qquad y_4 = y_5(3), \qquad y_5 = y_6(3),$$
$$y_6 = y_7(3), \qquad y_7 = y_8(3), \qquad y_8 = y_9(3), \qquad y_9 = y_4(3).$$

Note that indices of edges in the boundary are rotated. The y-variables are given by

$$\begin{aligned}
y_1(3) &= y_1^{-1}(1+y_2)^{-1},\\
y_2(3) &= y_2^{-1}(1+y_1(1+y_2))(1+y_3(1+y_2)),\\
y_3(3) &= y_3^{-1}(1+y_2)^{-1},\\
y_4(3) &= y_4(1+y_1^{-1}(1+y_2)^{-1})^{-1},\\
y_5(3) &= y_5(1+y_1(1+y_2)),\\
y_6(3) &= y_6(1+y_2^{-1})^{-1}(1+y_1^{-1}(1+y_2)^{-1})^{-1},\\
y_7(3) &= y_7(1+y_3^{-1}(1+y_2)^{-1})^{-1},\\
y_8(3) &= y_8(1+y_3(1+y_2)),\\
y_9(3) &= y_9(1+y_2^{-1})^{-1}(1+y_3^{-1}(1+y_2)^{-1})^{-1}.
\end{aligned}$$

A solution of the periodicity conditions is

$$y_1 = \tfrac{1}{2}, \qquad y_2 = 3, \qquad y_3 = \tfrac{1}{2}, \qquad y_4 = \sqrt{3}, \qquad y_5 = \tfrac{1}{\sqrt{3}},$$
$$y_6 = \tfrac{2}{\sqrt{3}}, \qquad y_7 = \sqrt{3}, \qquad y_8 = \tfrac{1}{\sqrt{3}}, \qquad y_9 = \tfrac{2}{\sqrt{3}}.$$

Shape parameters of three tetrahedra evaluated at the solution above are

$$Z(0) = -y_2(0) = -y_2 = -3,$$

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$$Z(1) = -y_1(1) = -y_1(1+y_2) = -2,$$
$$Z(2) = -y_3(2) = -y_3(1+y_2) = -2.$$

Substituting shape parameters to the Rogers dilogarithm $L(x)$, we have (with $Z(i)'' = (1 - Z(i))^{-1}$)

$$L(Z(0)'') + L(Z(1)'') + L(Z(2)'') = \frac{\pi^2}{6},$$

which is the complexified volume of the 3-manifold. This is identified with the central charge of $\hat{\mathrm{sl}}(2)$ WZW model at the level 4 (see Remark 0.3).

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References

[ADJM12] E. Andriyash, F. Denef, D. L. Jafferis and G. W. Moore, *Wall-crossing from supersymmetric galaxies*, J. High Energy Phys. **01** (2012), 115, arXiv:1008.0030.

[Ago11] I. Agol, "*Ideal triangulations of pseudo-Anosov mapping tori*", in *Topology and Geometry in Dimension Three*, Contemporary Mathematics **560**, American Mathematical Society, Providence, RI, 2011, 1–17.

[AGT10] L. F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, Lett. Math. Phys. **91** (2010), 167–197.

[Ami09] C. Amiot, *Cluster categories for algebras of global dimension 2 and quivers with potential*, Ann. Inst. Fourier **59**(6) (2009), 2525–2590.

[BFZ05] A. Bernstein, S. Fomin and A. Zelevinsky, *Cluster algebras III: upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), 1–52.

[CDW] M. Culler, N. M. Dunfield and J. R. Weeks, *Snappy, a computer program for studying the geometry and topology of 3-manifolds*, http://snappy.computop.org.

[CNV] S. Cecotti, A. Neitzke and C. Vafa, *R-twisting and 4d/2d correspondences*, preprint, arXiv:1006.3435.

[DS94] J. L. Dupont and C.-H. Sah, *Dilogarithm identities in conformal field theory and group homology*, Comm. Math. Phys. **161**(2) (1994), 265–282.

[EF12] R. Eager and S. Franco, *Colored BPS pyramid partition functions, quivers and cluster transformations*, J. High Energy Phys. **09** (2012), 038, arXiv:1112.1132.

[FG06] V. Fock and A. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. **103**(1) (2006), 1–211.

[FG07] V. Fock and A. Goncharov, "*Dual Teichmüller and lamination spaces*", in *Handbook of Teichmüller Theory, Vol. I, no. 1*, Eur. Math. Soc., Zurich, 2007, 647–684.

[FG09] V. V. Fock and A. B. Goncharov, *The quantum dilogarithm and representations of quantum cluster varieties*, Invent. Math. **175**(2) (2009), 223–286.

[Fom10] S. Fomin, "*Total positivity and cluster algebras*", in *Proceedings of the International Congress of Mathematicians, Vol. II*, Hindustan Book Agency, New Delhi, 2010, 125–145.

[FST08] S. Fomin, M. Shapiro and D. Thurston, *Cluster algebras and triangulated surfaces. Part I: cluster complexes*, Acta Math. **201**(1) (2008), 83–146.

[FZ02] S. Fomin and A. Zelevinsky, *Cluster algebras I: foundations*, J. Amer. Math. Soc. **15**(2) (2002), 497–529.

[FZ03] S. Fomin and A. Zelevinsky, *Y-systems and generalized associahedra*, Ann. of Math. (2) **158**(3) (2003), 977–1018.

[GMN] D. Gaiotto, G. W. Moore and A. Neitzke, *Framed BPS states*, Adv. Theor. Math. Phys. **17**(2) (2013), 241–397.

[GSV03] M. Gekhtman, M. Z. Shapiro and A. D. Vainshtein, *Cluster algebras and poisson geometry*, Mosc. Math. J. **3** (2003), 899–934.

[GT96] F. Gliozzi and R. Tateo, *Thermodynamic Bethe ansatz and three-fold triangulations*, Internat. J. Modern Phys. A **11**(22) (1996), 4051–4064.

[Gué06] F. Guéritaud, *On canonical triangulations of once-punctured torus bundles and two-bridge link complements*, Geom. Topol. **10** (2006), 1239–1284.

[Ked08] R. Kedem, *Q-systems as cluster algebras*, J. Phys. A **41** (2008), 194011.

[Kel10] B. Keller, "*Cluster algebras, quiver representations and triangulated categories*", in *Triangulated Categories*, London Mathematical Society Lecture Note Series **375**, Cambridge University Press, Cambridge, 2010, 76–160.

[Kel11] B. Keller, "*On cluster theory and quantum dilogarithm identities*", in *Representations of Algebras and Related Topics*, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, 85–116.

[Kel13] B. Keller, *The periodicity conjecture for pairs of Dynkin diagrams*, Ann. of Math. (2) **177**(1) (2013), 111–170.

[Kim12] Y. Kimura, *Quantum unipotent subgroup and dual canonical basis*, Kyoto J. Math. **52**(2) (2012), 277–331.

[KN11] R. Kashaev and T. Nakanishi, *Classical and quantum dilogarithm identities*, SIGMA **7**(102) (2011), 29, arXiv:1104.4630.

[KNS11] A. Kuniba, T. Nakanishi and J. Suzuki, *T-systems and Y-systems in integrable systems*, J. Phys. A **44** (2011), 103001.

[KS] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson–Thomas invariants and cluster transformations*, preprint, arXiv:0811.2435.

[KT15] T. Kitayama and Y. Terashima, *Torsion functions on moduli spaces in view of the cluster algebra*, Geom. Dedicata **175** (2015), 125–143.

[KY11] B. Keller and D. Yang, *Derived equivalences from mutations of quivers with potential*, Adv. Math. **226**(3) (2011), 2118–2168.

[Lec10] B. Leclerc, "*Cluster algebras and representation theory*", in *Proceedings of the International Congress of Mathematicians, Vol. IV*, Hindustan Book Agency, New Delhi, 2010, 2471–2488.

[Nag] K. Nagao, *Mapping class group, Donaldson–Thomas theory and S-duality*, http://www.math.nagoya-u.ac.jp/∼kentaron/MCG_DT.pdf.

[Nag11] K. Nagao, *Quantum dilogarithm idetities*, RIMS Kokyuroku Bessatsu **B28** (2011), 165–170.

[Nag13] K. Nagao, *Donaldson–Thomas theory and cluster algebras*, Duke Math. J. **162**(7) (2013), 1313–1367.

[Nak11a] H. Nakajima, *Quiver varieties and cluster algebras*, Kyoto J. Math. **51**(1) (2011), 71–126.

[Nak11b] T. Nakanishi, *Dilogarithm identities for conformal field theories and cluster algebras: simply laced case*, Nagoya Math. J. **202** (2011), 23–43.

[Nak11c] T. Nakanishi, "*Periodicities in cluster algebras and dilogarithm identities*", in *Representations of Algebras and Related Topics*, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, 407–443.

[Neu92] W. D. Neumann, "*Combinatorics of triangulations and the Chern–Simons invariant for hyperbolic 3-manifolds*", in *Topology '90 (Columbus, OH, 1990)*, Ohio State Univ. Math. Res. Inst. Publ. **1**, de Gruyter, Berlin, 1992, 243–271.

[NRT93] W. Nahm, A. Recknagel and M. Terhoeven, *Dilogarithm identities in conformal field theory*, Modern Phys. Lett. A **8**(19) (1993), 1835–1847.

[NZ85] W. D. Neumann and D. Zagier, *Volumes of hyperbolic* 3*-manifolds*, Topology **24** (1985), 307–332.

[Pla11] P.-G. Plamondon, *Cluster characters for cluster categories with infinite-dimensional morphism spaces*, Adv. Math. **227**(1) (2011), 1–39.

[Tes] J. Teschner, *Quantization of the Hitchin moduli spaces, Liouville theory, and the geometric Langlands correspondence I*, Adv. Theor. Math. Phys. **15**(2) (2011), 471–564.

[Tes07] J. Teschner, "*An analog of a modular functor from quantized Teichmüller theory*", in *Handbook of Teichmüller Theory, Vol. I, no. 1*, Eur. Math. Soc., Zurich, 2007, 685–760.

[Thu79] W. P. Thurston, *The geometry and topology of three-manifolds*, 1978–1979.

[TY11] Y. Terashima and M. Yamazaki, $\mathrm{SL}(2,R)$ *Chern–Simons, Liouville, and Gauge theory on duality walls*, J. High Energy Phys. **1108** (2011), 135, arXiv:1103.5748.

[TY13] Y. Terashima and M. Yamazaki, *Semiclassical analysis of the 3d/3d relation*, Phys. Rev. **D88**(2) (2013), 026011.

[TY14] Y. Terashima and M. Yamazaki, $3d\ N=2$ *theories from cluster algebras*, Progr. Theoret. Exp. Phys. **023** (2014), B01.

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