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SCHRÖDINGER PROPAGATOR ON WIENER AMALGAM SPACES IN THE FULL RANGE

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Abstract. Using the technique of Gabor analysis, we characterize the boundedness of $e^{i\Delta}: W_m^{p_1,q_1} \to W^{p_2,q_2}$ with modulation and translation operators, where $0 < p_i, q_i \le \infty$ and m is a v-moderate weight. The sharp exponents for the boundedness are also characterized in the case of power weight.

§1. Introduction

Consider the following Cauchy problem:

$$\begin{cases} i\partial_t u + \Delta u = F(u), \\ u(0,x) = u_0. \end{cases}$$

The formal solution is given by

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}F(u(s))ds.$$

If we want to establish the local existence with the initial data u_0 belonging to certain function space X, a crucial step is to consider the initial data in function spaces, which are invariant under linear propagator, that is, to establish the corresponding norm estimate for Schrödinger propagator on X of the following type:

$$\|e^{i\Delta}f\|_X \lesssim \|f\|_X. \tag{1.1}$$

In the classical research of Schrödinger equation, the initial data u_0 are usually assumed to belong to an L^2-based function spaces, such as the L^2-based Sobolev space $(I-\Delta)^{\frac{-s}{2}}L^2$ or Besov space $B_s^{2,q}$. An important reason is that only $p=2$ allows the estimate (1.1) to hold on $X = (I-\Delta)^{\frac{-s}{2}}L^p$ or $X = B_s^{p,q}$.

The situation changes if we replace the classical dyadic decomposition in the definition of Besov spaces by uniform decomposition. More precisely, if we use modulation space $X = M_s^{p,q}$ in the estimate (1.1), the following boundedness is valid [1], [21]:

$$\|e^{i\Delta}f\|_{M_s^{p,q}} \le C_{p,q}\|f\|_{M_s^{p,q}}, \quad p,q \in (0,\infty]. \tag{1.2}$$

Due to this advantage of modulation space, many researchers have begun to use modulation space to study partial differential equations. We refer the reader to the pioneer works [2], [20] and to [3], [17] for some recent progress.

Modulation spaces were introduced first by Feichtinger [7] in 1983. It has been regarded as a basic and important class of function spaces in the field of time–frequency (see [9]). Comparing with the classical Besov space $B_s^{p,q}$, modulation space $M_s^{p,q}$ (see [18] for

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the equivalent norm) can be regarded as the Besov-type space associated with uniform decomposition on the frequency domain. Another important function space that is closely related to the modulation space (see Lemma 2.6) is the Wiener amalgam space $W_s^{p,q}$ (see §2.1). This space can be regarded as the Triebel-type space associated with uniform decomposition (see [18] for the equivalent norm).

Due to the boundedness in (1.2) and the fact that both modulation and Wiener amalgam space are defined based on the uniform decomposition, an interesting question is to establish the corresponding boundedness on $W_s^{p,q}$ of Schrödinger propagator. For this direction, Cunanan and Sugimoto [5, Th. 1.1] proved the boundedness for $1 \le p,q \le \infty$, $s > d|1/p - 1/q|$. Conversely, Cordero and Nicola [4, Prop. 6.1] found some necessary conditions for this boundedness. Finally, a complete characterization was established by Kato and Tomita as follows.

THEOREM A [16, Th. 1.1]. *Suppose* $1 \le p,q \le \infty$. *Then* $e^{i\Delta} : W_{1\otimes v_s}^{p,q} \to W^{p,q}$ *is bounded if and only if* $s \ge d|1/p-1/q|$ *with the strict inequality when* $p \ne q$.

Here, we write the power weights $v_s(\xi) := (1+|\xi|^2)^{\frac{s}{2}}$. From this theorem, one can find that the Schrödinger propagator is unbounded on $W_s^{p,q}$ for $p \ne q$. This is quite different from the case on modulation spaces. We also refer to [13] for the sharp estimate on $W_s^{p,q}$ of a class of unimodular Fourier multipliers. In fact, because of the weaker separation property of Triebel-type spaces, the behaviors of $e^{i\Delta}$ on $W_s^{p,q}$ are more difficult to study than those on modulation space. For instance, the boundedness $e^{i\Delta} : M_s^{p_1,q_1} \to M^{p_2,q_2}$ of full range $p_i, q_i \in (0,\infty]$ has been established in [22]. However, the corresponding boundedness result on Wiener amalgam space of full range is still unknown. The main goal of this paper is to fill this gap. To achieve this goal, our strategy is to first establish an equivalent characterization for the case of general weight. To avoid the fact that $\mathcal{S}(\mathbb{R}^d)$ is not dense in some endpoint spaces, such as the case $p=\infty$ or $q=\infty$, we only consider the action of $e^{i\Delta}$ on Schwartz function spaces. For the sake of simplicity, we use the statement "$e^{i\Delta} : W_m^{p_1,q_1} \to W^{p_2,q_2}$" or $e^{i\Delta} \in \mathcal{L}(W_m^{p_1,q_1}, W^{p_2,q_2})$ to express the meaning that

$$\|e^{i\Delta} f\|_{W^{p_2,q_2}} \le C\|f\|_{W_m^{p_1,q_1}} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^d).$$

Here, we use m to denote the weight function belonging to the class $\mathscr{P}(\mathbb{R}^{2d})$ (see §2.1 for the precise definition). We write the dilation operator $\mathcal{D}_{\lambda_1,\lambda_2} m(x,\xi) = m(\lambda_1 x, \lambda_2 \xi)$. For a sequence $\vec{a} = \{a_{k,n}\}$, denote by $(\mathcal{T}\vec{a})_{k,n} = a_{k-n,n}$ the coordinate transformation of $\vec{a}$. Our first theorem gives an equivalent characterization of boundedness.

THEOREM 1.1 (Equivalent characterization). *Let* $p_i, q_i \in (0,\infty]$ *for* $i = 1,2$. *Assume that* $m \in \mathscr{P}(\mathbb{R}^{2d})$. *We have the following equivalent relation:*

$$e^{i\Delta} \in \mathcal{L}(W_m^{p_1,q_1}(\mathbb{R}^d), W^{p_2,q_2}(\mathbb{R}^d)) \Longleftrightarrow \mathcal{T} \in \mathcal{L}(l^{(p_1,q_1)}_{\mathcal{D}_{2,1/2}m}(\mathbb{Z}^{2d}), l^{(p_2,q_2)}(\mathbb{Z}^{2d})).$$

Thanks to the above equivalent relation, one can turn to studying the corresponding discrete inequality without having to consider the Schrödinger propagator directly. Our second theorem is a sharp exponents characterization for the case of $m = 1 \otimes v_s$. It is known that the boundedness $e^{i\Delta} \in \mathcal{L}(W_{1\otimes v_s}^{p_1,q_1}(\mathbb{R}^d), W^{p_2,q_2}(\mathbb{R}^d))$ is the most interesting case, especially in the crossing field of harmonic analysis and PDEs.

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THEOREM 1.2 (Sharp exponents characterization). *Let $s \in \mathbb{R}$, $p_i, q_i \in (0,\infty]$ for $i = 1,2$. Denote $A = d(1/p_2 - 1/q_1)$ and $B = d(1/q_2 - 1/p_1)$. The boundedness*

$$e^{i\Delta} \in \mathcal{L}(W^{p_1,q_1}_{1\otimes v_s}(\mathbb{R}^d), W^{p_2,q_2}(\mathbb{R}^d))$$

holds if and only if $1/p_2 \le 1/p_1$ and

$$s \ge A \vee B \vee (A+B) \vee 0$$

with the strict inequality if one of the following cases happens:

1. $A > 0 \ge B$;
2. $B > 0 \ge A$;
3. $A, B > 0$, $p_1 = p_2$.

Using this theorem, one can conclude directly the following boundedness of Schrödinger propagator on Wiener amalgam spaces without potential loss.

COROLLARY 1.3. $e^{i\Delta} \in \mathcal{L}(W^{p_1,q_1}(\mathbb{R}^d), W^{p_2,q_2}(\mathbb{R}^d))$ *holds if and only if*

$$q_1 \le p_2 \text{ and } p_1 \le q_2 \wedge p_2.$$

REMARK 1.4. Some remarks about our main theorems are listed as follows:

1. The proof of Theorem 1.1 is based on the Gabor analysis of Wiener amalgam space and the magic formula in Lemma 3.1 associated with Schrödinger propagator and the time–frequency shift. We point out that this formula was also used in [15] for giving some conservation quantity associated with Schrödinger propagator.
2. Theorem 1.2 is an essential extension of Theorem A in [16]. In our theorem of full range, more endpoint cases create new difficulties. The equivalent characterization in Theorem 1.1 allows us to see the nature more clearly and solve it.
3. All the conclusions in Theorems 1.1 and 1.2 can be extended to the fixed time Schrödinger propagator $e^{it_0\Delta}$ for any fixed $t_0 \in \mathbb{R}$.
4. The method of this paper is also applicable in the study of the following boundedness:
$$e^{i\Delta} \in \mathcal{L}(X_1, X_2), \text{ with } X_j = W^{p_j,q_j}_{m_j} \text{ or } M^{p_j,q_j}_{m_j}, j = 1,2.$$
5. Although the magic formula in Lemma 3.1 seems to hold only for very special unimodular Fourier multipliers, the similar equivalent characterizations as in Theorem 1.1 of general unimodular Fourier multipliers are still expected to be valid with appropriate modification. However, the exponents characterizations as in Theorem 1.2 should be much more difficult for more general unimodular Fourier multipliers.

The rest of this paper is organized as follows. In §2, we first introduce some basic definitions and properties of the function spaces used throughout this paper. We also prepare some useful conclusions of the Gabor analysis on Wiener amalgam space. Section 3 is devoted to the equivalent characterization of the boundedness $e^{i\Delta}: W^{p_1,q_1}_m \to W^{p_2,q_2}$. As mentioned above, the key tools are the Gabor analysis and the magic formula of Schrödinger propagator. The sharp exponents characterization is proved in §4. In some critical case, the boundedness of Schrödinger propagator is found to be equivalent to the boundedness of fractional integral operators.

Throughout the context, we adopt the notation $X \lesssim Y$ to denote the statement that $X \leqslant CY$, and the notation $X \sim Y$ means the statement $X \lesssim Y \lesssim X$, where the positive

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constants C might be different from line to line. Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^d)$ be the Schwartz space, and let $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^d)$ be the space of tempered distributions. The Fourier transform $\mathscr{F} f$ and the inverse Fourier transform $\mathscr{F}^{-1} f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ are defined by

$$\mathscr{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi} dx, \ \ \mathscr{F}^{-1} f(x) = \check{f}(x) = \int_{\mathbb{R}^d} f(\xi) e^{2\pi i x\cdot\xi} d\xi.$$

§2. Preliminaries

2.1 Function spaces

In order to introduce the function spaces, we first recall some definitions of weight. Recall that a weight is a positive and locally integral function on $\mathbb{R}^{2d}$. A weight function m is called v-moderate if there exists another weight function v such that

$$m(z_1 + z_2) \leq C_v^m v(z_1) m(z_2), \qquad z_1, z_2 \in \mathbb{R}^{2d},$$

where v belongs to the class of submultiplicative weight, that is, v satisfies

$$v(z_1 + z_2) \leq v(z_1) v(z_2), \qquad z_1, z_2 \in \mathbb{R}^{2d}.$$

Moreover, in this paper, we assume that v has at most polynomial growth. If the associated weight v is implicit, we call that m is moderate and use the notation $\mathscr{P}(\mathbb{R}^{2d})$ to denote the cone of all wights which are moderate in $\mathbb{R}^{2d}$. We also assume that every $m \in \mathscr{P}(\mathbb{R}^{2d})$ is continuous since there exists a continuous weight m_1 such that $m \sim m_1$. We refer to [6] for the origin of the v-moderate weights. See also [14] for more properties of these weights.

Definition 2.1 (Continuous mixed-norm spaces). Let $m \in \mathscr{P}(\mathbb{R}^{2d})$, $p, q \in (0, \infty]$. The weighted mixed-norm space $L_m^{p,q}(\mathbb{R}^{2d})$ consists of all Lebesgue measurable functions on $\mathbb{R}^{2d}$ such that the (quasi-)norm

$$\begin{aligned} \|F\|_{L_m^{p,q}(\mathbb{R}^{2d})} =& \|Fm\|_{L^{p,q}(\mathbb{R}^{2d})} = \|F(x,\xi)m(x,\xi)\|_{L^{p,q}_{x,\xi}} \\ =& \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x,\xi)|^p m(x,\xi)^p dx \right)^{q/p} d\xi \right)^{1/q} \end{aligned}$$

is finite, with the usual modification when $p = \infty$ or $q = \infty$. We also use the notation

$$\begin{aligned} \|F\|_{L_m^{(p,q)}(\mathbb{R}^{2d})} =& \|Fm\|_{L^{(p,q)}(\mathbb{R}^{2d})} = \|F(x,\xi)m(x,\xi)\|_{L^{q,p}_{\xi,x}} \\ =& \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x,\xi)|^q m(x,\xi)^q d\xi \right)^{p/q} dx \right)^{1/p}. \end{aligned}$$

Then the mixed-norm space $L_m^{(p,q)}(\mathbb{R}^{2d})$ can be defined by the similar way as above.

In order to introduce the definitions of modulation and Wiener amalgam spaces, we recall some definitions and notations in time–frequency analysis. For $x, \xi \in \mathbb{R}^d$, the translation operator T_x and the modulation operator M_ξ are given by

$$T_x f(t) = f(t - x) \text{ and } M_\xi f(t) = e^{2\pi i \xi \cdot t} f(t).$$

For $z := (x, \xi)$, we also use the notation $\pi(z) := M_\xi T_x$, which is also known as the time–frequency shift on the phase plane.

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DEFINITION 2.2. Let $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, the short-time Fourier transform (STFT) of $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window g is defined by

$$V_g f(x,\xi) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)} e^{-2\pi i \xi \cdot t} dt = \langle f, M_\xi T_x g\rangle = \langle f, \pi(z) g\rangle,$$

where the integral makes sense for nice function f.

The so-called fundamental identity of time–frequency analysis is as follows:

$$V_g f(x,\xi) = e^{-2\pi i x\cdot\xi} V_{\hat{g}} \hat{f}(\xi, -x), \quad (x,\xi) \in \mathbb{R}^{2d}. \tag{2.1}$$

It can be observed that the STFT of a distribution f takes both the decay and smooth properties into account. The modulation space can be regarded as a collection of functions sharing the same decay and smooth properties. Now, we recall the definition of modulation space.

DEFINITION 2.3 (Modulation space). Let $0 < p,q \leq \infty$, $m \in \mathscr{P}(\mathbb{R}^{2d})$. Given a nonzero window function $\phi \in \mathcal{S}(\mathbb{R}^d)$, the (weighted) modulation space $M^{p,q}_m(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the (quasi-)norm

$$\|f\|_{M^{p,q}_m(\mathbb{R}^d)} := \|V_\phi f\|_{L^{p,q}_m(\mathbb{R}^{2d})} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_\phi f(x,\xi) m(x,\xi)|^p dx\right)^{q/p} d\xi\right)^{1/q}$$

is finite. If $m \equiv 1$, we write $M^{p,q}$ for short. We also write $M^{p,q}_s$ for the case $m = 1 \otimes v_s$.

Recall that the above definition of $M^{p,q}_m$ is independent of the choice of window function ϕ. We refer the readers to [9] for the case $(p,q) \in [1,\infty]^2$ and to [8, Th. 3.1] for full range $(p,q) \in (0,\infty]^2$. More precisely, a class of admissible windows denoted by $\mathfrak{M}^{p,q}_v$ was found in [8] such that every window function $g \in \mathfrak{M}^{p,q}_v$ yields the equivalent quasi-norm on $M^{p,q}_m$.

DEFINITION 2.4 (The space of admissible windows). Let $0 < p,q \leq \infty$, $r = \min\{1,p\}$, and $s = \min\{1,p,q\}$. Let m be v-moderate. For $r_1, s_1 > 0$, denote

$$w_{r_1,s_1}(x,w) = v(x,\omega)(1+|x|)^{r_1}(1+|\omega|)^{s_1}.$$

Then $\mathfrak{M}^{p,q}_v$ the admissible windows for the modulation space $M^{p,q}_m$ can be defined as

$$\mathfrak{M}^{p,q}_v := \bigcup_{\substack{r_1 > d/r \\ s_1 > d/s \\ 1 \leq p_1 < \infty}} M^{p_1}_{w_{r_1,s_1}}.$$

Next, we recall the definition of the $W^{p,q}_m$ space.

DEFINITION 2.5 (Wiener amalgam space). Let $0 < p,q \leq \infty$, $m \in \mathscr{P}(\mathbb{R}^{2d})$. Given a nonzero window function $\phi \in \mathcal{S}(\mathbb{R}^d)$, the (weighted) Wiener amalgam space $W^{p,q}_m(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{W^{p,q}_m(\mathbb{R}^d)} := \|V_\phi f\|_{L^{(p,q)}_m(\mathbb{R}^{2d})} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_\phi f(x,\xi) m(x,\xi)|^q d\xi\right)^{p/q} dx\right)^{1/p}$$

is finite. We write briefly $W^{p,q}$ for the case $m(x,\xi) \equiv 1$. We also use the notation $W^{p,q}_s$ for the case $m = 1 \otimes v_s$.

Recall that there is a close connection between modulation and Wiener amalgam spaces.

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LEMMA 2.6. *Denote $\widetilde{m}(x,\xi) := m(\xi,-x)$. We have $W_m^{p,q} = \mathscr{F}M_{\widetilde{m}}^{q,p}$. More precisely, the following relation is valid:*

$$\|f\|_{W_m^{p,q}} \sim \|V_\phi f(x,\xi)m(x,\xi)\|_{L_{\xi,x}^{q,p}} = \|V_{\check{\phi}}\check{f}(x,\xi)m(\xi,-x)\|_{L_{x,\xi}^{q,p}} \sim \|\check{f}\|_{M_{\widetilde{m}}^{q,p}}.$$

Due to the above relation, many conclusions can be automatically converted from modulation spaces to Wiener amalgam spaces. For instance, the definition of $W_m^{p,q}$ is independent of the window $g \in \mathscr{F}\mathfrak{M}_{\widetilde{v}}^{q,p}$, where $\widetilde{v}(x,\xi) := v(\xi,-x)$.

2.2 Gabor analysis on modulation and Wiener amalgam spaces

In this subsection, we recall an important time–frequency tool on modulation and Wiener amalgam spaces. In fact, the functions or distributions in modulation and Wiener amalgam spaces can be characterized by the summability and decay properties of their Gabor coefficients. We first list some important operators with their basic properties.

DEFINITION 2.7. Assume that $g,\gamma \in L^2(\mathbb{R}^d)$, $\alpha,\beta > 0$, and $\Gamma = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$. The coefficient operator or analysis operator $C_g^{\alpha,\beta}$ is defined by

$$C_g^{\alpha,\beta} f = \{\langle f, M_{\beta n}T_{\alpha k}g\rangle\}_{k,n\in\mathbb{Z}^d}.$$

The synthesis operator or reconstruction operator $D_\gamma^{\alpha,\beta}$ is defined by

$$D_\gamma^{\alpha,\beta}\vec{c} = \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} c_{k,n} M_{\beta n}T_{\alpha k}\gamma.$$

The Gabor frame operator $S_{g,\gamma}^{\alpha,\beta}$ is defined by

$$S_{g,\gamma}^{\alpha,\beta} f = D_\gamma^{\alpha,\beta} C_g^{\alpha,\beta} f = \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} \langle f, M_{\beta n}T_{\alpha k}g\rangle M_{\beta n}T_{\alpha k}\gamma.$$

We also use C_g, D_γ, and $S_{g,\gamma}$ for short, if the parameters α and β are implicit.

We remark that the definitions of C_g and D_γ can be extended if the window functions g and γ are taken suitably. Next, we give the definitions of discrete mixed-norm spaces.

DEFINITION 2.8 (Discrete mixed-norm spaces). Let $0 < p,q \le \infty$, $m \in \mathscr{P}(\mathbb{R}^{2d})$. The space $l_m^{p,q}(\mathbb{Z}^{2d})$ consists of all sequences $\vec{a} = \{a_{k,n}\}_{k,n\in\mathbb{Z}^d}$ for which the (quasi-)norm

$$\|\vec{a}\|_{l_m^{p,q}(\mathbb{Z}^{2d})} = \left(\sum_{n\in\mathbb{Z}^d}\left(\sum_{k\in\mathbb{Z}^d} |a_{k,n}|^p m(k,n)^p\right)^{q/p}\right)^{1/q}$$

is finite, with the usual modification when $p=\infty$ or $q=\infty$. We also use the notation

$$\|\vec{a}\|_{l_m^{(p,q)}(\mathbb{Z}^{2d})} = \left(\sum_{k\in\mathbb{Z}^d}\left(\sum_{n\in\mathbb{Z}^d} |a_{k,n}|^q m(k,n)^q\right)^{p/q}\right)^{1/p}.$$

Then the mixed-norm space $l_m^{(p,q)}(\mathbb{Z}^{2d})$ can be defined by the similar way. For a continuous function F on $\mathbb{R}^{2d}$, we define

$$\|F\|_{l^{p,q}(\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d)} = \|(F(\alpha k,\beta n))_{k,n}\|_{l^{p,q}(\mathbb{Z}^{2d})}, \quad \|F\|_{l^{(p,q)}(\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d)} = \|(F(\alpha k,\beta n))_{k,n}\|_{l^{(p,q)}(\mathbb{Z}^{2d})}.$$

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Based on the admissible window class mentioned above, we recall the boundedness of C_g and D_g, which works on the full range $p,q \in (0,\infty]$ (see [8] for more details).

LEMMA 2.9 (See Theorems 3.5 and 3.6 in [8]). *Assume that m is v-moderate, $p,q \in (0,\infty]$, and g belongs to the subclass $M^{p_1}_{\omega_{r_1,s_1}}$ of $\mathfrak{M}^{p,q}_v$. For all lattice constants $\alpha,\beta > 0$, we have*

$$\|C_g f(k,n) m(\alpha k,\beta n)\|_{l^{p,q}} = \|V_g f(\alpha k,\beta n) m(\alpha k,\beta n)\|_{l^{p,q}} \lesssim \|V_g f\|_{L^{p,q}_m} \sim \|f\|_{M^{p,q}_m}$$

and

$$\|D_g \vec{c}\|_{M^{p,q}_m} \lesssim \|c_{k,n} m(\alpha k,\beta n)\|_{l^{p,q}}$$

independently of p,q,m.

Now, we recall the Gabor characterization of modulation space in [8, Th. 3.7].

LEMMA 2.10 (Gabor characterization of $M^{p,q}_m$ with $0 < p,q \le \infty$). *Assume that m is v-moderate on $\mathbb{R}^{2d}$, $p,q \in (0,\infty]$, $g,\gamma \in \mathfrak{M}^{p,q}_v$, and that the Gabor frame operator $S_{g,\gamma} = D_\gamma C_g = I$ on $L^2(\mathbb{R}^d)$. Then*

$$f = \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} \langle f, M_{\beta n}T_{\alpha k} g\rangle M_{\beta n}T_{\alpha k}\gamma = \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} \langle f, M_{\beta n}T_{\alpha k}\gamma\rangle M_{\beta n}T_{\alpha k} g$$

with unconditional convergence in $M^{p,q}_m$ if $p,q<\infty$, and with weak-star convergence in $M^\infty_{1/v}$ otherwise. Furthermore, there are constants $A,B>0$ such that for all $f \in M^{p,q}_m$,

$$A\|f\|_{M^{p,q}_m} \le \|V_g f \cdot m\|_{l^{p,q}(\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d)} \le B\|f\|_{M^{p,q}_m}.$$

Next, we list the corresponding results on Wiener amalgam spaces, which can be verified by using Lemmas 2.9 and 2.10 and the fundamental identity (2.1) directly. We omit the proof here.

LEMMA 2.11. *Assume that m is v-moderate, $p,q \in (0,\infty]$, and g belongs to the subclass $\mathscr{F}M^{p_1}_{\omega_{r_1,s_1}}$ of $\mathscr{F}\mathfrak{M}^{q,p}_{\widetilde{v}}$, where $\widetilde{v}(x,\xi) := v(\xi,-x)$. For all lattice constants $\alpha,\beta>0$, we have*

$$\|C_g f(k,n) m(\alpha k,\beta n)\|_{l^{q,p}_{n,k}} = \|V_g f(\alpha k,\beta n) m(\alpha k,\beta n)\|_{l^{q,p}_{n,k}} \lesssim \|V_g f(x,w) m(x,w)\|_{L^{q,p}_{w,x}} \sim \|f\|_{W^{p,q}_m}$$

and

$$\|D_g \vec{c}\|_{W^{p,q}_m} \lesssim \|c(k,n) m(\alpha k,\beta n)\|_{l^{q,p}_{n,k}}$$

independently of p,q,m.

LEMMA 2.12 (Gabor frames for Wiener spaces $W^{p,q}_m$ with $0 < p,q \le \infty$). *Assume that m is v-moderate on $\mathbb{R}^{2d}$, $p,q \in (0,\infty]$, $g,\gamma \in \mathscr{F}\mathfrak{M}^{q,p}_{\widetilde{v}}$ with $\widetilde{v}(x,\xi) := v(\xi,-x)$, and that the Gabor frame operator $S_{g,\gamma} = D_\gamma C_g = I$ on $L^2(\mathbb{R}^d)$. Then*

$$f = \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} \langle f, M_{\beta n}T_{\alpha k} g\rangle M_{\beta n}T_{\alpha k}\gamma = \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} \langle f, M_{\beta n}T_{\alpha k}\gamma\rangle M_{\beta n}T_{\alpha k} g$$

with unconditional convergence in $W^{p,q}_m$ if $p,q<\infty$, and with weak-star convergence in $W^\infty_{1/v}$ otherwise. Furthermore, there are constants $A,B>0$ such that for all $f \in W^{p,q}_m$,

$$A\|f\|_{W^{p,q}_m} \le \|V_g f \cdot m\|_{l^{(p,q)}(\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d)} \le B\|f\|_{W^{p,q}_m}.$$

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The following well-known theorem provides a way to find the Gabor frame of $L^2(\mathbb{R}^d)$. Recall that $\|g\|_{W(L^\infty,L^1)(\mathbb{R}^d)} = \sum_{n\in\mathbb{Z}^d}\|g\chi_{Q+n}\|_{L^\infty}$ with $Q=[0,1]^d$.

LEMMA 2.13 (Walnut [19]). *Suppose that $g\in W(L^\infty,L^1)(\mathbb{R}^d)$ satisfies*

$$A \le \sum_{k\in\mathbb{Z}^d}|g(x-\alpha k)|^2 \le B \qquad a.e.$$

for constants $A,B\in(0,\infty)$. Then there exists a constant β_0 depending on α such that $\mathcal{G}(g,\alpha,\beta) := \{T_{\alpha k}M_{\beta n}g\}_{k,n\in\mathbb{Z}^d}$ is a Gabor frame of $L^2(\mathbb{R}^d)$ for all $\beta\le\beta_0$.

In order to find the dual window in a suitable function space, we recall the following conclusion.

LEMMA 2.14 (See Theorem 4.2 in [10]). *Assume that $g\in M_v^1(\mathbb{R}^d)$ and that $\{T_{\alpha k}M_{\beta n}g\}_{k,n\in\mathbb{Z}^d}$ is a Gabor frame for $L^2(\mathbb{R}^d)$. Then the Gabor frame operator $S_{g,g}^{\alpha,\beta}$ is invertible on $M_v^1(\mathbb{R}^d)$. As a consequence, $S_{g,g}^{\alpha,\beta}$ is invertible on all modulation spaces $M_m^{p,q}(\mathbb{R}^d)$ for $1\le p,q\le\infty$ and $m\in\mathscr{P}(\mathbb{R}^{2d})$.*

Based on the above lemmas, we give a lemma used in our proof of Theorem 1.1.

LEMMA 2.15. *Suppose that $p_i,q_i\in(0,\infty]$, $m_i\in\mathscr{P}(\mathbb{R}^d)$, $i=1,2$. For any $g_1,g_2\in\mathcal{S}(\mathbb{R}^d)\backslash\{0\}$, there exists a constant $N\in\mathbb{N}$ such that for all $f\in\mathcal{S}(\mathbb{R}^d)$,*

$$\|f\|_{W^{p_i,q_i}_{m_i}} \sim \|V_{g_i}f(\frac{k}{N},\frac{n}{N})m_i(\frac{k}{N},\frac{n}{N})\|_{l^{(p_i,q_i)}(\mathbb{Z}^{2d})}.$$

Proof. For nonzero Schwartz functions g_1 and g_2, there exists a constant $\alpha = 1/\widetilde{N}$ with $\widetilde{N}\in\mathbb{N}$ such that

$$0<A_0\le\sum_{k\in\mathbb{Z}^d}|g_i\left(x-\frac{k}{\widetilde{N}}\right)|^2\le B_0<\infty, \quad x\in\mathbb{R}^d.$$

Using Lemma 2.13, one can find a constant $\beta=\alpha/L$ with $L\in\mathbb{N}$ such that $\mathcal{G}(g_i,\alpha,\beta) := \{T_{\alpha k}M_{\beta n}g_i\}_{k,n\in\mathbb{Z}^d}$ is a Gabor frame of $L^2(\mathbb{R}^d)$ for $i=1,2$, respectively. Denote by $\gamma_i = (S^{\alpha,\beta}_{g_i,g_i})^{-1}g_i$ be the canonical dual window function of g_i for $i=1,2$. Using Lemma 2.14, we find that $\gamma_i\in M^1_{v_s\otimes v_s}$ for any sufficiently large $s>0$. It follows that $\gamma_i\in\mathscr{F}M^1_{v_s\otimes v_s}\subset\mathscr{F}\mathfrak{M}^{q_i,p_i}_{\widetilde{v_i}}$, where $\widetilde{v_i}(x,\xi) := v_i(\xi,-x)$, $i=1,2$. We also have $S_{g_i,\gamma_i} = D_{\gamma_i}C_{g_i} = I$ on $L^2(\mathbb{R}^d)$.

Using Lemma 2.12, one can find two constants $A,B>0$ such that for every $f\in\mathcal{S}(\mathbb{R}^d)$,

$$A\|f\|_{W^{p_i,q_i}_{m_i}} \le \|V_{g_i}f\cdot m_i\|_{l^{(p_i,q_i)}(\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d)} \le B\|f\|_{W^{p_i,q_i}_{m_i}}, \qquad i=1,2.$$

Denote $N = L\widetilde{N}$. Note that $\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d\subset\beta\mathbb{Z}^d\times\beta\mathbb{Z}^d = \mathbb{Z}^d/N\times\mathbb{Z}^d/N$. We obtain

$$A\|f\|_{W^{p_i,q_i}_{m_i}} \le \|V_{g_i}f\cdot m_i\|_{l^{(p_i,q_i)}(\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d)} \le \|V_{g_i}f(\frac{k}{N},\frac{n}{N})m_i(\frac{k}{N},\frac{n}{N})\|_{l^{(p_i,q_i)}(\mathbb{Z}^{2d})}.$$

On the other hand, the sampling property (see Lemma 2.11) yields that

$$\|V_{g_i}f\left(\frac{k}{N},\frac{n}{N}\right)m_i\left(\frac{k}{N},\frac{n}{N}\right)\|_{l^{(p_i,q_i)}(\mathbb{Z}^{2d})} \le B_1\|f\|_{W^{p_i,q_i}_{m_i}}.$$

The desired conclusion follows by the above two estimates. ∎

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§3. Characterization of boundedness of $e^{i\Delta}$ on Wiener amalgam spaces

In this section, we give the proof of Theorem 1.1, showing that the boundedness $e^{i\Delta} \in \mathcal{L}(W_m^{p_1,q_1}, W^{p_2,q_2})$ is equivalent to the corresponding boundedness on discrete mixed spaces of coordinate transformation $\mathcal{T}$. To achieve this goal, we first give a calculation of STFT associated with Schördinger operator. See also [15].

LEMMA 3.1 (STFT of $e^{-i\pi|D|^2}$). *For $f \in \mathscr{S}'$, $\phi \in \mathscr{S}$, we have*

$$V_{e^{-i\pi|D|^2}\phi}(e^{-i\pi|D|^2}f)(x,\omega) = e^{-i\pi|\omega|^2} \cdot V_\phi f(x-\omega,\omega).$$

Proof. Note that $e^{-i\pi|D|^2}\phi \in \mathscr{S}$. Using the foundation identity of time–frequency analysis, we find that

$$\begin{aligned}
V_{e^{-i\pi|D|^2}\phi}(e^{-i\pi|D|^2}f)(x,\omega) &= e^{-2\pi i x\cdot\omega} V_{\widehat{e^{-i\pi|D|^2}\phi}}(\widehat{e^{-i\pi|D|^2}f})(\omega,-x) \\
&= e^{-2\pi i x\cdot\omega} \langle \widehat{e^{-i\pi|D|^2}f}, M_{-x}T_\omega \widehat{e^{-i\pi|D|^2}\phi} \rangle \\
&= e^{-2\pi i x\cdot\omega} \langle e^{-i\pi|\xi|^2}\widehat{f}(\xi), M_{-x}T_\omega e^{-i\pi|\xi|^2}\widehat{\phi}(\xi) \rangle \\
&= e^{-2\pi i x\cdot\omega + i\pi|\omega|^2} \langle \widehat{f}, M_{\omega-x}T_\omega \widehat{\phi} \rangle \\
&= e^{-2\pi i x\cdot\omega + i\pi|\omega|^2} V_{\widehat{\phi}}\widehat{f}(\omega,\omega-x) = e^{-i\pi|\omega|^2} \cdot V_\phi f(x-\omega,\omega).
\end{aligned}$$

∎

For convenience in the proof of Theorem 1.1, we would like to use the boundedness of $e^{-i\pi|D|^2}$ rather than that of the standard Schrödinger propagator $e^{i\Delta}$. The following lemma gives the relationship between the boundedness of these two operators.

LEMMA 3.2 (Boundedness of $e^{-i\pi|D|^2}$ and $e^{i\Delta}$). *Let $p_i, q_i \in (0,\infty]$ for $i = 1,2$. Assume that $m \in \mathscr{P}(\mathbb{R}^{2d})$. We have the following equivalent relation:*

$$e^{i\Delta} \in \mathcal{L}(W_m^{p_1,q_1}(\mathbb{R}^d), W^{p_2,q_2}(\mathbb{R}^d)) \Longleftrightarrow e^{-i\pi|D|^2} \in \mathcal{L}(W_{\mathcal{D}_{2,1/2}m}^{p_1,q_1}(\mathbb{R}^d), W^{p_2,q_2}(\mathbb{R}^d)).$$

Proof. By using the scaling property of the Fourier transform and a direct calculation, we have

$$\begin{aligned}
e^{i\Delta}f &= \mathscr{F}^{-1}(e^{-4\pi i|\omega|^2}\hat{f}(\omega)) = \mathscr{F}^{-1}(e^{-i\pi|2\omega|^2}\hat{f}(\omega)) = 2^d\mathscr{F}^{-1}(e^{-i\pi|2\omega|^2}\widehat{\mathcal{D}_2 f}(2\omega)) \\
&= \mathcal{D}_{1/2}\mathscr{F}^{-1}(e^{-i\pi|\omega|^2}\widehat{\mathcal{D}_2 f}(\omega)) = \mathcal{D}_{1/2}e^{-i\pi|D|^2}\mathcal{D}_2 f.
\end{aligned}$$

Note that

$$V_{\mathcal{D}_{1/2}\phi}\mathcal{D}_{1/2}f(x,\omega) = 2^d V_\phi f(x/2, 2\omega). \tag{3.1}$$

For any $f \in \mathscr{S}$, we have

$$\begin{aligned}
\|e^{i\Delta}f\|_{W^{p_2,q_2}} &= \|\mathcal{D}_{1/2}e^{-i\pi|D|^2}\mathcal{D}_2 f\|_{W^{p_2,q_2}} = \|V_{\mathcal{D}_{1/2}\phi}(\mathcal{D}_{1/2}e^{-i\pi|D|^2}\mathcal{D}_2 f)(x,\omega)\|_{L^{(p_2,q_2)}} \\
&= 2^d\|V_\phi(e^{-i\pi|D|^2}\mathcal{D}_2 f)(x/2,2\omega)\|_{L^{(p_2,q_2)}} \sim \|V_\phi(e^{-i\pi|D|^2}\mathcal{D}_2 f)(x,\omega)\|_{L^{(p_2,q_2)}} \\
&= \|e^{-i\pi|D|^2}\mathcal{D}_2 f\|_{W^{p_2,q_2}}.
\end{aligned}$$

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On the other hand, we find

$$\begin{aligned}\|f\|_{W_m^{p_1,q_1}} &= \|\mathcal{D}_{1/2}\mathcal{D}_2 f\|_{W_m^{p_1,q_1}} = \|V_{\mathcal{D}_{1/2}\phi}(\mathcal{D}_{1/2}\mathcal{D}_2 f)(x,\omega)m(x,\omega)\|_{L^{(p_1,q_1)}}\\ &= 2^d\|V_\phi(\mathcal{D}_2 f)(x/2,2\omega)m(x,\omega)\|_{L^{(p_1,q_1)}} \sim \|V_\phi(\mathcal{D}_2 f)(x,\omega)m(2x,\omega/2)\|_{L^{(p_1,q_1)}}\\ &= \|\mathcal{D}_2 f\|_{W^{p_1,q_1}_{\mathcal{D}_{2,1/2}m}}.\end{aligned}$$

From the above two estimates, for any $f \in \mathscr{S}$, we have

$$\begin{aligned}e^{i\Delta} \in \mathcal{L}(W_m^{p_1,q_1}, W^{p_2,q_2}) &\Longleftrightarrow \|e^{i\Delta}f\|_{W^{p_2,q_2}} \lesssim \|f\|_{W_m^{p_1,q_1}} \quad \text{for any } f\in\mathscr{S}\\ &\Longleftrightarrow \|e^{-i\pi|D|^2}\mathcal{D}_2 f\|_{W^{p_2,q_2}} \lesssim \|\mathcal{D}_2 f\|_{W^{p_1,q_1}_{\mathcal{D}_{2,1/2}m}} \quad \text{for any } f\in\mathscr{S}\\ &\Longleftrightarrow e^{-i\pi|D|^2} \in \mathcal{L}(W^{p_1,q_1}_{\mathcal{D}_{2,1/2}m}, W^{p_2,q_2}). \qquad \square\end{aligned}$$

Now, we turn to the characterization of the boundedness of $e^{-i\pi|D|^2}$.

PROPOSITION 3.3 (Equivalent characterization for $e^{-i\pi|D|^2}$). *Let $p_i, q_i \in (0,\infty]$ for $i=1,2$. Assume that $m\in\mathscr{P}(\mathbb{R}^{2d})$. The following equivalent relation is valid:*

$$e^{-i\pi|D|^2} \in \mathcal{L}(W_m^{p_1,q_1}(\mathbb{R}^d), W^{p_2,q_2}(\mathbb{R}^d)) \Longleftrightarrow \mathcal{T} \in \mathcal{L}(l_m^{(p_1,q_1)}(\mathbb{Z}^{2d}), l^{(p_2,q_2)}(\mathbb{Z}^{2d})).$$

Proof. We first verify the "$\Longrightarrow$" direction. For this purpose, we choose a nonnegative function $h\in\mathcal{S}(\mathbb{R}^d)\backslash\{0\}$ satisfying $\operatorname{supp}\hat{h}\subset B(0,1/8)$ and $\|h\|_{L^2}=1$. For any truncated (only finite nonzero terms) nonnegative sequence $\vec{a}=\{a_{k,n}\}_{(k,n)\in\mathbb{Z}^{2d}}$, we define

$$f = D_h\vec{a} = \sum_{(k,n)\in\mathbb{Z}^{2d}} a_{k,n}\pi(k,n)h.$$

By a direct calculation, we find that

$$\begin{aligned}V_h f(k,n) &= \sum_{j,l} a_{j,l}\langle \pi(j,l)h, \pi(k,n)h\rangle = \sum_j a_{j,n}\langle \pi(j,n)h,\pi(k,n)h\rangle\\ &= \sum_j a_{j,n}\langle T_j h, T_k h\rangle \ge a_{k,n}\langle T_k h, T_k h\rangle = a_{k,n}\|h\|_{L^2} = a_{k,n}.\end{aligned}$$

Combining this with Lemma 2.11, the sampling property, and Lemma 3.1, we obtain

$$\begin{aligned}\|e^{-i\pi|D|^2}f\|_{W^{p_2,q_2}} &\gtrsim \|V_{e^{-i\pi|D|^2}h}(e^{-i\pi|D|^2}f)(k,n)\|_{l^{(p_2,q_2)}}\\ &= \|e^{-i\pi|n|^2}\cdot V_h f(k-n,n)\|_{l^{(p_2,q_2)}}\\ &\ge \|a_{k-n,n}\|_{l^{(p_2,q_2)}} = \|\mathcal{T}\vec{a}\|_{l^{(p_2,q_2)}(\mathbb{Z}^{2d})}.\end{aligned}$$

On the other hand, by the boundedness of reconstruction operator (see Lemma 2.11), we obtain

$$\|f\|_{W_m^{p_1,q_1}} = \|D_h\vec{a}\|_{W_m^{p_1,q_1}} \lesssim \|\vec{a}\|_{l_m^{(p_1,q_1)}(\mathbb{Z}^{2d})}.$$

The "$\Longrightarrow$" direction follows by the above two estimates.

Next, we turn to the "$\Longleftarrow$" direction. Using Lemma 2.15, we can find a constant $N\in\mathbb{N}$ such that for any $f\in\mathcal{S}(\mathbb{R}^d)$,

$$\|e^{-i\pi|D|^2}f\|_{W^{p_2,q_2}} \sim \|V_{e^{-i\pi|D|^2}h}(e^{-i\pi|D|^2}f)\left(\frac{k}{N},\frac{n}{N}\right)\|_{l^{(p_2,q_2)}(\mathbb{Z}^{2d})}$$

and

$$\|f\|_{W_m^{p_1,q_1}} \sim \|V_h f\left(\frac{k}{N},\frac{n}{N}\right) m\left(\frac{k}{N},\frac{n}{N}\right)\|_{l^{(p_1,q_1)}(\mathbb{Z}^{2d})}.$$

From this, in order to obtain the desired conclusion, we only need to verify the following inequality:

$$\|V_{e^{-i\pi|D|^2}h}(e^{-i\pi|D|^2}f)\left(\frac{k}{N},\frac{n}{N}\right)\|_{l^{(p_2,q_2)}(\mathbb{Z}^{2d})} \lesssim \|V_h f\left(\frac{k}{N},\frac{n}{N}\right) m\left(\frac{k}{N},\frac{n}{N}\right)\|_{l^{(p_1,q_1)}(\mathbb{Z}^{2d})}.$$

Using Lemma 3.1, this is equivalent to

$$\|V_h f\left(\frac{k-n}{N},\frac{n}{N}\right)\|_{l^{(p_2,q_2)}(\mathbb{Z}^{2d})} \lesssim \|V_h f\left(\frac{k}{N},\frac{n}{N}\right) m\left(\frac{k}{N},\frac{n}{N}\right)\|_{l^{(p_1,q_1)}(\mathbb{Z}^{2d})}.$$

For this constant $N \in \mathbb{N}$, we construct a decomposition of $\mathbb{Z}^d \times \mathbb{Z}^d$ by

$$\mathbb{Z}^d \times \mathbb{Z}^d = \bigcup_{(j,l)\in\Lambda} \Gamma_{j,l},$$

where $\Gamma_{j,l} := \{(Nk+j, Nn+l), n,k \in \mathbb{Z}^d\}$ and $\Lambda := \{(j,l) \in \mathbb{N}^d \times \mathbb{N}^d : 0 \le \|j\|_{l^\infty}, \|l\|_{l^\infty} < N\}$. Write

$$\begin{aligned}\|V_h f\left(\frac{k-n}{N},\frac{n}{N}\right)\|_{l^{(p_2,q_2)}(\mathbb{Z}^{2d})} &\le \sum_{(j,l)\in\Lambda} \|V_h f\left(\frac{k-n}{N},\frac{n}{N}\right)\|_{l^{(p_2,q_2)}(\Gamma_{j,l})}\\ &= \sum_{(j,l)\in\Lambda} \|V_h f\left(k-n+\frac{j-l}{N}, n+\frac{l}{N}\right)\|_{l^{(p_2,q_2)}(\mathbb{Z}^{2d})}.\end{aligned}$$

Using the assumption $\mathcal{T} \in \mathcal{L}(l_m^{(p_1,q_1)}(\mathbb{Z}^{2d}), l^{(p_2,q_2)}(\mathbb{Z}^{2d}))$, the last term above can be dominated by

$$\begin{aligned}&\sum_{(j,l)\in\Lambda} \|V_h f\left(k+\frac{j-l}{N}, n+\frac{l}{N}\right) m(k,n)\|_{l^{(p_1,q_1)}(\mathbb{Z}^{2d})}\\ \lesssim &\sum_{(j,l)\in\Lambda} \|V_h f\left(k+\frac{j-l}{N}, n+\frac{l}{N}\right) m\left(k+\frac{j-l}{N}, n+\frac{l}{N}\right)\|_{l^{(p_1,q_1)}(\mathbb{Z}^{2d})}\\ \lesssim &\sum_{(j,l)\in\Lambda} \|V_h f\left(\frac{Nk+j}{N}, \frac{Nn+l}{N}\right) m\left(\frac{Nk+j}{N}, \frac{Nn+l}{N}\right)\|_{l^{(p_1,q_1)}(\mathbb{Z}^{2d})}\\ \lesssim &\|V_h f\left(\frac{k}{N},\frac{n}{N}\right) m\left(\frac{k}{N},\frac{n}{N}\right)\|_{l^{(p_1,q_1)}(\mathbb{Z}^{2d})},\end{aligned}$$

where we use the fact that

$$m(k,n) \sim m\left(k+\frac{j-l}{N}, n+\frac{l}{N}\right) \quad \text{for all} \quad (j,l) \in \Lambda.$$

We have now completed this proof. ∎

Proof of Theorem 1.1. The desired conclusion follows directly by Lemma 3.2 and Proposition 3.3. ∎

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§4. Sharp exponents characterization of $e^{i\Delta}$ on Wiener amalgam spaces

Thanks to the equivalent characterization in Theorem 1.1, we first consider the sharp exponents characterization for $\mathcal{T} \in \mathcal{L}(l^{(p_1,q_1)}_{1\otimes v_s}(\mathbb{Z}^{2d}), l^{(p_2,q_2)}(\mathbb{Z}^{2d}))$. The following lemmas play an important role in our proof.

LEMMA 4.1 (see Lemma 4.4 in [11]). *$l^{q,s} \subset l^p$ if and only if $s \ge d(1/p - 1/q) \vee 0$ with strict inequality if $1/p > 1/q$.*

LEMMA 4.2. *Let $0 < q \le p \le \infty$, $s \in \mathbb{R}$. We have $\mathcal{T} \in \mathcal{L}(l^{(q,p)}_{1\otimes v_s}(\mathbb{Z}^{2d}), l^{(p,q)}_{1\otimes v_s}(\mathbb{Z}^{2d}))$.*

Proof. Write

$$\begin{aligned}\|\mathcal{T}\vec{a}\|_{l^{(p,q)}_{1\otimes v_s}} &= \left(\sum_k \left(\sum_n |a_{k-n,n}|^q \langle n\rangle^{sq}\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \\ &= \left(\sum_k \left(\sum_n |a_{n,k-n}|^q \langle k-n\rangle^{sq}\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}.\end{aligned}$$

Applying the Minkowski inequality, the last term can be dominated from above by

$$\left(\sum_n \left(\sum_k |a_{n,k-n}|^p \langle k-n\rangle^{sp}\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} = \left(\sum_n \left(\sum_k |a_{n,k}|^p \langle k\rangle^{sp}\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} = \|\vec{a}\|_{l^{(q,p)}_{1\otimes v_s}}. \qquad \square$$

For the convenience of the readers, we introduce some notations used in the proof of Proposition 4.3. Denote

$$Y = \{(p_1,p_2,q_1,q_2,s) \in (0,\infty]^4 \times \mathbb{R} : \mathcal{T} \in \mathcal{L}(l^{(p_1,q_1)}_{1\otimes v_s}(\mathbb{Z}^{2d}), l^{(p_2,q_2)}(\mathbb{Z}^{2d}))\},$$

and let X be the set as follows:

$$X = \{(p_1,p_2,q_1,q_2,s) \in (0,\infty]^4 \times \mathbb{R} : 1/p_2 \le 1/p_1,\ s \ge A \vee B \vee (A+B) \vee 0\}.$$

Define the sets X_j, $j = 0,1,2,3$, as

$$\begin{aligned} X_0 &= \{(p_1,p_2,q_1,q_2,s) \in X : A, B \le 0\}, \\ X_1 &= \{(p_1,p_2,q_1,q_2,s) \in X : A > 0, B \le 0\}, \\ X_2 &= \{(p_1,p_2,q_1,q_2,s) \in X : A \le 0, B > 0\}, \\ X_3 &= \{(p_1,p_2,q_1,q_2,s) \in X : A, B > 0\}. \end{aligned}$$

Note that X is now the union of the mutually disjoint sets X_j, $j = 0,1,2,3$. Denote $Y_j = Y \cap X_j$, $j = 0,1,2,3$. In the proof of Proposition 4.3, our strategy is to prove $Y \subset X$ first, and then to characterize the set Y_j, $j = 0,1,2,3$. Therefore, the desired set Y can be characterized by

$$Y = Y \cap X = Y \cap \left(\bigcup_{j=0}^{3} X_j\right) = \bigcup_{j=0}^{3} Y_j.$$

PROPOSITION 4.3 (Sharp exponents for $\mathcal{T}$). *Let $s \in \mathbb{R}$, $p_i, q_i \in (0,\infty]$ for $i = 1,2$. The boundedness*

$$\mathcal{T} \in \mathcal{L}(l^{(p_1,q_1)}_{1\otimes v_s}(\mathbb{Z}^{2d}), l^{(p_2,q_2)}(\mathbb{Z}^{2d})) \tag{4.1}$$

holds if and only if $1/p_2 \le 1/p_1$ and s satisfies the conditions in Theorem 1.2.

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Proof. Recall the definition of Y mentioned above. This proof is actually to characterize the set Y. It can be achieved by the following steps.

Step 1. The goal of this part is to verify that $Y \subset X$. In fact, if we can verify the following claim:

$$\mathcal{T} \in \mathcal{L}(l^{(p_1,q_1)}_{1\otimes v_s}(\mathbb{Z}^{2d}), l^{(p_2,q_2)}(\mathbb{Z}^{2d})) \Longrightarrow \begin{cases} l^{p_1} \subset l^{p_2}, \ \ l^{q_1}_{v_s} \subset l^{p_2}, \ \ l^{p_1}_{v_s} \subset l^{q_2}, \\ s \geq d(1/p_2 - 1/q_1 + 1/q_2 - 1/p_1), \end{cases} \tag{4.2}$$

then the desired conclusion follows by Lemma 4.1. Now, we turn to the proof of this claim. If (4.1) holds, we have

$$\left(\sum_k \left(\sum_n |a_{k-n,n}|^{q_2}\right)^{\frac{p_2}{q_2}}\right)^{\frac{1}{p_2}} \lesssim \left(\sum_k \left(\sum_n |a_{k,n}|^{q_1} \langle n\rangle^{sq_1}\right)^{\frac{p_1}{q_1}}\right)^{\frac{1}{p_1}}. \tag{4.3}$$

The embedding relations $l^{p_1} \subset l^{p_2}$, $l^{q_1}_{v_s} \subset l^{p_2}$, $l^{p_1}_{v_s} \subset l^{q_2}$ follows by taking

$$a_{k,n} = \begin{cases} b_k, & n = 0, \\ 0, & n \neq 0, \end{cases} \quad \text{and} \quad a_{k,n} = \begin{cases} b_n, & k = 0, \\ 0, & k \neq 0, \end{cases} \quad \text{and} \quad a_{k,n} = \begin{cases} b_k, & k+n = 0, \\ 0, & k+n \neq 0, \end{cases}$$

in the inequality (4.3), respectively.

Moreover, for large constant N, take

$$a_{k,n} = \begin{cases} 1, & |k| \leq 2N, |n| \leq N, \\ 0, & \text{others.} \end{cases}$$

Notice that $s \geq 0$ deduced by (4.2) and Lemma 4.1. One can verify that

$$\begin{aligned} N^{d(1/q_2+1/p_2)} &\lesssim \|(a_{k-n,n})_{k,n}\|_{l^{(p_2,q_2)}} \\ &\lesssim \|(a_{k,n}\langle n\rangle^s)_{k,n}\|_{l^{(p_1,q_1)}} \lesssim N^{s+d(1/q_1+1/p_1)}. \end{aligned}$$

Letting $N \to \infty$, we obtain the desired conclusion $s \geq d(1/p_2 - 1/q_1 + 1/q_2 - 1/p_1)$.

Step 2. The goal of this part is to verify that $Y_0 = Z_0$, where $Z_0 := \{(p_1,p_2,q_1,q_2,s) \in (0,\infty]^4 \times \mathbb{R} : 1/p_2 \leq 1/p_1, A \leq 0, B \leq 0, s \geq 0\}$. It is obvious to see that $Z_0 = X_0$. To achieve this goal, we only need to verify that $Z_0 \subset Y$. In this case, we have

$$1/p_2 \leq 1/p_1, \ \ 1/p_2 \leq 1/q_1, \ \ 1/q_2 \leq 1/p_1, \ \ A \vee B \vee (A+B) \vee 0 = 0. \tag{4.4}$$

From this and Lemma 4.1, we have $l^{p_1} \subset l^{q_2}$ and $l^{q_1}_{v_s} \subset l^{p_2}$. We only need to verify $\mathcal{T} \in \mathcal{L}(l^{(p_1,q_1)}(\mathbb{Z}^{2d}), l^{(p_2,q_2)}(\mathbb{Z}^{2d}))$ under the above assumptions (4.4). In fact, we have

$$\|\mathcal{T}\vec{a}\|_{l^{(p_2,q_2)}} \lesssim \|\mathcal{T}\vec{a}\|_{l^{(p_2,p_1)}} \lesssim \|\vec{a}\|_{l^{(p_1,p_2)}} \lesssim \|\vec{a}\|_{l^{(p_1,q_1)}_{1\otimes v_s}},$$

where we use the embedding relation $l^{p_1} \subset l^{q_2}$ and $l^{q_1}_{v_s} \subset l^{p_2}$ in the first and third inequality respectively, and Lemma 4.2 in the second inequality.

Step 3. The goal of this part is to verify that $Y_1 = Z_1$, where $Z_1 := \{(p_1,p_2,q_1,q_2,s) \in (0,\infty]^4 \times \mathbb{R} : 1/p_2 \leq 1/p_1, A > 0 \geq B, s > A\}$. Actually, by the definition of X_1, $Z_1 = \{(p_1,p_2,q_1,q_2,s) \in X_1 : s > A \vee B \vee (A+B) \vee 0 = d(1/p_2 - 1/q_1)\}$.

For $Y_1 \subset Z_1$, we only need to prove $s > d(1/p_2 - 1/q_1)$. Using (4.2), we conclude that

$$(p_1,p_2,q_1,q_2,s) \in Y \cap X_1 \Longrightarrow l^{q_1}_{v_s} \subset l^{p_2}, \ \ 1/p_2 > 1/q_1.$$

Form this and Lemma 4.1, we have

$$s > d(1/p_2 - 1/q_1) = A \vee B \vee (A+B) \vee 0.$$

On the other hand, if $(p_1,p_2,q_1,q_2,s)\in Z_1$, by Lemma 4.1, we have

$$1/p_2\le 1/p_1,\quad l^{p_1}\subset l^{q_2},\quad l^{q_1}_{v_s}\subset l^{p_2}.$$

From this, we conclude that

$$\|\mathcal{T}\vec{a}\|_{l^{(p_2,q_2)}}\lesssim\|\mathcal{T}\vec{a}\|_{l^{(p_2,p_1)}}\lesssim\|\vec{a}\|_{l^{(p_1,p_2)}}\lesssim\|\vec{a}\|_{l^{(p_1,q_1)}_{1\otimes v_s}},$$

where we use the embedding relation $l^{p_1}\subset l^{q_2}$ and $l^{q_1}_{v_s}\subset l^{p_2}$ in the first and third inequalities, respectively, and Lemma 4.2 in the second inequality.

Step 4. The goal of this part is to verify that $Y_2=Z_2$, where $Z_2:=\{(p_1,p_2,q_1,q_2,s)\in(0,\infty]^4\times\mathbb{R}:1/p_2\le 1/p_1,B>0\ge A,s>B\}$. Actually, $Z_2=\{(p_1,p_2,q_1,q_2,s)\in X_2:s>A\vee B\vee(A+B)\vee 0=d(1/q_2-1/p_1)\}$.

For $Y_2\subset Z_2$, we only need to prove $s>=d(1/q_2-1/p_1)$. Using (4.2), we conclude that

$$(p_1,p_2,q_1,q_2,s)\in Y\cap X_2\Longrightarrow l^{p_1}_{v_s}\subset l^{q_2},\quad 1/q_2>1/p_1.$$

Form this and Lemma 4.1, we have

$$s>d(1/q_2-1/p_1)=A\vee B\vee(A+B)\vee 0.$$

On the other hand, if $(p_1,p_2,q_1,q_2,s)\in Z_2$, by Lemma 4.1, we have

$$1/p_2\le 1/p_1,\quad l^{p_1}_{v_s}\subset l^{q_2},\quad l^{q_1}\subset l^{p_2}.$$

From this, we conclude that

$$\|\mathcal{T}\vec{a}\|_{l^{(p_2,q_2)}}\lesssim\|\mathcal{T}\vec{a}\|_{l^{(p_2,p_1)}_{1\otimes v_s}}\lesssim\|\vec{a}\|_{l^{(p_1,p_2)}_{1\otimes v_s}}\lesssim\|\vec{a}\|_{l^{(p_1,q_1)}_{1\otimes v_s}},$$

where we use the embedding relation $l^{p_1}_{v_s}\subset l^{q_2}$ and $l^{q_1}\subset l^{p_2}$ in the first and third inequalities, respectively, and Lemma 4.2 in the second inequality.

Step 5. The goal of this part is to verify that $Y_3=Z_3$, where $Z_3=Z_{3,1}\cup Z_{3,2}$ with

$$\begin{aligned}Z_{3,1}&:=\{(p_1,p_2,q_1,q_2,s)\in(0,\infty]^4\times\mathbb{R}:A>0,B>0,s>A+B,p_1\le p_2\},\\ Z_{3,2}&:=\{(p_1,p_2,q_1,q_2,s)\in(0,\infty]^4\times\mathbb{R}:A>0,B>0,s=A+B,p_1<p_2\}.\end{aligned}$$

By the definition of X_3, we have $Z_3=X_3\setminus\{p_1=p_2,s=A+B\}$.

For $Y_3\subset Z_3$, we only need to prove that in X_3 the case $\{p_1=p_2,s=A+B\}$ is not true when $\mathcal{T}$ is bounded. We prove this by contradiction, assuming that $p_1=p_2=p,s=A+B$. For a large constant N, take

$$a_{k,n}=\begin{cases}b_n, & |k|\le 2N,|n|\le N,\\ 0, & \text{others}.\end{cases}$$

Using the inequality (4.3) and the following estimates

$$\begin{aligned}\left(\sum_k\left(\sum_n|a_{k-n,n}|^{q_2}\right)^{\frac{p}{q_2}}\right)^{\frac{1}{p}}&\gtrsim\left(\sum_{|k|\le N}\left(\sum_{|n|\le N}|a_{k-n,n}|^{q_2}\right)^{\frac{p}{q_2}}\right)^{\frac{1}{p}}\\ &=\left(\sum_{|k|\le N}\left(\sum_{|n|\le N}|b_n|^{q_2}\right)^{\frac{p}{q_2}}\right)^{\frac{1}{p}}\sim N^{d/p}\left(\sum_{|n|\le N}|b_n|^{q_2}\right)^{\frac{1}{q_2}}\end{aligned}$$

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and

$$\left(\sum_k\left(\sum_n|a_{k,n}|^{q_1}\langle n\rangle^{sq_1}\right)^{\frac{p}{q_1}}\right)^{\frac{1}{p}}=\left(\sum_{|k|\le 2N}\left(\sum_{|n|\le N}|a_{k,n}|^{q_1}\langle n\rangle^{sq_1}\right)^{\frac{p}{q_1}}\right)^{\frac{1}{p}}$$
$$=\left(\sum_{|k|\le 2N}\left(\sum_{|n|\le N}|b_n|^{q_1}\langle n\rangle^{sq_1}\right)^{\frac{p}{q_1}}\right)^{\frac{1}{p}}\sim N^{d/p}\left(\sum_{|n|\le N}|b_n|^{q_1}\langle n\rangle^{sq_1}\right)^{\frac{1}{q_1}},$$

we obtain the embedding relation $l^{q_1}_{v_s}\subset l^{q_2}$ by letting $N\to\infty$. Using this embedding relation and Lemma 4.1 with the fact $1/q_2>1/q_1$, we find $s>d(1/q_2-1/q_1)=A+B$. This is a contradiction.

We turn to prove that $Z_3\subset Y_3$. This proof is divided into two parts: $Z_{3,1}\subset Y_3$ and $Z_{3,2}\subset Y_3$.

First, let us prove that $Z_{3,1}\subset Y_3$. In this case, one can write $s=s_1+s_2$ with $s_1>A=d(1/p_2-1/q_1)$ and $s_2>B=d(1/q_2-1/p_1)$. From this and $A,B>0$, by Lemma 4.1, we obtain the embedding relations $l^{p_1}_{v_{s_2}}\subset l^{q_2}$ and $l^{q_1}_{v_{s_1}}\subset l^{p_2}$. Then the desired conclusion follows by

$$\|\mathcal{T}\vec{a}\|_{l^{(p_2,q_2)}}\lesssim\|\mathcal{T}\vec{a}\|_{l^{(p_2,p_1)}_{1\otimes v_{s_2}}}\lesssim\|\vec{a}\|_{l^{(p_1,p_2)}_{1\otimes v_{s_2}}}\lesssim\|\vec{a}\|_{l^{(p_1,q_1)}_{1\otimes v_{s_1+s_2}}}=\|\vec{a}\|_{l^{(p_1,q_1)}_{1\otimes v_s}},$$

where we use the embedding relation $l^{p_1}_{v_{s_2}}\subset l^{q_2}$ and $l^{q_1}_{v_{s_1}}\subset l^{p_2}$ in the first and third inequalities, respectively, and Lemma 4.2 in the second inequality.

Finally, we turn to the proof of $Z_{3,2}\subset Y_3$. In this case, we have $q_2<p_1<p_2<q_1$ and $s=d(1/p_2-1/q_1+1/q_2-1/p_1)$. We first consider the endpoint case $q_1=\infty$ and $s=d(1/p_2+1/q_2-1/p_1)$, then the final conclusion follows by using the interpolation argument. In this endpoint case, we write (4.3) as

$$\left(\sum_k\left(\sum_n|a_{k-n,n}|^{q_2}\right)^{\frac{p_2}{q_2}}\right)^{\frac{1}{p_2}}\lesssim\left(\sum_k\left(\sup_n|a_{k,n}|\langle n\rangle^s\right)^{p_1}\right)^{\frac{1}{p_1}}. \tag{4.5}$$

Denote by $b_k=\sup_n|a_{k,n}|\langle n\rangle^s$. We have

$$|a_{k-n,n}|=|a_{k-n,n}|\langle n\rangle^s\langle n\rangle^{-s}\le b_{k-n}\langle n\rangle^{-s}.$$

Then the inequality (4.5) is true if the following one is valid:

$$\left(\sum_k\left(\sum_n\left|\frac{b_{k-n}}{\langle n\rangle^s}\right|^{q_2}\right)^{\frac{p_2}{q_2}}\right)^{\frac{1}{p_2}}\lesssim\left(\sum_k|b_k|^{p_1}\right)^{\frac{1}{p_1}}.$$

This is equivalent to

$$\|\mathcal{I}_\lambda\vec{b}\|_{l^{r_2}}=\left\|\left(\sum_n\frac{b_{k-n}}{\langle n\rangle^{d-\lambda}}\right)_k\right\|_{l^{r_2}}\lesssim\|\vec{b}\|_{l^{r_1}}, \tag{4.6}$$

where $r_2=p_2/q_2$, $r_1=p_1/q_2$, $\lambda=d(1/r_1-1/r_2)$, and $\mathcal{I}_\lambda$ denotes the fractional integral operator of discrete form. Note that $r_1,r_2\in(1,\infty)$, $\lambda\in(0,d)$. The inequality (4.6) follows by the boundedness $\mathcal{I}_\lambda\in\mathcal{L}(l^{r_1},l^{r_2})$. We refer to [12] for more details about the boundedness of fractional integral operators.

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Let $r = q_1/2$, $\theta = 1/2$, $\widetilde{s} = d(1/\widetilde{p_2} - 1/\widetilde{q_1} + 1/\widetilde{q_2} - 1/\widetilde{p_1})$ with

$$\widetilde{q_1} = \infty,\ \frac{1}{\widetilde{p_1}} = 2\left(\frac{1}{p_1} - \frac{1}{q_1}\right),\ \frac{1}{\widetilde{p_2}} = 2\left(\frac{1}{p_2} - \frac{1}{q_1}\right),\ \frac{1}{\widetilde{q_2}} = 2\left(\frac{1}{q_2} - \frac{1}{q_1}\right).$$

So we have $\widetilde{q_2} < \widetilde{p_1} < \widetilde{p_2} < \widetilde{q_1} = \infty$, and then $\mathcal{T} \in \mathcal{L}(l_{v_{\widetilde{s}}}^{(\widetilde{p_1},\infty)}(\mathbb{Z}^{2d}), l^{(\widetilde{p_2},\widetilde{q_2})}(\mathbb{Z}^{2d}))$. Moreover, we have the relations

$$\frac{1-\theta}{\widetilde{p_1}} + \frac{\theta}{r} = \frac{1}{p_1},\ \frac{1-\theta}{\widetilde{p_2}} + \frac{\theta}{r} = \frac{1}{p_2},\ \frac{1-\theta}{\widetilde{q_1}} + \frac{\theta}{r} = \frac{1}{q_1},\ \frac{1-\theta}{\widetilde{q_2}} + \frac{\theta}{r} = \frac{1}{q_2}.$$

Then the final conclusion $\mathcal{T} \in \mathcal{L}(l_{v_s}^{(p_1,q_1)}(\mathbb{Z}^{2d}), l^{(p_2,q_2)}(\mathbb{Z}^{2d}))$ follows by the complex interpolation between the endpoint case $\mathcal{T} \in \mathcal{L}(l_{v_{\widetilde{s}}}^{(\widetilde{p_1},\infty)}(\mathbb{Z}^{2d}), l^{(\widetilde{p_2},\widetilde{q_2})}(\mathbb{Z}^{2d}))$ proved above and the obvious fact $\mathcal{T} \in \mathcal{L}(l^{(r,r)}, l^{(r,r)})$.

Accordingly, from above steps, we obtain that

$$Y = \bigcup_{j=0}^{3} Y_j = \bigcup_{j=0}^{3} Z_j.$$

Observe that $\bigcup_{j=0}^{3} Z_j$ is precisely the set consists of elements satisfying that: $1/p_2 \le 1/p_1$ and s satisfies the conditions in Theorem 1.2. So we complete the proof. ∎

Proof of Theorem 1.2. Using Theorem 1.1 and the fact $\mathcal{D}_{2,1/2}(1 \otimes v_s) \sim 1 \otimes v_s$, we obtain

$$e^{i\Delta} \in \mathcal{L}(W_{1\otimes v_s}^{p_1,q_1}(\mathbb{R}^d), W^{p_2,q_2}(\mathbb{R}^d)) \Longleftrightarrow \mathcal{T} \in \mathcal{L}(l_{1\otimes v_s}^{(p_1,q_1)}(\mathbb{Z}^{2d}), l^{(p_2,q_2)}(\mathbb{Z}^{2d})),$$

the desired conclusion follows by Proposition 4.3. ∎

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