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A COUNTEREXAMPLE TO A CONTINUED FRACTION CONJECTURE

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Abstract It is known that if $a \in \mathbb{C} \setminus (-\infty, -\frac{1}{4}]$ and $a_n \to a$ as $n \to \infty$, then the infinite continued fraction with coefficients $a_1, a_2, \dots$ converges. A conjecture has been recorded by Jacobsen *et al.*, taken from the unorganized portions of Ramanujan's notebooks, that if $a \in (-\infty, -\frac{1}{4})$ and $a_n \to a$ as $n \to \infty$, then the continued fraction diverges. Counterexamples to this conjecture for each value of a in $(-\infty, -\frac{1}{4})$ are provided. Such counterexamples have already been constructed by Glutsyuk, but the examples given here are significantly shorter and simpler.

Keywords: continued fractions; Möbius transformations; iteration; dynamics

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1. Introduction

For each $n \in \mathbb{N}$, let a_n be a non-zero complex number and let t_n be the Möbius transformation $t_n(z) = a_n/(1 + z)$; then the continued fraction

$$\boldsymbol{K}(a_n \mid 1) = \cfrac{a_1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{1 + \cdots}}} \tag{1.1}$$

is considered to converge if the sequence with the nth term equal to the n-fold composition $t_1 \cdots t_n(0)$ converges within the extended complex plane $\mathbb{C}_\infty$. We identify the continued fraction (1.1) with the sequence $t_1, t_2, \dots$ of Möbius transformations. A problem derived from the private notebooks of Ramanujan is posed in [**4**, p. 38], which asks whether, for a given complex number $a \neq -\frac{1}{4}$ and a sequence $a_1, a_2, \dots$ that converges to a, the continued fraction $\boldsymbol{K}(a_n \mid 1)$ diverges if and only if $a \in (-\infty, -\frac{1}{4})$. In this paper it is demonstrated that $\boldsymbol{K}(a_n \mid 1)$ may or may not converge if $a \in (-\infty, -\frac{1}{4})$, thereby proving the conjecture to be false. Glutsyuk has already provided such examples in [**3**], but the methods here are significantly shorter and simpler. Our conclusions are summarized in a theorem, whose proof is postponed until § 3.

Theorem 1.1. *If $a \in (-\infty, -\frac{1}{4})$, then there are sequences a_n of real numbers that converge to a for which $\boldsymbol{K}(a_n \mid 1)$ converges and there are sequences a_n of real numbers that converge to a for which $\boldsymbol{K}(a_n \mid 1)$ diverges.*

2. Iteration of a single Möbius transformation

To understand the dynamics of the sequence $t_1 \cdots t_n$, where $t_n(z) = a_n/(1+z)$ and $a_n \to a$ as $n \to \infty$, one must first understand the dynamics of the sequence formed through iterating the Möbius map $t(z) = a/(1+z)$. The theory of iteration of a single Möbius transformation is well known (see, for example, [**1**] or [**5**]) and it is independent of continued fractions. We elaborate briefly on this theory.

The conjugacy type of a given Möbius transformation $f(z) = (Az+B)/(Cz+D)$ may be determined from the conjugation-invariant quantity $T(f) = (A+D)^2/(AD-BC)$: if $T(f) \in [0,4)$, then f is *elliptic*; if $T(f) = 4$, then f is *parabolic*; otherwise f is *loxodromic*. Therefore, t is elliptic if $a \in (-\infty, -\frac{1}{4})$, parabolic if $a = -\frac{1}{4}$, and loxodromic otherwise.

If t is loxodromic and $a_n \to a$ as $n \to \infty$, it follows from the general theory (see [**2**] or [**6**]) that $\boldsymbol{K}(a_n \mid 1)$ converges. If t is parabolic and $a_n \to a$ as $n \to \infty$, $\boldsymbol{K}(a_n \mid 1)$ may converge or it may diverge, and it is easy to construct examples of both circumstances. This leaves the situation of Theorem 1.1, when t is elliptic. Elliptic maps are by definition conjugate to Möbius maps of the form $z \mapsto \mathrm{e}^{\mathrm{i}\theta} z$, where $\theta \in (0, 2\pi)$, hence $t^n(0)$ diverges (since 0 is not a fixed point of t), that is, $\boldsymbol{K}(a \mid 1)$ diverges. Thus, for one part of Theorem 1.1 we may choose a_n to be the constant sequence $a, a, \dots$. The other part of Theorem 1.1 is proved in § 3.

3. Proof of Theorem 1.1

We need a preliminary lemma.

Lemma 3.1. *The subset of $(-\infty, -\frac{1}{4})$ consisting of those numbers $a \in (-\infty, -\frac{1}{4})$ for which $t(z) = a/(1+z)$ is a map of finite order is a dense subset of $(-\infty, -\frac{1}{4})$.*

Proof. Let t be conjugate to $g(z) = \mathrm{e}^{\mathrm{i}\theta} z$, $\theta \in (0, 2\pi)$; then

$$-1/a = T(t) = T(g) = 4\cos^2 \tfrac{1}{2}\theta. \tag{3.1}$$

The maps t and g are of finite order if and only if θ is a rational multiple of π, and rational multiples of π are dense in $(0, 2\pi)$. The result is assured by continuity of the correspondence (3.1). □

Proof of Theorem 1.1. We construct a sequence a_n that converges to $a \in (-\infty, -\frac{1}{4})$ for which $\boldsymbol{K}(a_n \mid 1)$ converges. By Lemma 3.1, we may choose a sequence $\alpha_1, \alpha_2, \dots$ in $(-\infty, -\frac{1}{4})$ that converges to a for which each map $s_n(z) = \alpha_n/(1+z)$ is of finite order. Let $\epsilon_1, \epsilon_2, \dots$ be a sequence in $(0,1)$ that converges to 0 for which $\sum \epsilon_n$ diverges. Define $t_n(z) = (1-\epsilon_n)\alpha_n/(1+z)$, for $n = 1, 2, \dots$. One may easily verify that

$$t_n s_n^{-2} t_n(z) = z + \epsilon_n. \tag{3.2}$$

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Since s_n is of finite order, the two equal quantities in (3.2) are also equal to the m-fold composition $t_n s_n \cdots s_n t_n(z)$, where $m = \text{order}(s_n)$.

For each n, choose an integer N_n such that $N_n\epsilon_n$ is greater than the maximum element from the finite set

$$\{|t_{n+1}s_{n+1}^q(0)| : 0 \leqslant q \leqslant \text{order}(s_{n+1}) - 2,\ t_{n+1}s_{n+1}^q(0) \neq \infty\}. \tag{3.3}$$

Let ϕ_n represent the string of maps $t_n, s_n, \dots, s_n, t_n$, in which s_n occurs order$(s_n) - 2$ times. The continued fraction corresponding to the sequence of Möbius maps

$$\phi_1, \dots, \phi_1, \phi_2, \dots, \phi_2, \dots \tag{3.4}$$

is the example we require, where the string ϕ_n occurs in the continued fraction N_n times. To see that (3.4) provides an example of the required form, notice that the coefficients a_n arise from the maps s_n or t_n, thus certainly $a_n \to a$ as $n \to \infty$. It remains to demonstrate that the continued fraction converges (to ∞). This is true as

$$(t_1 s_1^{-2} t_1)^{N_1} \cdots (t_n s_n^{-2} t_n)^{N_n}(z) = z + \sum_{i=1}^{n} N_i \epsilon_i,$$

by (3.2), hence

$$\begin{aligned}(t_1 s_1^{-2} t_1)^{N_1} \cdots (t_n s_n^{-2} t_n)^{N_n} &(t_{n+1} s_{n+1}^{-2} t_{n+1})^p t_{n+1} s_{n+1}^q(0) \\ &= t_{n+1}s_{n+1}^q(0) + p\epsilon_{n+1} + \sum_{i=1}^{n} N_i\epsilon_i \\ &> \sum_{i=1}^{n-1} N_i \epsilon_i,\end{aligned}$$

by (3.3), where $0 \leqslant p < N_{n+1}$ and $0 \leqslant q \leqslant \text{order}(s_{n+1}) - 2$. Therefore, the continued fraction converges to ∞. □

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