THE REPRESENTATIONS OF GL(3, q), GL(4, q), PGL(3, q), AND PGL(4, q)

ROBERT STEINBERG

1. Introduction. This paper is a result of an investigation into general methods of determining the irreducible characters of GL(n, q), the group of all non-singular linear substitutions with marks in GF(q), and of the related groups, SL(n, q), PGL(n, q), PSL(n, q), the corresponding group of determinant unity, projective group, projective group of determinant unity, respectively. This investigation is not complete, but the general problem was answered partially in [9]. In [3], [7], [6], [1], Frobenius, Schur, Jordan, and Brinkmann gave the characters of PSL(2, p); SL(2, q), GL(2, q); SL(2, q), GL(2, q); PSL(3, q), respectively. In this paper in §2 and §3, the characters of GL(2, q) and GL(3, q) are determined, and, from them, those of PGL(2, q) and PGL(3, q) deduced. In §4, an outline of the determination of the characters of GL(4, q) is given together with the degrees and frequencies of the characters of GL(4, q) and PGL(4, q) and a table of the rational characters of GL(4, q).

The simple properties of the underlying geometry, PG($n-1$, q), of which PGL(n, q) is the collineation group, are used throughout the work. The most powerful and frequent tool used in the determination of the characters is the Frobenius method[1] of induced representations [5] which enables one to construct a representation of a group if a representation of a subgroup is known. The explicit formula for the character in this case is $\chi(G) = \frac{m}{g_G} \Sigma\psi(G')$, where m is the index of the subgroup, g_G is the number of elements of the group similar to G, ψ is the character of the subgroup, and the summation is made over all elements G' which are similar to G and lie in the subgroup. Of fundamental use in the application of this method are the $q-1$ linear characters of GL(n, q) which correspond to the powers of the determinants of the matrices which define the elements of GL(n, q). Also very useful are pseudo-characters—linear combinations of irreducible characters with negative coefficients permissible—and the fact that a pseudo-character, $\chi(G)$, is an irreducible character if and only if $\Sigma|\chi(G)|^2 = g$ and $\chi(E) > 0$, where E is the unit element of the group.

The descent from the characters of GL(n, q) to those of PGL(n, q) is immediate because of the following two theorems due to Frobenius [4], [5]:

If $\mathfrak{H}$ is a normal subgroup of a group $\mathfrak{G}$, then every character of $\mathfrak{G}/\mathfrak{H}$ is also a character of $\mathfrak{G}$.

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[1] See [8] for a complete account of the properties of group characters used here.

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In order that a character of $\mathfrak{G}$ may belong to the group $\mathfrak{G}/\mathfrak{H}$, it is necessary and sufficient that it have the same value for all elements of $\mathfrak{H}$. Then, it has also equal values for every two elements of $\mathfrak{G}$ which are equivalent mod $\mathfrak{H}$.

In our case, $\mathfrak{G}$ is the group GL(n, q), $\mathfrak{H}$ is the cyclic group of the $q - 1$ scalar matrices, and $\mathfrak{G}/\mathfrak{H}$ is the group PGL(n, q). For this reason, and also because the group GL(n, q) is easier to handle, its characters are first determined and then those of PGL(n, q) obtained from them.

In what follows, $\chi_q^{(r)}$ for example, will denote a character of degree q, the superscript being used to distinguish between two characters of the same degree. GL$(1, 2; q)$ denotes the subgroup $\begin{pmatrix} A_1 & 0 \\ * & A_2 \end{pmatrix}$ of GL$(3, q)$; $\rho, \sigma, \tau, \omega$ are primitive elements of GF(q), GF(q^2), GF(q^3), GF(q^4) respectively, such that $\rho = \sigma^{q+1} = \tau^{q^2+q+1} = \omega^{q^3+q^2+q+1}$ and $\sigma = \tau^{q^2+1}$.

2. The characters of GL$(2, q)$ **and** PGL$(2, q)$. The group GL$(2, q)$ is of order $q(q-1)^2(q+1)$ and each of its elements is similar to a matrix of one of the following four types [2]:

$$A_1: \begin{pmatrix} \rho^a & \\ & \rho^a \end{pmatrix},\quad A_2: \begin{pmatrix} \rho^a & \\ 1 & \rho^a \end{pmatrix},\quad A_3: \begin{pmatrix} \rho^a & \\ & \rho^b \end{pmatrix}_{a \neq b},\quad B_1: \begin{pmatrix} \sigma^a & \\ & \sigma^{aq} \end{pmatrix}_{a \neq \text{mult. } (q+1)}$$

The number of classes of each type and the number of elements in each class is given by Table I. The total number of classes is $(q-1)(q+1) = k$.

TABLE I

Element	Number of classes	Number of elements in each class
A_1	$q - 1$	1
A_2	$q - 1$	$(q-1)(q+1)$
A_3	$\frac{1}{2}(q-1)(q-2)$	$q(q+1)$
B_1	$\frac{1}{2}q(q-1)$	$q(q-1)$

Now, if we consider each matrix as a linear transformation of PG$(1, q)$, we get a representation of degree $q + 1$ representing the permutation of the points of PG$(1, q)$. The character of any element of GL$(2, q)$ is just the number of points left fixed by it. This permutation group is doubly transitive and hence splits into the unit representation and an irreducible representation [9] of degree q. Multiplication of each of these characters by each of the $q - 1$ linear characters given by the powers of the determinants gives us $q - 1$ irreducible characters of degree 1 and $q - 1$ of degree q. (See Table I.)

We next consider the subgroup GL$(1, 1; q) = \begin{pmatrix} A_1 & 0 \\ * & B_1 \end{pmatrix}$ of index $q + 1$. Clearly, any character of A_1 or GL$(1, q)$ multiplied by any character of B_1 or GL$(1, q)$ is a character of GL$(1, 1; q)$. If we use the linear characters of GL$(1, 1; q)$ obtainable in this way as a basis for Frobenius's method of induced

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characters, we get $\frac{1}{2}(q-1)(q-2)$ irreducible characters of degree $q+1$ of $GL(2, q)$. (See Table I.)

Finally, the linear characters of the cyclic subgroup $\begin{pmatrix}\sigma & \\ & \sigma^q\end{pmatrix}^a$ of index $q(q-1)$ induce in $GL(2, q)$ the following representations $\Psi_{q(q-1)}{}^{(n)}$ of degree $q^2 - q$, all of which are reducible:

$$A_1: (q^2 - q)\ \epsilon^{na(q+1)},\quad A_2: 0,\quad A_3: 0,\quad B_1: \epsilon^{na} + \epsilon^{naq},$$

where $\epsilon^{q^2-1} = 1$ and $n = 1, 2, \ldots, q-1$. But, if we form $\chi_q{}^{(o)}\chi_{q+1}{}^{(o,n)} - \chi_{q+1}{}^{(o,n)} - \psi_{q(q-1)}{}^{(n)}$, we get an irreducible character provided $n \neq$ mult. $(q+1)$. We thus have $\frac{1}{2}q(q-1)$ irreducible characters of degree $q-1$ and this completes the list since we now have in all $(q-1)(q+1) = k$ characters. They are shown in Table II.

TABLE II
Characters of $GL(2, q)$

Element	$\chi_1{}^{(n)}$	$\chi_q{}^{(n)}$	$\chi_{q+1}{}^{(m,n)}$	$\chi_{q-1}{}^{(n)}$
	$n = 1, 2, \ldots, q-1$ $\epsilon^{q-1} = 1$	$n = 1, 2, \ldots, q-1$ $\epsilon^{q-1} = 1$	$m, n = 1, 2, \ldots, q-1$; $m \neq n$; $(m, n) \equiv (n, m)$ $\epsilon^{q-1} = 1$	$n = 1, 2, \ldots, q^2-2$; $n \neq$ mult. $(q+1)$ $\epsilon^{q^2-1} = 1$
A_1	ϵ^{2na}	$q\epsilon^{2na}$	$(q+1)\epsilon^{(m+n)a}$	$(q-1)\epsilon^{na(q+1)}$
A_2	ϵ^{2na}	0	$\epsilon^{(m+n)a}$	$-\epsilon^{na(q+1)}$
A_3	$\epsilon^{n(a+b)}$	$\epsilon^{n(a+b)}$	$\epsilon^{ma+nb} + \epsilon^{na+mb}$	0
B_1	ϵ^{na}	$-\epsilon^{na}$	0	$-(\epsilon^{na} + \epsilon^{naq})$

The theorems of Frobenius [4], [5] mentioned in the introduction immediately give us the characters of $PGL(2, q)$. For q even they are as in Table III. For q odd, there are in addition the two characters

$$A_1: 1,\quad A_2: 1,\quad A_3: (-1)^{a+b},\quad B_1: (-1)^a,$$

and

$$A_1: q,\quad A_2: 0,\quad A_3: (-1)^{a+b},\quad B_1: (-1)^{a+1}.$$

TABLE III
Characters of $PGL(2, q)$

Element	χ_1	χ_q	$\chi_{q+1}{}^{(n)}$	$\chi_{q-1}{}^{(n)}$
			$n = 1, 2, \ldots, [\frac{1}{2}(q-1)]$ $\epsilon^{q-1} = 1$	$n = 1, 2, \ldots, [\frac{1}{2}(q+1)]$ $\epsilon^{q+1} = 1$
A_1	1	q	$q+1$	$q-1$
A_2	1	0	1	-1
A_3	1	1	$\epsilon^{n(b-a)} + \epsilon^{-n(b-a)}$	0
B_1	1	-1	0	$-(\epsilon^{na} + \epsilon^{naq})$

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3. The characters of GL(3, q) **and** PGL(3, q). The group GL(3, q) is of order $q^3(q-1)^3(q+1)(q^2+q+1)$ and each of its elements similar to one of the following types [2]:

$$A_2: \begin{pmatrix} \rho^a & & \\ & \rho^a & \\ & & \rho^a \end{pmatrix}, \quad A_2: \begin{pmatrix} \rho^a & & \\ 1 & \rho^a & \\ & & \rho^a \end{pmatrix}, \quad A_3: \begin{pmatrix} \rho^a & & \\ 1 & \rho^a & \\ & 1 & \rho^a \end{pmatrix}, \quad A_4: \begin{pmatrix} \rho^a & & \\ & \rho^a & \\ & & \rho^b \end{pmatrix},$$

$$A_5: \begin{pmatrix} \rho^a & & \\ 1 & \rho^a & \\ & & \rho^b \end{pmatrix}, \quad A_6: \begin{pmatrix} \rho^a & & \\ & \rho^b & \\ & & \rho^c \end{pmatrix}, \quad B_1: \begin{pmatrix} \rho^a & & \\ & \sigma^b & \\ & & \sigma^{bq} \end{pmatrix}, \quad C_1: \begin{pmatrix} \tau^a & & \\ & \tau^{aq} & \\ & & \tau^{aq^2} \end{pmatrix},$$

where $a \neq$ mult. (q^2+q+1) in C. The number of elements in each class and the number of classes of each type are given in Table IV. The total number of classes is $q(q-1)(q+1) = k$.

TABLE IV

Element	Number of Classes	Elements in each Class
A_1	$q-1$	1
A_2	$q-1$	$(q-1)(q+1)(q^2+q+1)$
A_3	$q-1$	$q(q-1)^2(q+1)(q^2+q+1)$
A_4	$(q-1)(q-2)$	$q^2(q^2+q+1)$
A_5	$(q-1)(q-2)$	$q^2(q-1)(q+1)(q^2+q+1)$
A_6	$\frac{1}{6}(q-1)(q-2)(q-3)$	$q^3(q+1)(q^2+q+1)$
B_1	$\frac{1}{2}q(q-1)$	$q^3(q-1)(q^2+q+1)$
C_1	$\frac{1}{3}q(q-1)(q+1)$	$q^3(q-1)^2(q+1)$

Here, as before, the permutation of the points of the underlying geometry gives us a double-transitive permutation group, in this case of degree q^2+q+1. We thus get the unit representation and an irreducible representation of degree q^2+q. The geometric entities each of which consists of a point and a line through it are also permuted by the elements of GL(3, q), and this furnishes us with a representation of degree $(q+1)(q^2+q+1)$. The orthogonality properties of group characters tell us that the character of this representation contains the unit character χ_1 once and χ_{q^2+q} twice and an irreducible character [9] of degree q^3. Multiplying each of the characters of degrees 1, q^2+q, q^3 by each of the $q-1$ linear characters given by the powers of the determinants, we obtain $q-1$ irreducible characters of each of these degrees, as in Table V.

TABLE V

Element	$\chi_1^{(n)}$	$\chi_{q^2+q}^{(n)}$	$\chi_{q^3}^{(n)}$
A_1	ϵ^{3na}	$(q^2+q)\epsilon^{3na}$	$q^3\epsilon^{3na}$
A_2	ϵ^{3na}	$q\epsilon^{3na}$	0
A_3	ϵ^{3na}	0	0
A_4	$\epsilon^{n(2a+b)}$	$(q+1)\epsilon^{n(2a+b)}$	$q\epsilon^{n(2a+b)}$
A_5	$\epsilon^{n(2a+b)}$	$\epsilon^{n(2a+b)}$	0
A_6	$\epsilon^{n(a+b+c)}$	$2\epsilon^{n(a+b+c)}$	$\epsilon^{n(a+b+c)}$
B_1	$\epsilon^{n(a+b)}$	0	$-\epsilon^{n(a+b)}$
C_1	ϵ^{na}	$-\epsilon^{na}$	ϵ^{na}

(where $n = 1, 2, \ldots q-1$ and $\epsilon^{q-1} = 1$).

We next consider the subgroup of index q^2+q+1:

$$\mathrm{GL}(1, 2; q) = \begin{pmatrix} A_1 & 0 \; 0 \\ * & A_2 \\ * & \end{pmatrix}.$$

It is clear that any character of A_1 (or GL(1, q)) multiplied by any character of A_2(or GL(2, q)) is a character of GL(1, 2; q). By multiplying linear characters of GL(1, q) by the characters of degree 1, q, $q+1$, $q-1$ of GL(2, q) determined in §2, we get characters of these degrees of GL(1, 2; q). These characters induce in GL(3, q) a set of characters from which we can extract $(q-1)(q-2)$ irreducible characters of degree q^2+q+1, $(q-1)(q-2)$ of degree $q(q^2+q+1)$, $\frac{1}{6}(q-1)(q-2)(q-3)$ of degree $(q+1)(q^2+q+1)$, $\frac{1}{2}q(q-1)^2$ of degree $(q-1)(q^2+q+1)$. See Table VI and Table VII.

TABLE VI

Element	$\chi_{q^2+q+1}^{(m,n)}$	$\chi_{q(q^2+q+1)}^{(m,n)}$
A_1	$(q^2+q+1)\,\epsilon^{(m+2n)a}$	$q(q^2+q+1)\,\epsilon^{(m+2n)a}$
A_2	$(q+1)\epsilon^{(m+2n)a}$	$q\epsilon^{(m+2n)a}$
A_3	$\epsilon^{(m+2n)a}$	0
A_4	$(q+1)\epsilon^{(m+n)a+nb}+\epsilon^{2na+mb}$	$(q+1)\epsilon^{(m+n)a+nb}+q\epsilon^{2na+mb}$
A_5	$\epsilon^{(m+n)a+nb}+\epsilon^{2na+mb}$	$\epsilon^{(m+n)a+nb}$
A_6	$\Sigma_{(a,\,b,\,c)}\epsilon^{ma+n(b+c)}$	$\Sigma_{(a,\,b,\,c)}\epsilon^{ma+n(b+c)}$
B_1	ϵ^{ma+nb}	$-\epsilon^{ma+nb}$
C_1	0	0

(where $m, n = 1, 2, \ldots q-1$; $m \neq n$ and $\epsilon^{q-1} = 1$).

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TABLE VII

Element	$\chi_{(q+1)(q^2+q+1)}^{(l,m,n)}$	$\chi_{(q-1)(q^2+q+1)}^{(m,n)}$
	$l, m, n, = 1, 2, \ldots, q-1; l \neq m \neq n \neq l;$ $\epsilon^{q-1} = 1$	$m = 1, 2, \ldots, q-1; n = 1, 2, \ldots, q^2-2;$ $n \neq$ mult. $(q+1)$ $\epsilon^{q^2-1} = 1$
A_1	$(q+1)(q^2+q+1)\epsilon^{(l+m+n)a}$	$(q-1)(q^2+q+1)\epsilon^{(m+n)a(q+1)}$
A_2	$(2q+1)\epsilon^{(l+m+n)a}$	$-\epsilon^{(m+n)a(q+1)}$
A_3	$\epsilon^{(l+m+n)a}$	$-\epsilon^{(m+n)a(q+1)}$
A_4	$(q+1)\Sigma_{(l,m,n)}\epsilon^{(l+m)a+nb}$	$(q-1)\,\epsilon^{(na+mb)(q+1)}$
A_5	$\Sigma_{(l,m,n)}\epsilon^{(l+m)a+nb}$	$-\epsilon^{(na+mb)(q+1)}$
A_6	$\Sigma_{(l,m,n)}\epsilon^{la+mb+nc}$	0
B_1	0	$-\epsilon^{ma(q+1)}(\epsilon^{nb}+\epsilon^{nbq})$
C_1	0	0

By $\Sigma_{(l,m,n)}\epsilon^{(l+m)a+nb}$, we mean the symmetric function in l, m, and n which has $\epsilon^{(l+m)a+nb}$ as its typical term.

Finally, we turn to the cyclic subgroup of order $(q-1)(q^2+q+1)$:

$$\begin{pmatrix} \tau & & \\ & \tau^q & \\ & & \tau^{q^2} \end{pmatrix}^a.$$

The linear characters of this subgroup induce the following in the group GL$(3,q)$:

A_1: $q^3(q-1)^2(q+1)\ \epsilon^{na(q^2+q+1)}$, A_2: 0, A_3: 0, A_4: 0,
A_5: 0, A_6: 0, B_1: 0, C_1: $\epsilon^{na}+\epsilon^{naq}+\epsilon^{naq^2}$.

If from this character we subtract $[\chi_{q^3}{}^{(0)} - \chi_{q^2+q}{}^{(0)} + \chi_1{}^{(0)}]\,\chi_{(q-1)(q^2+q+1)}{}^{(0,n)}$, we get:

A_1: $(q-1)^2(q+1)\epsilon^{na(q^2+q+1)}$, A_2: $-(q-1)\epsilon^{na(q^2+q+1)}$, A_3: $\epsilon^{na(q^2+q+1)}$
A_4: 0, A_5: 0, A_6: 0, B_1: 0, C_1: $\epsilon^{na}+\epsilon^{naq}+\epsilon^{naq^2}$.

This is an irreducible character if $n \neq$ mult. (q^2+q+1). Since $(n) \equiv (nq) \equiv (nq^2)$, we thus get $\frac{1}{3}\,q(q-1)(q+1)$ irreducible characters of degree $(q-1)^2(q+1)$.

This completes the list of characters since we have now obtained $q(q-1)(q+1) = k$ irreducible characters.

In obtaining the characters of PGL$(3, q)$, again two cases must be distinguished: $q = 3t+1$ or $q \neq 3t+1$. The revision of classes and characters in each case is straightforward and we shall content ourselves with a list of the number of characters of each degree. (See Table VIII.)

TABLE VIII
Characters of PGL$(3, q)$

Degree	1	q^2+q	q^3	q^2+q+1	$q(q^2+q+1)$	$(q+1)\times(q^2+q+1)$	$(q-1)\times(q^2+q+1)$	$(q-1)^2\times(q+1)$
Frequency								
$q=3t+1$	3	3	3	$q-4$	$q-4$	$\frac{1}{6}(q^2-5q+10)$	$\frac{1}{2}q(q-1)$	$\frac{1}{3}(q-1)(q+2)$
$q \neq 3t+1$	1	1	1	$q-2$	$q-2$	$\frac{1}{6}(q-2)(q-3)$	$\frac{1}{2}q(q-1)$	$\frac{1}{3}q(q+1)$

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4. The characters of GL(4, q) **and** PGL(4, q). The group GL(4, q) is of order $q^6(q-1)^4(q+1)^2(q^2+1)(q^2+q+1)$ and each of its elements is similar to one of the following twenty-two types [2]:

$$A_1\colon \begin{pmatrix} \rho^a &&& \\ & \rho^a && \\ && \rho^a & \\ &&& \rho^a \end{pmatrix}, \quad A_2\colon \begin{pmatrix} \rho^a &&& \\ 1 & \rho^a && \\ && \rho^a & \\ &&& \rho^a \end{pmatrix}, \quad A_3\colon \begin{pmatrix} \rho^a &&& \\ 1 & \rho^a && \\ && \rho^a & \\ && 1 & \rho^a \end{pmatrix},$$

$$A_4\colon \begin{pmatrix} \rho^a &&& \\ 1 & \rho^a && \\ & 1 & \rho^a & \\ &&& \rho^a \end{pmatrix}, \quad A_5\colon \begin{pmatrix} \rho^a &&& \\ 1 & \rho^a && \\ & 1 & \rho^a & \\ && 1 & \rho^a \end{pmatrix}, \quad A_6\colon \begin{pmatrix} \rho^a &&& \\ & \rho^a && \\ && \rho^a & \\ &&& \rho^b \end{pmatrix},$$

$$A_7\colon \begin{pmatrix} \rho^a &&& \\ 1 & \rho^a && \\ && \rho^a & \\ &&& \rho^b \end{pmatrix}, \quad A_8\colon \begin{pmatrix} \rho^a &&& \\ 1 & \rho^a && \\ & 1 & \rho^a & \\ &&& \rho^b \end{pmatrix}, \quad A_9\colon \begin{pmatrix} \rho^a &&& \\ & \rho^a && \\ && \rho^b & \\ &&& \rho^b \end{pmatrix},$$

$$A_{10}\colon \begin{pmatrix} \rho^a &&& \\ 1 & \rho^a && \\ && \rho^b & \\ &&& \rho^b \end{pmatrix}, \quad A_{11}\colon \begin{pmatrix} \rho^a &&& \\ 1 & \rho^a && \\ && \rho^b & \\ && 1 & \rho^b \end{pmatrix}, \quad A_{12}\colon \begin{pmatrix} \rho^a &&& \\ & \rho^a && \\ && \rho^b & \\ &&& \rho^c \end{pmatrix},$$

$$A_{13}\colon \begin{pmatrix} \rho^a &&& \\ 1 & \rho^a && \\ && \rho^b & \\ &&& \rho^c \end{pmatrix}, \quad A_{14}\colon \begin{pmatrix} \rho^a &&& \\ & \rho^b && \\ && \rho^c & \\ &&& \rho^d \end{pmatrix}, \quad B_1\colon \begin{pmatrix} \rho^a &&& \\ & \rho^a && \\ && \sigma^b & \\ &&& \sigma^{bq} \end{pmatrix},$$

$$B_2\colon \begin{pmatrix} \rho^a &&& \\ 1 & \rho^a && \\ && \sigma^b & \\ &&& \sigma^{bq} \end{pmatrix}, \quad B_3\colon \begin{pmatrix} \rho^a &&& \\ & \rho^b && \\ && \sigma^c & \\ &&& \sigma^{cq} \end{pmatrix}, \quad C_1\colon \begin{pmatrix} \sigma^a &&& \\ & \sigma^{aq} && \\ && \sigma^a & \\ &&& \sigma^{aq} \end{pmatrix},$$

$$C_2\colon \begin{pmatrix} \sigma^a &&& \\ & \sigma^{aq} && \\ 1 && \sigma^a & \\ & 1 && \sigma^{aq} \end{pmatrix}, C_3\colon \begin{pmatrix} \sigma^a &&& \\ & \sigma^{aq} && \\ && \sigma^b & \\ &&& \sigma^{bq} \end{pmatrix}, D_1\colon \begin{pmatrix} \rho^a &&& \\ & \tau^b && \\ && \tau^{bq} & \\ &&& \tau^{bq^2} \end{pmatrix}, E_1\colon \begin{pmatrix} \omega^a &&& \\ & \omega^{aq} && \\ && \omega^{aq^2} & \\ &&& \omega^{aq^3} \end{pmatrix}$$

Now, we shall make use of the underlying geometry to obtain five irreducible characters. To do this, we consider the following five geometric entities: the PG(3, q); a point; a line; a point and a line through it; a point, a line through it, and a plane through the line. It will be noted that these five entities correspond to the five partitions of 4: (4), (13), (2^2), $(1^2 2)$, (1^4), respectively. In fact, GL(4, q), GL(1, 3; q), GL(2, 2; q), GL(1, 1, 2; q) and GL(1, 1, 1, 1; q) are the subgroups of GL(4, q) which leave fixed one of each of these entities, respectively. Each of these sets of entities will be permuted by the elements of

TABLE IX

Characters of GL(4 q) and PGL(4, q)

Element	Unit	Point	$q(q^2+q+1)$	Line	$q^2(q^2+1)$	Point-line	$q^3(q^2+q+1)$	Point-Line-Plane	q^6
A_1	1	$(q+1)(q^2+1)$	$q(q^2+q+1)$	$(q^2+1)(q^2+q+1)$	$q^2(q^2+1)$	$(q+1)(q^2+1)(q^2+q+1)$	$q^3(q^2+q+1)$	$(q+1)^2(q^2+1)(q^2+q+1)$	q^6
A_2	1	q^2+q+1	q^2+q	$2q^2+q+1$	q^2	q^3+3q^2+2q+1	q^3	$3q^3+5q^2+3q+1$	0
A_3	1	$q+1$	q	q^2+q+1	q^2	q^2+2q+1	0	$2q^2+3q+1$	0
A_4	1	$q+1$	q	$q+1$	0	$2q+1$	0	$3q+1$	0
A_5	1	1	0	1	0	1	0	1	0
A_6	1	q^2+q+2	q^2+q+1	$2q^2+2q+2$	q^2+q	q^3+4q^2+4q+3	q^3+q^2+q	$4q^3+8q^2+8q+4$	q^3
A_7	1	$q+2$	$q+1$	$2q+2$	q	$4q+3$	q	$8q+4$	0
A_8	1	2	1	2	0	3	0	4	0
A_9	1	$2q+2$	$2q+1$	q^2+2q+3	q^2+1	$2q^2+6q+4$	q^2+2q	$6q^2+12q+6$	q^2
A_{10}	1	$q+2$	$q+1$	$q+3$	1	$3q+4$	q	$6q+6$	0
A_{11}	1	2	1	3	1	4	0	6	0
A_{12}	1	$q+3$	$q+2$	$2q+4$	$q+1$	$5q+7$	$2q+1$	$12q+12$	q
A_{13}	1	3	2	4	1	7	1	12	0
A_{14}	1	4	3	6	2	12	3	24	1
B_1	1	$q+1$	q	2	$-q+1$	$q+1$	-1	0	$-q$
B_2	1	1	0	2	1	1	-1	0	0
B_3	1	2	1	2	0	2	-1	0	-1
C_1	1	0	-1	q^2+1	q^2+1	0	$-q^2$	0	q^2
C_2	1	0	-1	1	1	0	0	0	0
C_3	1	0	-1	2	2	0	-1	0	1
D_1	1	1	0	0	-1	0	0	0	1
D_2	1	0	-1	0	0	0	1	0	-1

GL(4, q) and in this way permutation representations of degree 1, $(q+1)(q^2+1)$, $(q^2+1)(q^2+q+1)$, $(q+1)(q^2+1)(q^2+q+1)$ and $(q+1)^2(q^2+1)(q^2+q+1)$ will be obtained. All except the first of these five characters are reducible, but they can be combined to give five irreducible characters as follows [9]:

$$1 = 1;\ (q+1)(q^2+1) - 1 = q(q^2+q+1);$$
$$(q^2+1)(q^2+q+1) - (q+1)(q^2+1) = q^2(q^2+1);$$
$$(q+1)(q^2+1)(q^2+q+1) - (q^2+1)(q^2+q+1) - (q+1)(q^2+1) + 1 = q^3(q^2+q+1);$$
$$(q+1)^2(q^2+1)(q^2+q+1) - 3(q+1)(q^2+1)(q^2+q+1) + (q^2+1)(q^2+q+1) + 2(q+1)(q^2+1) - 1 = q^6.$$

Multiplication of each of these characters by the $q-1$ linear characters given by the powers of the determinants gives $q-1$ irreducible characters of each of these degrees. Table IX lists the basic characters and shows the "fixed entity" situation.

We next consider characters induced by those of subgroup GL(1, 3; q) of index $(q+1)(q^2+1)$. In a manner analogous to those obtained of GL(3, q) from GL(1, 2; q), we get irreducible characters of the degrees and frequencies[2] shown in Table X:

TABLE X

Degree	Frequency
$(q+1)(q^2+1)$	$(q-1)(q-2)$
$q(q+1)^2(q^2+1)$	$(q-1)(q-2)$
$q^3(q+1)(q^2+1)$	$(q-1)(q-2)$
$(q+1)(q^2+1)(q^2+q+1)$	$\frac{1}{2}(q-1)(q-2)(q-3)$
$q(q+1)(q^2+1)(q^2+q+1)$	$\frac{1}{2}(q-1)(q-2)(q-3)$
$(q+1)^2(q^2+1)(q^2+q+1)$	$\frac{1}{24}(q-1)(q-2)(q-3)(q-4)$
$(q-1)(q+1)(q^2+1)(q^2+q+1)$	$\frac{1}{4}q(q-1)^2(q-2)$
$(q-1)^2(q+1)^2(q^2+1)$	$\frac{1}{3}q(q-1)^2(q+1)$

In the same way, the subgroup GL(2, 2; q) yields the irreducible characters shown in Table XI:

TABLE XI

Degree	Frequency
$(q^2+1)(q^2+q+1)$	$\frac{1}{2}(q-1)(q-2)$
$q^2(q^2+1)(q^2+q+1)$	$\frac{1}{2}(q-1)(q-2)$
$q(q^2+1)(q^2+q+1)$	$(q-1)(q-2)$
$(q-1)(q^2+1)(q^2+q+1)$	$\frac{1}{2}q(q-1)^2$
$q(q-1)(q^2+1)(q^2+q+1)$	$\frac{1}{2}q(q-1)^2$
$(q-1)^2(q^2+1)(q^2+q+1)$	$\frac{1}{8}q(q-1)(q+1)(q-2)$.

[2]The actual characters of GL(4, q) with a more detailed account of the methods are available in [10].

As a bi-product of the set of characters of degree $(q-1)^2(q^2+1)(q^2+q+1)$ we obtain $\frac{1}{2}q(q-1)$ characters of this degree each of which is the sum of two irreducible characters which are not among those that we have already obtained. Let us denote them by $\chi^{(n)}$, $n = 1, 2, \ldots, \frac{1}{2}q(q-1)$.

Finally, the linear characters of the cyclic subgroup of order q^4-1,

$$\begin{pmatrix} \omega & & & \\ & \omega^q & & \\ & & \omega^{q^2} & \\ & & & \omega^{q^3} \end{pmatrix}^a,$$

induce in GL$(4, q)$ a set of characters of degree $q^6(q-1)^3(q+1)(q^2+q+1)$. Each of these is reducible, but by a suitable use of the characters already obtained, i.e., by multiplication, addition and subtraction, a set of $\frac{1}{4}q^2(q-1)(q+1)$ irreducible characters of degree $(q-1)^3(q+1)(q^2+q+1)$ can be extracted from them. Again there is a bi-product: $\frac{1}{2}q(q-1)$ pseudocharacters of degree $(q-1)^3(q+1)(q^2+q+1)$ each of which is the difference of two irreducible characters. Denote them by $\psi^{(n)}$. Then, if the proper correlation is made be-

TABLE XII

Characters of PGL$(4, q)$

Degrees	Frequencies		
	$q = 4t$ or $4t+2$	$q = 4t+1$	$q = 4t+3$
1	1	4	2
$(10)(111)$	1	4	2
$(10)^2(101)$	1	4	2
$(10)^3(111)$	1	4	2
$(10)^6$	1	4	2
$(11)(101)$	$1-2$	$1-5$	$1-3$
$(10)(11)^2(101)$	$1-2$	$1-5$	$1-3$
$(10)^3(11)(101)$	$1-2$	$1-5$	$1-3$
$(11)(101)(111)$	$\frac{1}{2}(1-2)(1-3)$	$\frac{1}{2}(1-6-13)$	$\frac{1}{2}(1-3)^2$
$(10)(11)(101)(111)$	$\frac{1}{2}(1-2)(1-3)$	$\frac{1}{2}(1-6-13)$	$\frac{1}{2}(1-3)^2$
$(11)^2(101)(111)$	$\frac{1}{24}(1-2)(1-3)(1-4)$	$\frac{1}{24}(1-5)(1-49)$	$\frac{1}{24}(1-3)(1-6-11)$
$(1-1)(11)(101)(111)$	$\frac{1}{4}(10)(1-1)(1-2)$	$\frac{1}{4}(1-1)^3$	$\frac{1}{4}(1-1)^3$
$(1-1)^2(11)^2(101)$	$\frac{1}{3}(10)(1-1)(11)$	$\frac{1}{3}(10)(1-1)(11)$	$\frac{1}{3}(10)(1-1)(11)$
$(101)(111)$	$\frac{1}{2}(1-2)$	$1-3$	$1-2$
$(10)^2(101)(111)$	$\frac{1}{2}(1-2)$	$1-3$	$1-2$
$(10)(101)(111)$	$1-2$	$2-6$	$2-4$
$(1-1)(101)(111)$	$\frac{1}{2}(10)(1-1)$	$\frac{1}{2}(1-1)^2$	$\frac{1}{2}(1-1)^2$
$(10)(1-1)(101)(111)$	$\frac{1}{2}(10)(1-1)$	$\frac{1}{2}(1-1)^2$	$\frac{1}{2}(1-1)^2$
$(1-1)^2(101)(111)$	$\frac{1}{8}(10)(11)(1-2)$	$\frac{1}{8}(1-1)(10-3)$	$\frac{1}{8}(11)(1-2-1)$
$(1-1)^2(111)$	$\frac{1}{2}(10)$	$1-1$	10
$(10)^2(1-1)^2(111)$	$\frac{1}{2}(10)$	$1-1$	10
$(1-1)^3(11)(111)$	$\frac{1}{4}(10)^2(11)$	$\frac{1}{4}(1-1)(11)^2$	$\frac{1}{4}(1-1)(11)^2$

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tween the $\chi^{(n)}$'s and the $\psi^{(n)}$'s, it turns out that $\frac{1}{2}(\chi^{(n)}+\psi^{(n)})$ and $\frac{1}{2}(\chi^{(n)}-\psi^{(n)})$ are irreducible characters. In this way we obtain $\frac{1}{2}\,q(q-1)$ irreducible characters of each of the degrees $q^2(q-1)^2(q^2+q+1)$ and $(q-1)^2(q^2+q+1)$. This completes the character list since we have now obtained $q^4-q=k$ of them.

In cutting down the characters of GL(4, q) to get those of PGL(4, q), three cases are distinct: q even, $q = 4t+1$, $q = 4t+3$. Table XII gives the degrees and frequencies in each of these cases. For convenience in notation, we shall mean by $\frac{1}{3}$ (10-11), for example, $\frac{1}{3}$ (q^3-q+1), etc.

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University of California at Los Angeles