Bull. Aust. Math. Soc. (First published online 2025), page 1 of 10*
doi:10.1017/S0004972725100397
*Provisional—final page numbers to be inserted when paper edition is published

THE FERMAT QUARTIC $X^4 + Y^4 = 2^m$ IN QUADRATIC NUMBER FIELDS

JAYANTH ATHIPATLA

(Received 21 June 2025; accepted 23 June 2025)

Abstract

In this paper, we generalise to the family of Fermat quartics $X^4 + Y^4 = 2^m, m \in \mathbb{Z}$, a result of Aigner ['Über die Möglichkeit von $x^4 + y^4 = z^4$ in quadratischen Körpern', *Jahresber. Deutsch. Math.-Ver.* **43** (1934), 226–228], which proves that there is only one quadratic field, namely $\mathbb{Q}(\sqrt{-7})$, that contains solutions to the Fermat quartic $X^4 + Y^4 = 1$. The $m \equiv 0 \pmod 4$ case is due to Aigner. The $m \equiv 2 \pmod 4$ case follows from a result of Emory ['The Diophantine equation $X^4 + Y^4 = D^2Z^4$ in quadratic fields', *Integers* **12** (2012), Article no. A65, 8 pages]. This paper focuses on the two cases $m \equiv 1, 3 \pmod 4$, classifying for $m \equiv 1 \pmod 4$ the infinitely many quadratic number fields that contain solutions, and proving for $m \equiv 3 \pmod 4$ that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-2})$ are the only quadratic number fields that contain solutions.

2020 *Mathematics subject classification*: primary 11D25; secondary 11R11, 11D45.

Keywords and phrases: quartic Diophantine equations, quadratic number fields, elliptic curves.

1. Introduction

The Fermat quartic $x^4 + y^4 = 2^m$ was shown to have no nontrivial rational solutions for any integer m by Lebesgue [6]. After Hilbert proved in his Zahlbericht [5, Theorem 169] that there are no solutions $X, Y \in \mathbb{Q}(\sqrt{-1})$ of $X^4 + Y^4 = 1$ in the Gaussian field $\mathbb{Q}(\sqrt{-1})$, Aigner showed in 1934 that only in $\mathbb{Q}(\sqrt{-7})$ do quadratic solutions to $X^4 + Y^4 = 1$ exist [1]. For instance,

$$\left(\frac{1+\sqrt{-7}}{2}\right)^4 + \left(\frac{1-\sqrt{-7}}{2}\right)^4 = 1.$$

Faddeev in 1960 and Mordell in 1967 reproved this result [4, 8]. We adapt the method used by Mordell and classify the quadratic number fields that contain solutions to the Fermat quartic $X^4 + Y^4 = 2^m$, $m \in \mathbb{Z}$. Since a solution of $X^4 + Y^4 = 2^m$ can be used to produce a solution to $X^4 + Y^4 = 2^{m+4k}$ for any $k \in \mathbb{Z}$, it is sufficient to restrict our discussion to $m = 1, 2, 3$.

Emory used the approach taken by Mordell in [8] to determine all quadratic number fields that contain solutions to $X^4 + Y^4 = d^2$ [3]. She found that the equation

$X^4 + Y^4 = d^2$ has nontrivial quadratic solutions precisely when $d = 1$, or d is a congruent number, a positive integer that can be expressed as the area of a right triangle with rational sides. Since 2 is not a congruent number [2], there are no nontrivial quadratic solutions to $X^4 + Y^4 = 2^m$ for $m = 2$. Of course, solutions such as $(X, Y) = (0, \sqrt{2})$ exist.

Thus, in this paper, we restrict ourselves to the two cases $m = 1$ and $m = 3$. For the remainder of this paper, $K = \mathbb{Q}(\sqrt{n})$ is a quadratic field with $n \in \mathbb{Z}$ squarefree. A *trivial solution* to the Fermat quartic $X^4 + Y^4 = 2^m$ occurs when $XY = 0$ or $X = \pm Y$. A *quadratic solution* $(X, Y) \in K \times K$ occurs when at least one of X, Y is irrational. A *quadratic conjugate pair* (X, Y) exists when $X = a + b\sqrt{n}$, $a, b \in \mathbb{Q}$, $ab \neq 0$, $Y = \pm(a - b\sqrt{n})$ and $X^4 + Y^4 = 2^m$.

THEOREM 1.1. *Let K be a quadratic number field and let $(X, Y) \in K \times K$ be a quadratic solution of the Fermat quartic $X^4 + Y^4 = 2$. Then, one of the following occurs:*

(1) *$X^4 = Y^4$ and $K = \mathbb{Q}(\sqrt{-1})$—for instance, $(X, Y) = (1, \sqrt{-1})$;*
(2) *$X^4 \neq Y^4$ and $X^2, Y^2 \in \mathbb{Q}$—in this case, (X, Y) generates a rational point (a, b) on the elliptic curve $y^2 = x^3 + 8x$;*
(3) *$X^4 \neq Y^4$ and at least one of X^2, Y^2 is irrational. In this case, (X, Y) is necessarily a quadratic conjugate pair—indeed, neither of X^2, Y^2 is rational—and (X, Y) generates a rational point (a, b) on the elliptic curve $y^2 = x^3 - 2x$.*

Conversely, given a rational point $(a, b) \in \mathbb{Q} \times \mathbb{Q}$ on one of these two elliptic curves:

(1) *for $y^2 = x^3 + 8x$, then $K = \mathbb{Q}(\sqrt{2a^3 - 24a^2 - b^2 - 48b - 64})$ contains the K-point on $X^4 + Y^4 = 2$,*
$$(X, Y) = \left(\frac{4a + b - 8}{2a + b + 8}, \frac{\sqrt{2a^3 - 24a^2 - b^2 - 48b - 64}}{2a + b + 8}\right);$$
(2) *for $y^2 = x^3 - 2x$, then $K = \mathbb{Q}(\sqrt{-a^3 - 3a^2 - 2a})$ contains the K-point on $X^4 + Y^4 = 2$,*
$$(X, Y) = \left(\frac{a + \sqrt{-a^3 - 3a^2 - 2a}}{b}, \frac{a - \sqrt{-a^3 - 3a^2 - 2a}}{b}\right).$$

Furthermore, each of these elliptic curves produces infinitely many different quadratic fields K that contain solutions to $X^4 + Y^4 = 2$.

THEOREM 1.2. *Let K be a quadratic number field and let $(X, Y) \in K \times K$ be a quadratic solution of the Fermat quartic $X^4 + Y^4 = 8$. Then, either:*

(1) *$K = \mathbb{Q}(\sqrt{2})$ with trivial solutions such as $(X, Y) = (\sqrt{2}, \sqrt{2})$; or*
(2) *$K = \mathbb{Q}(\sqrt{-2})$ with trivial solutions such as $(X, Y) = (\sqrt{-2}, \sqrt{-2})$.*

REMARK 1.3. Quadratic solutions to $X^4 + Y^4 = 2$ arise from the infinitely many rational points on the two elliptic curves $y^2 = x^3 + 8x$ and $y^2 = x^3 - 2x$, which belong to the same isogeny class with LMFDB label 256.b [11]. The isogenies are described in [10, Ch. 3, Section 4]. Quadratic solutions to $X^4 + Y^4 = 8$ arise from the elliptic curve $y^2 = x^3 - 8x$, which has only one rational point, namely $(0, 0)$. Thus, there are no nontrivial solutions to $X^4 + Y^4 = 8$.

In a slightly different direction, but still using Mordell's approach, A. Li has determined the quadratic solutions to $X^4 + 2^m Y^4 = 1$ [7]. None of the solutions Li describes are quadratic conjugate pairs. Furthermore, although it is beyond the scope of this paper, we would be remiss if we did not point out that Faddeev and Mordell each also addressed the question of solutions to $X^4 + Y^4 = 1$ within cubic number fields [4, 8], and that there continues to be work that examines Fermat quartics over algebraic number fields more generally, much of it using methods far beyond the scope of this paper. So that we do not expand our scope, we mention only [12] by N. X. Tho.

We adopt the following convention: rationals are represented by lower case letters; generic elements of a quadratic field K are represented by upper case letters.

1.1. Outline and preliminary results. Let n be a squarefree integer, $K = \mathbb{Q}(\sqrt{n})$ be a quadratic number field and suppose that $X, Y \in K$ satisfy

$$X^4 + Y^4 = 2^m, \quad m = 1, 3. \tag{1.1}$$

We follow Mordell's approach [8], whose first step in addressing $X^4 + Y^4 = 1$ was to use the rational parametrisation of the unit circle, writing X^2 and Y^2 in terms of a parameter T. Using this well-known parametrisation, and additionally noting that when m is odd, we have $(X^2 - Y^2)^2 + (X^2 + Y^2)^2 = 2^{m+1}$, we determine, for $X, Y \in K$ satisfying (1.1), that

$$X^2 = 2^{(m-1)/2}\frac{T^2 + 2T - 1}{T^2 + 1}, \quad Y^2 = 2^{(m-1)/2}\frac{T^2 - 2T - 1}{T^2 + 1}. \tag{1.2}$$

Solving for T yields

$$T = \frac{X^2 - Y^2}{2^{(m+1)/2} - (X^2 + Y^2)} \in K. \tag{1.3}$$

Notice that $T \in \mathbb{Q}$ if and only if $(X^2, Y^2) \in \mathbb{Q} \times \mathbb{Q}$, and $T = 0$ if and only if $X^2 = Y^2$.

Following Mordell, our next step is to treat the two cases, T rational and T irrational, separately. When T is rational, our argument follows Mordell closely. However, in the second case, we find that $T \notin \mathbb{Q}$ is not quite strong enough to follow Mordell. To follow Mordell's approach, we need $T^2 \notin Q$, but then we can continue with some necessary adjustments. The end result is that if $T^2 \notin Q$, there are no quadratic solutions. This leaves the case $T \notin Q$ and $T^2 \in Q$, which implies that our quadratic solution is a quadratic conjugate pair. Thus, in our final result, we use an *ad hoc* approach to classify all quadratic conjugate pairs.

2. Case T is rational

Adopt the notation in Section 1.1 and let $(X, Y) \in K \times K$ be a quadratic solution to $X^4 + Y^4 = 2^m$, $m = 1, 3$. Suppose T is rational. Then, because of (1.2), we have $X^2, Y^2 \in \mathbb{Q}$. Note that for $a, b \in \mathbb{Q}$, $(a + b\sqrt{n})^2 \in \mathbb{Q}$ implies that $ab = 0$. Thus, setting $X = a + b\sqrt{n}$ and $Y = a_1 + b_1\sqrt{n} \in K$, there are four cases: $(X, Y) = (a, a_1), (a, b_1\sqrt{n}), (a\sqrt{n}, b_1)$ or $(b\sqrt{n}, b_1\sqrt{n})$, which we condense into two, that is, $X/Y \in \mathbb{Q}$ or $X/Y \notin \mathbb{Q}$.

PROPOSITION 2.1. *If $(X, Y) \in K \times K$ is a quadratic solution of $X^4 + Y^4 = 2^m$ such that $T \in \mathbb{Q}$ and $X/Y \in \mathbb{Q}$, then $X^4 = Y^4$. If $(X, Y) \in K \times K$ is a quadratic solution of $X^4 + Y^4 = 2^m$ such that $X^4 = Y^4$, then $K = \mathbb{Q}(\sqrt{-1})$ when $m = 1$ and $K = \mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{-2})$ when $m = 3$.*

PROOF. Let X, Y satisfying $X^4 + Y^4 = 2^m$ lie in a quadratic number field K. Suppose that $T \in \mathbb{Q}$ and that $u = X/Y \in \mathbb{Q}$. Then, using (1.2), we deduce that $u^2 = (T^2 + 2T - 1)/(T^2 - 2T - 1)$. Setting $v = u(T^2 - 2T - 1) \in \mathbb{Q}$ yields $v^2 = T^4 - 6T^2 + 1$, which is a curve with rational point $(T, v) \in \mathbb{Q} \times \mathbb{Q}$. Following the proof of [9, Theorem 2 on page 77], we make a change of variables. First, we set $v = T^2 - 1 - 2x$. Note that $x \neq 1$, because if $x = 1$, then we have $v = T^2 - 3$, which would contradict $v^2 = T^4 - 6T^2 + 1$. Next, we set $T = y/(1 - x)$. In this way, the point $(T, u) \in \mathbb{Q} \times \mathbb{Q}$ yields a rational point (x, y) with $x \neq 1$ on the elliptic curve $y^2 = x^3 - x$, which has LMFDB label 32.a3 and Mordell–Weil group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. This elliptic curve has three rational points $(0, 0), (1, 0), (-1, 0)$, but only $(x, y) = (0, 0), (-1, 0)$ satisfy $x \neq 1$. Since $y = 0$ in these two cases, we find that $T = 0$. Since $T = 0$, we have $u^2 = 1$, which means that $X^2 = Y^2$ and thus $X^4 = Y^4$.

Assume now that $X^4 = Y^4$. Then, $X^4 + Y^4 = 2$ reduces to $X^4 = 1$ and thus $K = \mathbb{Q}(\sqrt{-1})$. Similarly, $X^4 + Y^4 = 8$ reduces to $X^4 = 4$ and thus $\mathbb{Q}(\sqrt{\pm 2})$. □

PROPOSITION 2.2. *There exist infinitely many quadratic fields $K = \mathbb{Q}(\sqrt{n})$ that contain solutions $(X, Y) \in K \times K$ to the Fermat quartic $X^4 + Y^4 = 2$ when $T \in \mathbb{Q}$, $X/Y \notin \mathbb{Q}$ and $X^4 \neq Y^4$. These solutions are derived from the infinitely many rational points on the elliptic curve $y^2 = x^3 + 8x$. For $X^4 + Y^4 = 8$, there exist no quadratic fields $K = \mathbb{Q}(\sqrt{n})$ that contain solutions $(X, Y) \in K \times K$ to the Fermat quartic $X^4 + Y^4 = 8$ when $T \in \mathbb{Q}$, $X/Y \notin \mathbb{Q}$ and $X^4 \neq Y^4$.*

PROOF. We first begin with $m = 1$ and the curve $X^4 + Y^4 = 2$. Without loss of generality, $(X, Y) = (a, b_1\sqrt{n})$ and, thus, $a^4 + b_1^4 n^2 = 2$ gives $j^2 + k^4 = 2$. Notice that if $k = 1$, then $j = \pm 1$, which contradicts $X^4 \neq Y^4$. Thus, $k \neq 1$, and using [9, proof of Theorem 2 on page 77] and the solution $(j, k) = (1, 1)$ to $j^2 + k^4 = 2$, we make the change of variables $j = v(w + 1)^{-2}$, $k = 1 + (w + 1)^{-1}$, which yields $v^2 = w^4 - 12w^2 - 24w - 14$. Since $k \neq 1$, the rational point (j, k) produces a rational point $(w, v) \in \mathbb{Q} \times \mathbb{Q}$. We then set $v = -w^2 + 2h + 2$ to determine that

$$w^2 - \frac{6}{h-2}w = \frac{h^2 + 2h + 9/2}{h-2}.$$

Completing the square for the quadratic in w determines that $((h-2)w-3)^2 = h^3 + h/2$. Replacing $(h-2)w-3$ with $y/8$ and letting $x = 4h$ yields a rational point on the elliptic curve $y^2 = x^3 + 8x$ with LMFDB label 256.b2, which has Mordell–Weil group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with the point $(1, 3)$ of infinite order.

Working backwards from any rational point $(a, b) \neq (8, -24)$ on the elliptic curve $y^2 = x^3 + 8x$ to our curve $X^4 + Y^4 = 2$ yields the solution

$$\left(\frac{4a+b-8}{2a+b+8}\right)^4 + \left(\frac{\sqrt{2a^3 - 24a^2 - b^2 - 48b - 64}}{2a+b+8}\right)^4 = 2$$

in the field $K = \mathbb{Q}(\sqrt{2a^3 - 24a^2 - b^2 - 48b - 64})$. This field is quadratic, namely $K \neq \mathbb{Q}$, for every rational point (a, b) on $y^2 = x^3 + 8x$ with the exception of $(a, b) = (1, -3)$. Note that for $(a, b) = (1, -3)$, we have $2a^3 - 24a^2 - b^2 - 48b - 64 = 49$ and thus $K = \mathbb{Q}$. To prove this field is otherwise quadratic, we use the fact that, as Lebesgue proved, $(X, Y) = (\pm 1, \pm 1)$ are the only rational solutions to $X^4 + Y^4 = 2$. Set

$$A(x, y) = \frac{4x+y-8}{2x+y+8}, \quad B(x, y) = \frac{\sqrt{2x^3 - 24x^2 - y^2 - 48y - 64}}{2x+y+8}.$$

Suppose that $B(a, b)$ is rational. This makes $(A(a, b), B(a, b))$ a rational point on $X^4 + Y^4 = 2$. Thus, $A(a, b) = \pm 1$ which, using $y^2 = x^3 + 8x$, means that $(a, b) = (0, 0), (1, -3), (8, 24)$. Recall that the point $(8, -24)$ was excluded. A quick calculation shows that $B(a, b) \neq \pm 1$ for $(a, b) = (0, 0)$ and $(a, b) = (8, 24)$. We record that $(A(1, -3), B(1, -3)) = (-1, 1)$.

We now use Falting's theorem and the fact that the Fermat quartic $X^4 + Y^4 = 2$ is a nonsingular curve of genus 3 to prove there are infinitely many different quadratic fields $\mathbb{Q}(\sqrt{2a^3 - 24a^2 - b^2 - 48b - 64}) \neq \mathbb{Q}$, where (a, b) is a rational point on $y^2 = x^3 + 8x$. Since each quadratic field K contains only finitely many solutions to $X^4 + Y^4 = 2$, this will follow once we establish the fact that there are infinitely many different complex ordered pairs $(A(a, b), B(a, b)) \in \mathbb{C} \times \mathbb{C}$.

Since $A(a, b) \in \mathbb{Q}$ and $B(a_1, b_1) \notin \mathbb{Q}$ for all rational points (a_1, b_1) on $y^2 = x^3 + 8x$ except $(a_1, b_1) = (1, -3)$, there is no possibility of $A(a, b) = B(a_1, b_1)$ unless $B(a_1, b_1) = 1$. If $B(a_1, b_1) = 1$, then $A(a_1, b_1) = \pm 1$ since $A(a_1, b_1)$ is rational. Recall that we have already described the finitely many points (a_1, b_1) on $y^2 = x^3 + 8x$ such that $A(a_1, b_1) = \pm 1$. Thus, we focus on proving there are infinitely many different ordered pairs $(A(a, b), B(a, b)) \in \mathbb{C} \times \mathbb{C} \setminus \{(\pm 1, 1)\}$.

Our only concern is the possibility that there might be infinitely many rational points (a_i, b_i) on $y^2 = x^3 + 8x$ such that $(A(a_i, b_i), B(a_i, b_i)) = (A(a_j, b_j), B(a_j, b_j)) \notin \{(\pm 1, 1)\}$. If this were the case, letting $k = A(a_i, b_i) \in \mathbb{Q}$ be the common value of the $A(a_i, b_i)$, we would have infinitely many intersections of $y^2 = x^3 + 8x$ with the level curve

$$k = \frac{4x+y-8}{2x+y+8}.$$

Recall that since $(A(a,b), B(a,b)) \notin \{(\pm 1, 1)\}$, we have $k \neq \pm 1$. Since $k \neq 1$, we may solve for y in the equation for the level curve and then use this to substitute for y in $y^2 = x^3 + 8x$. The result is a cubic equation in x with no more than three rational roots. As a result, the infinitely many rational points on $y^2 = x^3 + 8x$ produce infinitely many different quadratic extensions K.

We now consider the case $m = 3$ and the curve $X^4 + Y^4 = 8$. Again without loss of generality, we have $(X, Y) = (a, b_1\sqrt{n})$ with $a, b_1 \in \mathbb{Q}$. Then, $a^4 + b_1^4 n^2 = 8$ and thus a nontrivial integer solution for $8x^4 - y^4 = z^2$. However, this curve is known to have no nontrivial integer solutions by Lebesgue [6] using the method of infinite descent, and thus there are no nontrivial quadratic fields that come out of this case. □

EXAMPLE 2.3. Taking the point $(a,b) = (1,3)$, which is a generator of the elliptic curve $y^2 = x^3 + 8x$, gives the quadratic field $\mathbb{Q}(\sqrt{-239})$ with solution

$$\left(-\frac{1}{13}\right)^4 + \left(\frac{\sqrt{-239}}{13}\right)^4 = 2.$$

EXAMPLE 2.4. Doubling the generator $(1,3)$ under the group law of the elliptic curve $y^2 = x^3 + 8x$ yields the point $(a,b) = (49/36, -791/216)$. Using this point gives the quadratic field $\mathbb{Q}(\sqrt{2750257})$ with solution

$$\left(-\frac{1343}{1525}\right)^4 + \left(\frac{\sqrt{2750257}}{1525}\right)^4 = 2.$$

3. Case T is irrational

Adopt the notation in Section 1.1 and let $(X, Y) \in K \times K$ be a quadratic solution to $X^4 + Y^4 = 2^m$, $m = 1, 3$. In this section, we prove two results. In the first result, we use an argument modelled after Mordell to prove that there are no quadratic solutions under the assumption that $T^2 \notin \mathbb{Q}$. As a result, we assume $T^2 \in \mathbb{Q}$ and since $(X, Y) \in K \times K$ with $T \notin \mathbb{Q}$, we conclude that (X, Y) is a quadratic conjugate pair. The second result uses the fact that (X, Y) is a quadratic conjugate pair on the Fermat quartic $X^4 + Y^4 = 2^m$ to produce a rational point on the elliptic curve $y^2 = x^3 - 2^m x$.

PROPOSITION 3.1. *Let $K = \mathbb{Q}(\sqrt{n})$ with $n \in \mathbb{Z}$ squarefree, and let $(X, Y) \in K \times K$ be a quadratic solution to the Fermat quartic $X^4 + Y^4 = 2^m$, $m = 1, 3$. If T, as defined by (1.3), is irrational, then (X, Y) is a quadratic conjugate pair.*

PROOF. We begin by supposing that $T \notin \mathbb{Q}$ and $T^2 \in \mathbb{Q}$. Thus, we have $T = r\sqrt{n} \notin \mathbb{Q}$ for some nonzero $r \in \mathbb{Q}$. Using (1.2), this means that $\sigma(X^2) = Y^2$, where $\sigma : K \to K$ denotes conjugation. Since σ is a multiplicative homomorphism, $\sigma(X)^2 = Y^2$, which means that $Y = \pm\sigma(X)$ and (X, Y) is a quadratic conjugate pair.

Now, we derive contradictions in the more complicated case in which $T^2 \notin \mathbb{Q}$. Following the argument Mordell used in [8], set $Z = (1 + T^2)XY \in K$. Using (1.2), we find that $Z^2 = 2^{m-1}(T^4 - 6T^2 + 1)$. Set $V = Z/(2^{(m-1)/2}) - T^2 + 3 \in K$. Since $Z = 2^{(m-1)/2}(V + T^2 - 3)$, we have $V^2 + 2VT^2 - 6V + 8 = 0$. If $V \in \mathbb{Q}$, then because $V^2 + 2VT^2 - 6V + 8 = 0$, we have $T^2 \in \mathbb{Q}$, which is a contradiction. Thus, $V \notin \mathbb{Q}$, and there exist $c, d \in \mathbb{Q}$ with $d \neq 0$ such that $V = c + dT$. Define the linear polynomial $V(z) = c + dz \in \mathbb{Q}[z]$ and observe that the polynomial

$$p(z) = V(z)^2 + 2V(z)z^2 - 6V(z) + 8 \in \mathbb{Q}[z]$$

has T as a root. Since $d \neq 0$, this polynomial is cubic, which means that $p(z) = F(z)G(z)$, where $F(z) \in \mathbb{Q}[z]$ is the monic irreducible quadratic polynomial for T and $G(z) \in \mathbb{Q}[z]$ is linear. At this point,

$$V(z)^2 + 2V(z)z^2 - 6V(z) + 8 = F(z)G(z). \tag{3.1}$$

Since $G(z)$ is linear, it has a rational root e, which means that $(x, y) = (e, V(e))$ is a rational point on the curve $y^2 + 2x^2y - 6y + 8 = 0$. This elliptic curve has minimal Weierstrass equation $y^2 = x^3 - x$ with LMFDB label 32.a3 and Mordell–Weil group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The four rational projective points on $y^2z + 2x^2y - 6yz^2 + 8z^3 = 0$ are $(x : y : z) = (0 : 1 : 0), (0 : 2 : 1), (0 : 4 : 1)$ and $(1 : 0 : 0)$. Thus, we determine that either $(e, V(e)) = (0, 2)$ or $(e, V(e)) = (0, 4)$. In both of these cases, the root of the linear polynomial $Q(z)$ is 0; therefore, $G(z) = hz$ for some $h \in \mathbb{Q}$. Thus, (3.1) becomes $(c + dz)^2 + 2(c + dz)z^2 - 6(c + dz) + 8 = F(z)hz$. Comparing the leading terms, we see that $h = 2d$. Comparing the constant terms, we see that $c^2 - 6c + 8 = 0$. Thus, $c = 2$ or 4, and solving for $F(z)$, we find that

$$F(z) = z^2 + z(2c + d^2)/2d + (c - 3),$$

which means that

$$T = \frac{-d^2 - 2c \pm \sqrt{d^4 + (48 - 12c)d^2 + 4c^2}}{4d}. \tag{3.2}$$

There are two cases $c = 2$ and $c = 4$ to consider. For $c = 2$, substituting T into (1.2) yields

$$X^2 = \pm\frac{2^{(m-1)/2}(d-2)^2}{\sqrt{d^4 + 24d^2 + 16}}, \quad Y^2 = \pm\frac{2^{(m-1)/2}(d+2)^2}{\sqrt{d^4 + 24d^2 + 16}},$$

where the $\pm$ signs agree with the $\pm$ sign in (3.2). Note that $X^4, Y^4 \in \mathbb{Q}$. If either of X^2 or $Y^2 \in \mathbb{Q}$, then both are rational and $T \in \mathbb{Q}$, which contradicts $T^2 \notin \mathbb{Q}$. Thus, we find that $X^4, Y^4 \in \mathbb{Q}$, while neither $X^2 \in \mathbb{Q}$ nor $Y^2 \in \mathbb{Q}$. Letting $X = r + s\sqrt{n}$ with $r, s \in \mathbb{Q}$, and expanding X^2 and X^4, yields $rs \neq 0$ and $rs(r^2 + s^2n) = 0$. So $n = -r^2/s^2$. Since n is a squarefree integer, $n = -1$ and $s = r$. Thus, $X = r(1 + \sqrt{-1})$ and $X^4 = -4r^4$. Similarly, $Y^4 = -4r_1^4$ for some $r_1 \in \mathbb{Q}$. However, this means that $X^4 + Y^4 < 0$, which contradicts $X^4 + Y^4 = 2^m$.

For $c = 4$, we substitute the values for T into (1.2) separately. In the first place, for $T = (-d^2 - 8 + \sqrt{d^2 + 64})/4d$, we find

$$X^2 = 2^{(m-1)/2} \cdot \frac{-4d - \sqrt{d^4 + 64}}{d^2 + 8}, \quad Y^2 = 2^{(m-1)/2} \cdot \frac{4d - \sqrt{d^4 + 64}}{d^2 + 8}.$$

Second, for $T = (-d^2 - 8 - \sqrt{d^2 + 64})/4d$, we find

$$X^2 = 2^{(m-1)/2} \cdot \frac{-4d + \sqrt{d^4 + 64}}{d^2 + 8}, \quad Y^2 = 2^{(m-1)/2} \cdot \frac{4d + \sqrt{d^4 + 64}}{d^2 + 8}.$$

Since Lebesgue [6] proved that the Fermat quartic $x^4 + 2^m y^4 = z^2$ has only nonzero integer solutions for $m \equiv 2 \pmod 4$ and $d \in \mathbb{Q}$ is nonzero, $\sqrt{d^4 + 64} \notin \mathbb{Q}$. Thus, $K = \mathbb{Q}(\sqrt{d^4 + 64})$ is a real-valued quadratic field. Consider the square of any element of K, namely $(a + b\sqrt{d^4 + 64})^2 = a^2 + b^2(d^4 + 64) + 2ab\sqrt{d^4 + 64}$. Notice that the rational summand is positive. This leads to a contradiction in the expressions for X^2 and Y^2, since $-4d$ and $4d$ have opposite signs. □

PROPOSITION 3.2. *Let $K = \mathbb{Q}(\sqrt{n})$ with $n \in \mathbb{Z}$ squarefree. There are no quadratic solutions $(X, Y) \in K \times K$ to the curve $X^4 + Y^4 = 8$ with (X, Y) a quadratic conjugate pair. However, there are infinitely many quadratic fields K that contain a quadratic conjugate pair solution to the Fermat quartic $X^4 + Y^4 = 2$, each derived from one of the infinitely many rational points on the elliptic curve $y^2 = x^3 - 2x$.*

PROOF. We begin with $m = 1$ and suppose there exists a solution $(X, Y) \in K \times K$ to $X^4 + Y^4 = 2$ with $X = a + b\sqrt{n}$, $Y = \pm(a - b\sqrt{n})$ for some $a, b \in \mathbb{Q}$. Expanding $(a + b\sqrt{n})^4 + (a - b\sqrt{n})^4 = 2$ produces $a^4 + 6a^2b^2n + b^4n^2 = 1$. The change of variable $b^2n = c - 3a^2$ yields $c^2 - 8a^4 = 1$. Then, the further change of variables

$$(a, c) = \left(\frac{x}{y}, \frac{y^2 - 2x^3}{y^2}\right) \implies b^2n = \frac{y^2 - 2x^3 - 3x^2}{y^2} = \frac{-x^3 - 3x^2 - 2x}{y^2}$$

yields a rational point on the the elliptic curve $y^2 = x^3 - 2x$ with LMFDB label 256.b1, Mordell–Weil group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with point $(2, 2)$ of infinite order.

Working backwards from any rational point $(a, b) \neq (0, 0)$ on the elliptic curve $y^2 = x^3 - 2x$ to $X^4 + Y^4 = 2$ yields the solution

$$\left(\frac{a + \sqrt{-a^3 - 3a^2 - 2a}}{b}\right)^4 + \left(\frac{a - \sqrt{-a^3 - 3a^2 - 2a}}{b}\right)^4 = 2$$

in the field $K = \mathbb{Q}(\sqrt{-a^3 - 3a^2 - 2a})$. Since, as Lebesgue proved, there are no nontrivial rational points on $X^4 + Y^4 = 2$, it follows that $\sqrt{-a^3 - 3a^2 - 2a} \notin \mathbb{Q}$ and K is a quadratic extension of $\mathbb{Q}$.

We now use Falting's theorem and the fact that $X^4 + Y^4 = 2$ is a nonsingular curve of genus 3 to prove there are infinitely many different quadratic fields $\mathbb{Q}(\sqrt{-a^3 - 3a^2 - 2a})$ with (a, b) a rational point on the elliptic curve $y^2 = x^3 - 2x$. Since each quadratic field K contains only finitely many solutions to $X^4 + Y^4 = 2$, this result will follow once we establish the fact that there are infinitely many different values of

$$A(x, y) = \frac{\sqrt{-x^3 - 3x^2 - 2x}}{y} \in \mathbb{C}.$$

Our concern is that there might be simultaneous values $k = A(x, y)$ that occur for infinitely many different points (a, b) on $y^2 = x^3 - 2x$. So we ask whether there can be any complex values $k \in \mathbb{C}$ such that the level curve $y^2k^2 = -x^3 - 3x^2 - 2x$ intersects $y^2 = x^3 - 2x$ at infinitely many real points (x_1, y_1). Since each intersection point yields a real solution to the polynomial $k^2(x^3 - 2x) = -x^3 - 3x^2 - 2x$, which is a polynomial with no more than three distinct roots, this does not occur.

Now, we let $m = 3$ and suppose there exists a solution $(X, Y) \in K \times K$ to the curve $X^4 + Y^4 = 8$ with $X = a + b\sqrt{n}$, $Y = \pm(a - b\sqrt{n})$ for some $a, b \in \mathbb{Q}$. Expanding $(a + b\sqrt{n})^4 + (a - b\sqrt{n})^4 = 8$ produces $a^4 + 6a^2b^2n + b^4n^2 = 4$. The change of variable $b^2n = c - 3a^2$ now yields $c^2 - 8a^4 = 4$. Using the rational point $(a, c) = (0, 2)$ and [9, Theorem 2 on page 77] yields a birational equivalence with the elliptic curve $y^2 = x^3 - 8x$ LMFDB label 256.c1, which has Mordell–Weil group $\mathbb{Z}/2\mathbb{Z}$ and only the trivial rational point $(0, 0)$. Working backwards from this point and the point at infinity, we obtain two points $(x : y : z) = (0 : 2 : 1)$ and $(0 : 1 : 0)$ on the projective version of our curve $y^2z^2 - 8x^4 = 4z^4$. The only rational point on $c^2 - 8a^4 = 4$ is $(a, c) = (0, 2)$. Since $a = 0$, this contradicts our assumption that a and b are both nonzero. □

EXAMPLE 3.3. Using the generator $(a, b) = (2, 2)$ of the elliptic curve $y^2 = x^3 - 2x$ gives a solution in the quadratic field $\mathbb{Q}(\sqrt{-6})$ to the curve $X^4 + Y^4 = 2$, namely

$$(1 + \sqrt{-6})^4 + (1 - \sqrt{-6})^4 = 1.$$

EXAMPLE 3.4. If we add the torsion point $(0, 0)$ and the nontorsion generator $(2, 2)$ under the group law of the elliptic curve $y^2 = x^3 - 2x$, we get the point $(-1, -1)$. Doubling this point gives us the point $(a, b) = (9/4, 21/8)$. Using the point (a, b) to find a solution to $X^4 + Y^4 = 2$ gives the quadratic field $\mathbb{Q}(\sqrt{-221})$ with solution

$$\left(\frac{6 + \sqrt{-221}}{7}\right)^4 + \left(\frac{6 - \sqrt{-221}}{7}\right)^4 = 2.$$

Acknowledgement

I sincerely thank my advisor, Dr. Griff Elder, for his guidance, support and the invaluable research opportunity.

References

[1] A. Aigner, 'Über die Möglichkeit von $x^4 + y^4 = z^4$ in quadratischen Körpern', *Jahresber. Dtsch. Math.-Ver.* **43** (1934), 226–228.

[2] K. Conrad, 'The congruent number problem', *Harv. College Math. Rev.* **2**(2) (2008), 58–74.

[3] M. Emory, 'The Diophantine equation $X^4 + Y^4 = D^2Z^4$ in quadratic fields', *Integers* **12** (2012), Article no. A65, 8 pages.

[4] D. K. Faddeev, 'Group of divisor classes on the curve defined by the equation $x^4 + y^4 = 1$', *Soviet Math. Dokl.* **1** (1960), 1149–1151.

[5] D. Hilbert, 'Die theorie der algebraischen zahlkörper', *Jahresber. Dtsch. Math.-Ver.* **4** (1894/95), 175–535.

[6] V. A Lebesgue, 'Resolution des équations biquadratiques $z^2 = x^4 \pm 2^m y^4, z^2 = 2^m x^4 - y^4, 2^m z^2 = x^4 \pm y^4$', *J. Math. Pures Appl. (9)* **18** (1853), 73–86.

[7] A. Li, 'The Diophantine equation $x^4 + 2^n y^4 = 1$ in quadratic number fields', *Bull. Aust. Math. Soc.* **104**(1) (2021), 21–28.
[8] L. J. Mordell, 'The Diophantine equation $x^4 + y^4 = 1$ in algebraic number fields', *Acta Arith.* **14** (1967/68), 347–355.
[9] L. J. Mordell, *Diophantine Equations*, Pure and Applied Mathematics, 30 (Academic Press, London–New York, 1969).
[10] J. H. Silverman, *The Arithmetic of Elliptic Curves*, 2nd edn, Graduate Texts in Mathematics, 106 (Springer, Dordrecht, 2009).
[11] The LMFDB Collaboration, *The L-functions and Modular Forms Database*, release 1.2.1. https://www.lmfdb.org.
[12] N. X. Tho, 'The equation $x^4 + 2^n y^4 = z^4$ in algebraic number fields', *Acta Math. Hungar.* **167**(1) (2022), 309–331.

JAYANTH ATHIPATLA, Department of Mathematics,
University of Nebraska Omaha, Omaha, NE 68182, USA
e-mail: jathipatla@unomaha.edu