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A NON-ABSOLUTELY SUMMING OPERATOR

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Abstract

In the case when $0 < p < 1$ it is proved, using a method of Macphail that the identity map i: $l_p \to l_p$ is not (r, s)-absolutely summing for any r, s.

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1. Introduction

Mitiagin and Pełczyński (1966) define a bounded linear operator T, between Banach spaces X and Y, to be (r, s)-absolutely summing, $1 \leqslant s \leqslant r \leqslant \infty$, if $\Sigma\|T(x_k)\|^r < \infty$ whenever (x_k) is a sequence in X such that

$$\sum |f(x_k)|^s < \infty \quad \text{for each } f \in X^*.$$

As usual, X^* denotes the continuous dual of X and we interpret, for example, the case $r = \infty$ as $\sup_k\|T(x_k)\| < \infty$.

For $0 < p < \infty$ we denote by l_p the space of real sequences $x = (x_k)$ such that $\Sigma|x_k|^p < \infty$. The following is a well-known result of Orlicz (1933):

THEOREM 1. *Let* $1 \leqslant p \leqslant \infty$ *and let* $r(p) = \max(p, 2)$. *Then the identity map* i: $l_p \to l_p$ *is* $(r(p), 1)$*-absolutely summing.*

More generally, Bennett (1973) has elucidated the absolutely summing properties of the inclusion map $l_p \to l_q$ where $1 \leqslant p \leqslant q \leqslant \infty$.

Now although the definition of Mitiagin and Pełczyński was formulated for Banach spaces it is still meaningful for p-normed spaces X and Y, provided X^* is non-trivial, for example, if $X = l_p$ with $0 < p < 1$. Thus we may consider the problem of completing Theorem 1 by examining the identity map $i\colon l_p \to l_p$ for $0 < p < 1$. The result that we give in Theorem 4 below indicates the completely different character of the case when $0 < p < 1$.

2. The main result

For the proof of Theorem 4 we employ two lemmas. The ideas in these lemmas are due to Macphail (1947) who needed them for another purpose. Since Macphail did not explicitly state the results in the form that we need, we modify his presentation. We use the following notation:

$$|S| = \sum_{k=1}^{m} \|x_k\|, \qquad |S|^* = \sup\left\| \sum_{k \in E} x_k \right\|,$$

where $S = (x_1, x_2, \dots, x_m) \in X^m$ and the supremum is taken over all subsets E of $\{1, 2, \dots, m\}$.

Also, if $0 < p < 1$ and $b = (b_k) \in l_p$ we denote the natural p-norm of b by

$$\|b\|_p = \sum_{k=1}^{\infty} |b_k|^p.$$

LEMMA 2. *For each $n \geqslant 1$ suppose that $|S_n| > 0$ and*

$$|S_n|^*/|S_n| < 4^{-n}$$

where $S_n = (x_{n1}, x_{n2}, \dots, x_{nq(n)})$, $x_{nk} \in X$, $q(n)$ being a natural number.

Then, if $c_n = 2^n/|S_n|$, the series

$$\sum_{i=1}^{\infty} b_i = c_1 x_{11} + \cdots + c_1 x_{1q(1)} + c_2 x_{21} + \cdots + c_2 x_{2q(2)} + \cdots$$

is unconditionally convergent in X.

LEMMA 3. *For each $n \geqslant 1$ there are sequences $R_1^n, \dots, R_n^n$ such that*

$$|T(n)|^*/|T(n)| \leqslant n^{-1/2},$$

where for $1 \leqslant i \leqslant n$,

$$|R_i^n(m)| = 1 \quad \text{for } 1 \leqslant m \leqslant 2^n,$$

$$|R_i^n(m)| = 0 \quad \text{for } m > 2^n,$$

and

$$T(n) = (R_1^n, R_2^n, \ldots, R_n^n).$$

We remark that the R_i^n of Lemma 3 are constructed using Rademacher functions. For example, $R_1^n = (-1, -1, \ldots, -1, 1, 1, \ldots, 1, 0, 0, 0, \ldots)$ with -1 in the first 2^{n-1} places, 1 in the next 2^{n-1} places and 0 thereafter. Note also that each $R_i^n \in l_p$ for $p > 0$.

We now give the main theorem.

THEOREM 4. *Let* $0 < p < 1$. *Then the identity map* i: $l_p \to l_p$ *is not* (r, s)-*absolutely summing for any* r, s.

PROOF. It is clear that we need only show that i is not $(\infty, 1)$-absolutely summing.

Take $T(n)$ as in Lemma 3 and define for $n \geqslant 1$, $S_n = T(4^{2n})$. It follows from Lemma 2 that

$$\sum_{i=1}^{\infty} b_i = c_1 R_1^{16} + \cdots + c_1 R_{16}^{16} + c_2 R_1^{256} + \cdots + c_2 R_{256}^{256} + \cdots$$

is unconditionally convergent in l_1, which implies that $\Sigma |f(b_i)| < \infty$ for each $f \in l_1^*$. But for $0 < p < 1$, each $b_i \in l_p$ and also l_p^* may be identified with l_∞. Consequently we have

$$\text{(1)} \qquad \sum |f(b_i)| < \infty \quad \text{for each } f \in l_p^*.$$

Now consider terms in Σb_i of the form $b_i = c_n R_1^{4^{2n}}$ and write $k = 4^{2n}$ for simplification. Then

$$\|b_i\|_p = |c_n|^p \|R_1^k\|_p = |c_n|^p \cdot 2^k$$

with $c_n = 2^n/|T(k)| = 2^n/k2^k$. Hence $\|b_i\|_p = 2^{(1-p)k - 3np}$ and since $(1-p)4^{2n} - 3np \to \infty$ $(n \to \infty)$ we have

$$\text{(2)} \qquad \sup_i \|b_i\|_p = \infty.$$

By (1) and (2) we see that i: $l_p \to l_p$ is not $(\infty, 1)$-absolutely summing, which completes the proof.

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