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Wirtinger's Inequalities on Time Scales

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Abstract. This paper is devoted to the study of Wirtinger-type inequalities for the Lebesgue Δ-integral on an arbitrary time scale $\mathbb{T}$. We prove a general inequality for a class of absolutely continuous functions on closed subintervals of an adequate subset of $\mathbb{T}$. By using this expression and by assuming that $\mathbb{T}$ is bounded, we deduce that a general inequality is valid for every absolutely continuous function on $\mathbb{T}$ such that its Δ-derivative belongs to $L^2_\Delta([a, b) \cap \mathbb{T})$ and at most it vanishes on the boundary of $\mathbb{T}$.

1 Introduction

The goals of the theory of dynamic equations on time scales, whose origin is in the Ph.D. dissertation of S. Hilger [14], are to study differential and difference equations jointly and moreover to solve equations on an arbitrary nonempty closed subset of the real numbers, called time scale, with mixed continuous and discrete parts. We refer the reader to the monographs [7, 8, 15] to see some of the concepts and properties related to this field of research.

On the other hand, Wirtinger-type inequalities play an important role in the development of variational techniques to solve boundary value problems, for instance, in ordinary or partial differential equations and difference equations. Many versions of this class of inequalities have been studied in depth both in the continuous case, see [3, 6, 11, 12, 16, 18, 19], and in the discrete one, see [1, 3, 17]. In [13, Theorem 1], as well as in [2, Theorem 6.8] or in [7, Theorem 6.33], a Wirtinger-type inequality for the Riemann Δ-integral on an arbitrary time scale is proved.

This paper is devoted to deriving a generalization of the classical Lebesgue one-dimensional quadratic Wirtinger-type inequality for the Lebesgue Δ-integral on an arbitrary time scale $\mathbb{T}$ with $a := \inf \mathbb{T}$ and $b := \sup \mathbb{T}$, which covers the one given either in [16, Theorem 5 with $\alpha = 2$] or in [19, Theorem 1] whenever $\mathbb{T} = \mathbb{R}$.

The proof of the more general Wirtinger-type inequality for the Lebesgue Δ-integral, which is valid on the whole time scale $\mathbb{T}$ except at most at b whenever it belongs to $\mathbb{T}$, is given in Section 2. The functions involved in our result are absolutely continuous functions on closed subintervals of an adequate subset of $\mathbb{T}$ which may be singular at the endpoints of $\mathbb{T}$ whenever they are dense.

By using the aforementioned inequality and by assuming that $\mathbb{T}$ is bounded, we deduce in Section 3 a Wirtinger-type inequality for absolutely continuous functions

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on $\mathbb{T}$ with Δ-derivative in $L^2_\Delta([a,b) \cap \mathbb{T})$, which generalizes the one given either in [16, Theorem 6 with $\alpha = 2$] or in [19, Theorem 2] for $\mathbb{T} = [a,b]$. Moreover, as an application of this inequality, we show particular Wirtinger-type inequalities, some of which unify the analogous ones known in the discrete and real analysis; unfortunately, we are not able to deduce from it the generalization to a bounded time scale of the classical Wirtinger-type inequality for periodic absolutely continuous functions with null Lebesgue integral.

Finally, we remark that these kinds of inequalities are applied in a great variety of boundary value problems on time scales, for instance, we cite [5] devoted to the proof of the existence of multiple positive solutions to a singular second order dynamic equation with homogeneous Dirichlet boundary conditions by employing variational techniques and critical point theory; the inequality that will be derived in Corollary 3.3 is an important key in the development of that paper since, under different behaviors of the nonlinear part of the equation, it allows us to guarantee that the operator used in the variational formulation of the considered problem is bounded from below and coercive and so, the critical point theory ensures the existence of a minimum which is a weak solution of that problem.

2 Main Result

In this section we deduce a quadratic Wirtinger-type inequality for a class of absolutely continuous functions on closed subintervals of the subset of $\mathbb{T}$ defined as

$$J := \begin{cases} [a,b] \cap \mathbb{T} & \text{if } a < \sigma(a) \text{ and } \rho(b) < b, \\ [a,b) \cap \mathbb{T} & \text{if } a < \sigma(a) \text{ and } \bar{\rho}(b) = b, \\ (a,b] \cap \mathbb{T} & \text{if } a = \bar{\sigma}(a) \text{ and } \rho(b) < b, \\ (a,b) \cap \mathbb{T} & \text{if } a = \bar{\sigma}(a) \text{ and } \bar{\rho}(b) = b, \end{cases}$$

with $\bar{\sigma}(a) = \sigma(a)$ if $a \in \mathbb{T}$ and $\bar{\sigma}(a) = a$ if $a \notin \mathbb{T}$ and $\bar{\rho}(b) = \rho(b)$ if $b \in \mathbb{T}$ and $\bar{\rho}(b) = b$ if $b \notin \mathbb{T}$.

For every $t_1, t_2 \in \mathbb{T}$ such that $t_1 < t_2$ and for every $f \in L^1_\Delta([t_1,t_2) \cap \mathbb{T})$ we denote

$$\int_{t_1}^{t_2} f(s)\,\Delta s = \int_{[t_1,t_2)\cap\mathbb{T}} f(s)\,\Delta s$$

for every set $A \subset \mathbb{T}$. We denote by $AC(A)$ the set of all absolutely continuous functions on closed subintervals of A, by $C^1_{rd}(A \cap [a, \bar{\rho}(b)])$ the set of all functions such that they are continuous on A, Δ-differentiable on $A \cap [a, \bar{\rho}(b)]$ and their Δ-derivatives belong to $C_{rd}(A \cap [a, \bar{\rho}(b)])$, and by $C^2_{rd}(A \cap [a, \bar{\rho}(b)))$ the set of all functions such that they and their Δ-derivatives are continuous on A and $A \cap [a, \bar{\rho}(b)]$ respectively, their Δ-derivatives are Δ-differentiable on $A \cap [a, \bar{\rho}(b))$ and their second Δ-derivatives belong to $C_{rd}(A \cap [a, \bar{\rho}(b)))$.

We define the Δ-differential operator $L\colon C^2_{rd}(J \cap [a, \bar{\rho}(b))) \to C_{rd}(J \cap [a, \bar{\rho}(b)))$ for every $v \in C^2_{rd}(J \cap [a, \bar{\rho}(b)))$ by

$$Lv := \left(p \cdot v^\Delta\right)^\Delta + q \cdot v^\sigma \quad \text{on } J \cap [a, \bar{\rho}(b)),$$

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with $q \in C_{rd}(J \cap [a, \bar{\rho}(b)])$, $p \in C^1_{rd}(J \cap [a, \bar{\rho}(b)])$ and $p > 0$ on J.

We shall work with solutions $v \in C^2_{rd}(J \cap [a, \bar{\rho}(b)))$ to the Δ-differential inequality

$$-Lv \geq \lambda_0 \cdot r \cdot v^\sigma \quad \text{on } J \cap [a, \bar{\rho}(b)), \tag{2.1}$$

where λ_0 is a real number and $r \in C_{rd}(J \cap [a, \bar{\rho}(b)])$ is positive on $J \cap [a, \bar{\rho}(b)]$.

It is easy to see that if $\rho(b) < b$, then, extending v^Δ to b by

$$v^\Delta(b) := \frac{[p \cdot v^\Delta - \mu \cdot (q + \lambda_0 \cdot r) \cdot v^\sigma](\rho(b))}{p(b)}, \tag{2.2}$$

the function $v^\Delta \colon J \to \mathbb{R}$ is continuous on J, $v^{\Delta\Delta} \in C_{rd}(J \cap [a, \rho(b)])$ and v is a solution to the Δ-differential inequality

$$-(p \cdot v^\Delta)^\Delta - q \cdot v^\sigma \geq \lambda_0 \cdot r \cdot v^\sigma \quad \text{on } J \cap [a, \rho(b)]. \tag{2.3}$$

For a solution $v \in C^2_{rd}(J \cap [a, \bar{\rho}(b)))$ to (2.1) which is non-negative on J and positive on $(a, b) \cap J$, we shall consider functions $u \in AC(J)$ such that the expressions below exist and are finite:

$$S_1(u, v) := \begin{cases} \left(\frac{p \cdot u^2 \cdot v^\Delta}{v}\right)(a) & \text{if } v(a) > 0, \\ \left(\frac{p \cdot u^2 \cdot v^\Delta}{v}\right)(\sigma(a)) & \text{if } v(a) = 0, \\ \lim_{\substack{t \to a^+ \\ t \in \mathbb{T}}} \left(\frac{p \cdot u^2 \cdot v^\Delta}{v}\right)(t) & \text{if } a = \bar{\sigma}(a), \end{cases} \tag{2.4}$$

$$S_2(u, v) := \begin{cases} \left(\frac{p \cdot (u^\sigma)^2 \cdot v^\Delta}{v^\sigma}\right)(\rho(b)) & \text{if } v(b) > 0, \\ \left(\frac{p \cdot u^2 \cdot v^\Delta}{v}\right)(\rho(b)) & \text{if } v(b) = 0, \\ \lim_{\substack{t \to b^- \\ t \in \mathbb{T}}} \left(\frac{p \cdot u^2 \cdot v^\Delta}{v}\right)(t) & \text{if } \bar{\rho}(b) = b. \end{cases} \tag{2.5}$$

We let

$$T_1(u, v) := \begin{cases} 0 & \text{if } a = \bar{\sigma}(a) \text{ or } v(a) > 0, \\ \{\mu \cdot [(q + \lambda_0 r) \cdot (u^\sigma)^2 - p \cdot (u^\Delta)^2]\}(a) & \text{if } v(a) = 0, \end{cases} \tag{2.6}$$

and

$$T_2(u, v) := \begin{cases} 0 & \text{if } \bar{\rho}(b) = b, \\ \left[\mu \cdot \left(q + \lambda_0 r\right) \cdot (u^\sigma)^2\right](\rho(b)) & \text{if } v(b) > 0, \\ \{\mu \cdot [(q + \lambda_0 r) \cdot (u^\sigma)^2 - p \cdot (u^\Delta)^2]\}(\rho(b)) & \text{if } v(b) = 0. \end{cases} \tag{2.7}$$

Note that the definition of J allows that all the functions we are using may be singular at the endpoints of $\mathbb{T}$ whenever they are dense.

Now we are able to prove the Wirtinger-type inequality achieved for the Lebesgue Δ-integral.

Theorem 2.1 *Let λ_0 be a real number, let $p \in C^1_{rd}(J \cap [a, \bar{\rho}(b)])$ be a positive function on J, let $q \in C_{rd}(J \cap [a, \bar{\rho}(b)])$, and let $r \in C_{rd}(J \cap [a, \bar{\rho}(b)])$ be a positive function on $J \cap [a, \bar{\rho}(b)]$ and let $v \in C^2_{rd}(J \cap [a, \bar{\rho}(b)))$ be a solution to (2.1) such that it is non-negative on J and positive on $(a, b) \cap J$.*

If $u \in AC(J)$ is such that $\big(q + \lambda_0 r\big)\cdot(u^\sigma)^2 \in L^1_\Delta([a, b)\cap\mathbb{T})$, $p\cdot\big(u^\Delta\big)^2 \in L^1_\Delta([a, b)\cap\mathbb{T})$ and the expressions (2.4) and (2.5) exist and are finite, then the following inequality holds:

$$\int_a^b \left[(q + \lambda_0 r)\cdot(u^\sigma)^2\right](t)\,\Delta t \le \int_a^b \left[p\cdot\big(u^\Delta\big)^2\right](t)\,\Delta t + S_1(u, v) - S_2(u, v) + T_1(u, v) + T_2(u, v), \tag{2.8}$$

where $T_1(u, v)$ and $T_2(u, v)$ are given in (2.6) and (2.7), respectively.

Moreover, equality holds if and only if v is a solution to the dynamic equation

$$-Lv = \lambda_0 \cdot r \cdot v^\sigma \quad \text{on } J \cap [a, \bar{\rho}(b)) \tag{2.9}$$

and there exists a constant $K \in \mathbb{R}$ such that $u(t) = K \cdot v(t)$ for all $t \in J$ with $v(t) > 0$.

Proof Since $u \in AC(J)$, from the product rule for Δ-differentiation, [7, Theorem 1.20(iii)], we achieve that the Δ-differential identity

$$p\cdot v\cdot v^\sigma\left[\left(\frac{u}{v}\right)^\Delta\right]^2 + \left[\frac{p\cdot u^2\cdot v^\Delta}{v}\right]^\Delta = p\cdot\big(u^\Delta\big)^2 + \frac{(u^\sigma)^2}{v^\sigma}\cdot Lv - q\cdot(u^\sigma)^2$$

is satisfied Δ-almost everywhere on $(a, \bar{\rho}(b)) \cap \mathbb{T}$. The previous equality holds true at a whenever $v(a) > 0$; moreover, if $\rho(b) < b$, as v is a solution to the Δ-differential inequality (2.3), then by extending v^Δ to b as in (2.2), we have that the previous equality remains valid at $\rho(b)$ whenever $v(b) > 0$.

By integrating the above identity over a subinterval $[t_1, t_2) \cap \mathbb{T}$ of J such that $t_1 \ge \sigma(a)$ if $v(a) = 0$ and $t_2 < b$ if $v(b) = 0$ or $\bar{\rho}(b) = b$, we obtain that

$$\int_{t_1}^{t_2} \left[p\cdot\big(u^\Delta\big)^2 - (q + \lambda_0 r)\cdot(u^\sigma)^2\right](t)\,\Delta t \ \ge \left[\frac{p\cdot u^2\cdot v^\Delta}{v}\right]_{t_1}^{t_2}, \tag{2.10}$$

where equality holds if and only if v is a solution to (2.9) and $(u/v)^\Delta = 0$ Δ-a.e. on $[t_1, t_2) \cap \mathbb{T}$, which is equivalent to the existence of a real constant k such that $u = k\cdot v$ on $[t_1, t_2] \cap \mathbb{T}$.

Because $u \in AC(J)$ and we know that for every $f : \mathbb{T} \to \mathbb{R}$ and $t \in [a, b) \cap \mathbb{T}$, it is true that

$$\int_t^{\sigma(t)} f(s)\,\Delta s = f(t)\cdot\mu(t),$$

by taking limits in (2.10) as $t_1 \to a^+$ and $t_2 \to b^-$, when necessary, we obtain the conclusion (2.8). ■

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3 Some Consequences

The aim of this section is to apply Theorem 2.1 to obtain a general Wirtinger-type inequality for absolutely continuous functions on the bounded time scale $\mathbb{T}$ with $a = \min \mathbb{T}$ and $b = \max \mathbb{T}$ such that their Δ-derivative belongs to $L^2_\Delta([a,b) \cap \mathbb{T})$ and at most they are null on the boundary of $\mathbb{T}$. Moreover, we illustrate the application of this expression to some particular conditions.

We will assume that the rd-continuity of p^Δ, q and r extends to $[a, \rho(b)] \cap \mathbb{T}$ and $v \in C^2_{rd}([a, \rho(b)) \cap \mathbb{T})$ is non-negative on $\mathbb{T}$ and positive on $(a,b) \cap \mathbb{T}$ and it is an eigenfunction corresponding to the eigenvalue $\lambda_0 \in \mathbb{R}$ of the boundary value problem

$$\begin{cases} Lv + \lambda_0 \cdot r \cdot v^\sigma = 0 & \text{on } [a, \rho(b)) \cap \mathbb{T}, \\ c_1 v(a) + c_2 v^\Delta(a) = 0, \\ c_3 v(b) + c_4 v^\Delta(\rho(b)) = 0, \end{cases} \tag{3.1}$$

where c_1, c_2, c_3, and c_4 are real constants such that $(c_1^2 + c_2^2)(c_3^2 + c_4^2) \neq 0$.

By defining

$$A_1 := \begin{cases} \left(\frac{p \cdot v^\Delta}{v}\right)(a) & \text{if } v(a) > 0, \\ 1 & \text{if } v(a) = 0, a = \sigma(a), \\ 0 & \text{if } v(a) = 0, a < \sigma(a), \end{cases} \tag{3.2}$$

$$A_2 := \begin{cases} 0 & \text{if } v(a) > 0 \text{ or } a = \sigma(a), \\ \left[\left(\frac{p \cdot v^\Delta}{v}\right)^\sigma + \mu \cdot (q + \lambda_0 r)\right](a), & \text{if } v(a) = 0, a < \sigma(a), \end{cases} \tag{3.3}$$

$$A_3 := \begin{cases} 0 & \text{if } v(a) > 0, \\ -(\mu \cdot p)(a) & \text{if } v(a) = 0. \end{cases} \tag{3.4}$$

$$B_1 := \begin{cases} 0 & \text{if } v(b) > 0 \text{ or } \rho(b) = b, \\ \left(\frac{p \cdot v^\Delta}{v}\right)(\rho(b)) & \text{if } v(b) = 0, \rho(b) < b, \end{cases} \tag{3.5}$$

$$B_2 := \begin{cases} \left[\left(\frac{p \cdot v^\Delta}{v^\sigma}\right) - \mu \cdot (q + \lambda_0 r)\right](\rho(b)) & \text{if } v(b) > 0, \rho(b) < b, \\ \left(\frac{p \cdot v^\Delta}{v}\right)(b), & \text{if } v(b) > 0, \rho(b) = b, \\ -[\mu \cdot (q + \lambda_0 r)](\rho(b)) & \text{if } v(b) = 0, \rho(b) < b, \\ 1 & \text{if } v(b) = 0, \rho(b) = b, \end{cases} \tag{3.6}$$

and

$$B_3 := \begin{cases} 0 & \text{if } v(b) > 0 \text{ or } \rho(b) = b, \\ (\mu \cdot p)(\rho(b)), & \text{if } 4v(b) = 0, \rho(b) < b, \end{cases} \tag{3.7}$$

we achieve that if a is right-dense and $v(a) > 0$ or a is right-scattered, then

$$S_1(u,v) + T_1(u,v) = \left[A_1 u^2 + A_2(u^\sigma)^2 + A_3(u^\Delta)^2\right](a), \tag{3.8}$$

and if b is left-dense and $v(b) > 0$ or b is left-scattered, then

$$T_2(u,v) - S_2(u,v) = -\left[B_1 u^2 + B_2(u^\sigma)^2 + B_3(u^\Delta)^2\right](\rho(b)), \tag{3.9}$$

where $S_1(u,v)$, $S_2(u,v)$, $T_1(u,v)$ and $T_2(u,v)$ are given in (2.4), (2.5), (2.6), and (2.7), respectively.

Theorem 3.1 *Let $q \in C_{rd}([a,\rho(b)] \cap \mathbb{T})$, let $p \in C^1_{rd}([a,\rho(b)] \cap \mathbb{T})$ and $r \in C_{rd}([a,\rho(b)] \cap \mathbb{T})$ be positive functions on $\mathbb{T}$ and $[a,\rho(b)] \cap \mathbb{T}$, respectively, and let $v \in C^2_{rd}([a,\rho(b)) \cap \mathbb{T})$ be an eigenfunction corresponding to the eigenvalue $\lambda_0 \in \mathbb{R}$ of* (3.1) *such that it is non-negative on $\mathbb{T}$, positive on $(a,b) \cap \mathbb{T}$, $v^\Delta(a) > 0$ if a is right-dense and $v(a) = 0$ and $v^\Delta(b) < 0$ if b is left-dense and $v(b) = 0$.*

Then every absolutely continuous function u on $\mathbb{T}$ such that $u^\Delta \in L^2_\Delta([a,b) \cap \mathbb{T})$ and $u(t) = 0$ for every $t \in \{a,b\}$ with t dense and $v(t) = 0$, satisfies the following Wirtinger-type inequality:

$$\begin{aligned}\int_a^b \left[(q+\lambda_0 r)\cdot(u^\sigma)^2\right](t)\,\Delta t \le \int_a^b &\left[p\cdot(u^\Delta)^2\right](t)\,\Delta t \\ &+ \left[A_1 u^2 + A_2(u^\sigma)^2 + A_3(u^\Delta)^2\right](a) \\ &- \left[B_1 u^2 + B_2(u^\sigma)^2 + B_3(u^\Delta)^2\right](\rho(b)),\end{aligned} \tag{3.10}$$

where A_1, A_2, A_3, B_1, B_2 and B_3 are given in (3.2), (3.3), (3.4), (3.5), (3.6), *and* (3.7), *respectively.*

Moreover, equality holds if and only if u is a constant multiple of v on $\mathbb{T}$.

Proof As a consequence of Theorem 2.1 and equalities (3.8) and (3.9), it only remains to show that inequality (3.10) is valid whenever there exists $t \in \{a,b\}$ such that t is dense and $v(t) = 0$. Because of the analogy between all cases, we only prove one of them.

Assume that a is right-dense and $v(a) = 0$ and b is left-dense and $v(b) > 0$ or b is left-scattered. Let $u\colon \mathbb{T} \to \mathbb{R}$ be an absolutely continuous function on $\mathbb{T}$ such that $u(a) = 0$.

First, suppose that u^Δ is continuous on $[a,\rho(b)] \cap \mathbb{T}$. As $v^\Delta(a) > 0$ and $u(a) = 0$, L'hôpital's rule, [8, Theorem 4.3], allows us to assert that

$$\begin{aligned}\lim_{\substack{t\to a^+\\ t\in\mathbb{T}}}\left(\frac{p\cdot u^2\cdot v^\Delta}{v}\right)(t) &= \lim_{\substack{t\to a^+\\ t\in\mathbb{T}}}\left[\frac{(p\cdot u^2\cdot v^\Delta)^\Delta}{v^\Delta}\right](t) \\ &= \lim_{\substack{t\to a^+\\ t\in\mathbb{T}}}[u^\Delta\cdot(u+u^\sigma)\cdot p](t) - \lim_{\substack{t\to a^+\\ t\in\mathbb{T}}}\left[\frac{(u^\sigma)^2\cdot(q+\lambda_0 r)\cdot v^\sigma}{v^\Delta}\right](t) \\ &= 0,\end{aligned}$$

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which implies that

$$S_1(u,v) + T_1(u,v) = 0 = [A_1 u^2 + A_2 (u^\sigma)^2 + A_3 (u^\Delta)^2](a), \tag{3.11}$$

where $S_1(u,v)$ and $T_1(u,v)$ are given in (2.4) and (2.6), respectively.

Therefore, equalities (3.9) and (3.11) and Theorem 2.1 ensure that (3.10) is true.

In order to end the proof, suppose that $u^\Delta \in L^2_\Delta([a,b) \cap \mathbb{T})$. The fact that the set of all continuous functions on $[a,b) \cap \mathbb{T}$ with compact support in $[a,b) \cap \mathbb{T}$ is dense in $L^2_\Delta([a,b) \cap \mathbb{T})$, see [4, Proposition 2.4], allows us to construct a sequence $\{g_n\}_{n\in\mathbb{N}}$ of continuous functions on $\mathbb{T}$ such that its restriction to $[a,b) \cap \mathbb{T}$ converges to u^Δ strongly in $L^2_\Delta([a,b) \cap \mathbb{T})$.

Thus it is that we deduce from the fundamental theorem of calculus for continuous functions [8, Theorem 5.36] that for every $n \in \mathbb{N}$, the function $u_n \colon \mathbb{T} \to \mathbb{R}$ defined as

$$u_n(t) := \int_a^t g_n(s)\, \Delta s \quad \text{for all } t \in \mathbb{T},$$

is absolutely continuous on $\mathbb{T}$ and $u_n^\Delta = g_n$ is continuous on $[a, \rho(b)] \cap \mathbb{T}$.

Thereby, as $u_n(a) = 0$ for every $n \in \mathbb{N}$, inequality (3.10) holds for every $n \in \mathbb{N}$, that is,

$$\begin{aligned} \int_a^b \left[(q + \lambda_0 r) \cdot (u_n^\sigma)^2 \right](t)\, \Delta t \le & \int_a^b \left[p \cdot (u_n^\Delta)^2 \right](t)\, \Delta t \\ & + \left[A_1 u_n^2 + A_2 (u_n^\sigma)^2 + A_3 (u_n^\Delta)^2 \right](a) \\ & - \left[B_1 u_n^2 + B_2 (u_n^\sigma)^2 + B_3 (u_n^\Delta)^2 \right](\rho(b)) \end{aligned} \tag{3.12}$$

is valid for all $n \in \mathbb{N}$.

Furthermore, since $u(a) = 0 = u_n(a)$ for every $n \in \mathbb{N}$, the fundamental theorem of calculus for Lebesgue Δ-integrable functions, [10, Theorem 4.1], and Hölder's inequality guarantee the existence of $k_1 > 0$ such that for every $n \in \mathbb{N}$ and $t \in \mathbb{T}$,

$$|u - u_n|(t) \le k_1 \cdot \|u^\Delta - g_n\|_{L^2_\Delta}.$$

Therefore, the strong convergence of $\{g_n\}_{n\in\mathbb{N}}$ to u^Δ in $L^2_\Delta([a,b) \cap \mathbb{T})$ and inequality (3.12) guarantee the validity of the inequality (3.10).

The last assertion in the theorem is a straight forward consequence of Theorem 2.1. ■

The last part of this section is devoted to showing some applications of the previous result.

Corollary 3.2 *Let $q \in C_{rd}([a, \rho(b)] \cap \mathbb{T})$, let $p \in C^1_{rd}([a, \rho(b)] \cap \mathbb{T})$ and $r \in C_{rd}([a, \rho(b)] \cap \mathbb{T})$ be positive functions on $\mathbb{T}$ and $[a, \rho(b)] \cap \mathbb{T}$, respectively, such that $p^\Delta(a) = 0$.*

If problem (3.1) with $c_2 = 0 = c_4$ has an eigenvalue $\lambda_0 \in \mathbb{R}$ and an eigenfunction $v \in C^2_{rd}([a, \rho(b)) \cap \mathbb{T})$ that is positive on $(a,b) \cap \mathbb{T}$, then every absolutely continuous

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function u on $\mathbb{T}$ such that $u^\Delta \in L^2_\Delta([a,b)\cap\mathbb{T})$ and $u(a) = 0 = u(b)$ satisfies the following Wirtinger-type inequality:

$$\int_a^b \left[(q+\lambda_0 r)\cdot(u^\sigma)^2\right](t)\,\Delta t \le \int_a^b \left[p\cdot(u^\Delta)^2\right](t)\,\Delta t.$$

Moreover, equality holds if and only if u is a constant multiple of v on $\mathbb{T}$.

If we set $p \equiv 1 \equiv r$ and $q \equiv 0$ in Corollary 3.2, then we know from [8, Theorem 7.15] that there exists the smallest positive eigenvalue of problem (3.1) with $c_2 = 0 = c_4$ and the eigenfunctions corresponding to it can be chosen positive on $(a,b)\cap\mathbb{T}$. Thus it is that Corollary 3.2 yields to the following property, which covers the one given in [1, Theorem 11.6.1] for the discrete case.

Corollary 3.3 *Every absolutely continuous function u on $\mathbb{T}$ such that*

$$u^\Delta \in L^2_\Delta([a,b)\cap\mathbb{T})$$

and $u(a) = 0 = u(b)$ satisfies the following Wirtinger-type inequality:

$$\int_a^b (u^\sigma)^2(t)\,\Delta t \le \frac{1}{\lambda_0}\int_a^b (u^\Delta)^2(t)\,\Delta t,$$

where λ_0 is the smallest positive eigenvalue of problem (3.1) with $c_2 = 0 = c_4$, $p \equiv 1 \equiv r$, and $q \equiv 0$.

Moreover, equality holds if and only if u is an eigenfunction corresponding to λ_0.

Corollary 3.4 *Let $q \in C_{rd}([a,\rho(b)]\cap\mathbb{T})$, let $p \in C^1_{rd}([a,\rho(b)]\cap\mathbb{T})$ and $r \in C_{rd}([a,\rho(b)]\cap\mathbb{T})$ be positive functions on $\mathbb{T}$ and $[a,\rho(b)]\cap\mathbb{T}$, respectively, such that $p^\Delta(a) = 0$.*

If problem (3.1) with $c_2 = 0 = c_3$ has an eigenvalue $\lambda_0 \in \mathbb{R}$ and an eigenfunction $v \in C^2_{rd}([a,\rho(b))\cap\mathbb{T})$ such that it is positive on $(a,b)\cap\mathbb{T}$, then, every absolutely continuous function u on $\mathbb{T}$ such that $u^\Delta \in L^2_\Delta([a,b)\cap\mathbb{T})$ and $u(a) = 0$ satisfies the following Wirtinger-type inequality:

$$\int_a^{\rho(b)} \left[(q+\lambda_0 r)\cdot(u^\sigma)^2\right](t)\,\Delta t \le \int_a^{\rho(b)} \left[p\cdot(u^\Delta)^2\right](t)\,\Delta t.$$

Moreover, equality holds if and only if u is a constant multiple of v on $\mathbb{T}$.

If we set $p \equiv 1 \equiv r$ and $q \equiv 0$ in Corollary 3.4, then we know from [8, Theorem 7.17] that there exists the smallest positive eigenvalue of problem (3.1) with $c_2 = 0 = c_3$ and the eigenfunctions corresponding to it can be chosen positive on $(a,b)\cap\mathbb{T}$. Therefore, as a consequence of Corollary 3.4 we obtain the following inequality, which generalizes the one given for the discrete case in [1, Theorem 11.6.2] and the one given for C^1-functions in the real case either in [16, Corollary 9] or in [19, Corollary 3].

Corollary 3.5 *Every absolutely continuous function u on* $\mathbb{T}$ *with* $u^\Delta \in L^2_\Delta([a,b)\cap\mathbb{T})$ *satisfies the following Wirtinger-type inequality:*

$$\int_a^{\rho(b)} (u^\sigma - u(a))^2(t)\,\Delta t \le \frac{1}{\lambda_0}\int_a^{\rho(b)} (u^\Delta)^2(t)\,\Delta t,$$

where λ_0 *is the smallest positive eigenvalue of problem* (3.1) *with* $c_2 = 0 = c_3$, $p \equiv 1 \equiv r$ *and* $q \equiv 0$.

Moreover, equality holds if and only if $u - u(a)$ *is an eigenfunction corresponding to* λ_0.

By choosing as $p \equiv 1$, $r \equiv 0$, $\lambda_0 = 0$, $c_2 = 0$, and $c_3, c_4 \in \mathbb{R}$ such that $v^\Delta(\rho(b)) \ge 0$, one can deduce from Theorem 3.1 the following result, which matches up to [6, Theorem 1.1] whenever $\mathbb{T} = [0,a]$ for some $a \in \mathbb{R}$.

Corollary 3.6 *Assume that problem* (3.1) *with* $p \equiv 1$, $r \equiv 0$, $\lambda_0 = 0$ *and* $q \in C_{rd}([a,\rho(b)]\cap\mathbb{T})$ *has a solution* $v \in C^2_{rd}([a,\rho(b))\cap\mathbb{T})$ *such that it is positive on* $(a,b)\cap\mathbb{T}$, $v(a) = 0$ *and* $v^\Delta(\rho(b)) \ge 0$. *Then every absolutely continuous function u on* $\mathbb{T}$ *such that* $u^\Delta \in L^2_\Delta([a,b)\cap\mathbb{T})$ *and* $u(a) = 0$ *satisfies the following Wirtinger-type inequality:*

$$\int_a^{\rho(b)} \left[q\cdot(u^\sigma)^2\right](t)\,\Delta t \le \int_a^{\rho(b)} (u^\Delta)^2(t)\,\Delta t.$$

Moreover, equality holds if and only if $u = K \cdot v$ *on* $[a,\rho(b)]\cap\mathbb{T}$ *for some constant* $K \in \mathbb{R}$, *with* $k = 0$ *whenever* $v^\Delta(\rho(b)) > 0$.

By changing the initial condition $u(a) = 0$ in Corollary 3.6 to the properties

$$\int_a^{\rho(b)} q(t)\,\Delta t \ge 0 \quad \text{and} \quad u^\sigma(a)\cdot\int_a^{\rho(b)} (q\cdot u^\sigma)(t)\,\Delta t \le 0$$

and by applying Corollary 3.6 to the absolutely continuous function defined as $w := u - u(a)$, we achieve the following result, which yields [6, Theorem 1.1*], by choosing $\mathbb{T} = [0,a]$ for some $a \in \mathbb{R}$.

Corollary 3.7 *Assume the hypotheses in Corollary* 3.6 *and* $\int_a^{\rho(b)} q(t)\,\Delta t \ge 0$. *Then every absolutely continuous function u on* $\mathbb{T}$ *such that* $u^\Delta \in L^2_\Delta([a,b)\cap\mathbb{T})$ *and*

$$u^\sigma(a)\cdot\int_a^{\rho(b)} (q\cdot u^\sigma)(t)\,\Delta t \le 0,$$

satisfies the following Wirtinger-type inequality:

$$\int_a^{\rho(b)} \left[q\cdot(u^\sigma)^2\right](t)\,\Delta t \le \int_a^{\rho(b)} (u^\Delta)^2(t)\,\Delta t.$$

Finally, we allow the eigenfunctions corresponding to the eigenvalues of problem (3.1) to change sign at one point in the interval. Suppose that $v \in C^2_{rd}([a, \rho(b)) \cap \mathbb{T})$ is such that $v(a) > 0$, $v(b) < 0$, v is non-increasing on $\mathbb{T}$ and decreasing on $(a, b) \cap \mathbb{T}$, then there exists a unique point $\bar{t} \in \mathbb{T}$ such that $(v \cdot v^\sigma)(\bar{t}) \le 0$; by reasoning as in Theorem 3.1 with function $w := u - u(\bar{t})$ and each of the subintervals $[a, \rho(\bar{t})) \cap \mathbb{T}$, $[\rho(\bar{t}), \bar{t}) \cap \mathbb{T}$, $[\bar{t}, \sigma(\bar{t})) \cap \mathbb{T}$ and $[\sigma(\bar{t}), b) \cap \mathbb{T}$, we obtain the following result, which for $\mathbb{T} = [0, a]$ with $a \in \mathbb{R}$ is given in [6, Theorem 1.2].

Corollary 3.8 *Assume that problem* (3.1) *with* $p \equiv 1$, $r \equiv 0$, $\lambda_0 = 0$ *and* $q \in C_{rd}([a, \rho(b)] \cap \mathbb{T})$ *with* $\int_a^b q(t)\,\Delta t \ge 0$ *has a solution* $v \in C^2_{rd}([a, \rho(b)) \cap \mathbb{T})$ *such that* $v(a) > 0$, $v(b) < 0$, v *is non-increasing on* $\mathbb{T}$ *and decreasing on* $(a, b) \cap \mathbb{T}$. *Then every absolutely continuous function* u *on* $\mathbb{T}$ *such that* $u^\Delta \in L^2_\Delta([a, b) \cap \mathbb{T})$ *and*

$$u^\sigma(a) \cdot \int_a^b (q \cdot u^\sigma)(t)\,\Delta t \le 0,$$

satisfies the following Wirtinger-type inequality:

$$\int_a^b \left[q \cdot (u^\sigma)^2 \right](t)\,\Delta t \le \int_a^b (u^\Delta)^2(t)\,\Delta t + \left[q \cdot \mu \cdot (u^\sigma)^2 \right](\bar{t}),$$

where $\bar{t} \in (a, b) \cap \mathbb{T}$ *is the unique point in* $\mathbb{T}$ *such that* $(v \cdot v^\sigma)(\bar{t}) \le 0$.

Moreover, equality holds if and only if $u = u(\bar{t}) + Kv$ *on* $([a, \rho(\bar{t})] \cup [\sigma(\bar{t}), b]) \cap \mathbb{T}$ *for some constant* $K \in \mathbb{R}$, *with* $u(\bar{t}) = 0$ *whenever either* $\int_a^b q(t)\,\Delta t > 0$ *or* $\int_a^b (q \cdot u^\sigma)(t)\,\Delta t < 0$ *and* $K = 0$ *whenever either* $v^\Delta(a) = 0$ *or* $v^\Delta(\rho(b) = 0$.

References

[1] R. P. Agarwal, *Difference Equations and Inequalities. Theory, Methods, and Applications.* Second edition. Monographs and Textbooks in Pure and Applied Mathematics 228. Marcel Dekker, New York, 2000.

[2] R. P. Agarwal, M. Bohner, and A. Peterson, *Inequalities on time scales: a survey.* Math. Inequal. Appl. **4**(2001), no. 4, 535–557.

[3] R. P. Agarwal, V. Čuljak, and J. Pečarić, *On discrete and continuous Wirtinger inequalities.* Appl. Anal. **70**(1998), no. 1-2, 195–204.

[4] R. P. Agarwal, V. O. Espinar, K. Perera, and D. R. Vivero, *Basic properties of Sobolev's spaces on time scales.* Adv. Difference Equ. 2006 Article 38121, 14 pp.

[5] ———, *Multiple positive solutions of singular Dirichlet problems on time scales via variational methods.* Nonlinear Anal. **67**(2007), no. 2, 368–381.

[6] P. R. Beesack, *Integral inequalities of Wirtinger type.* Duke Math. J. **25**(1958), 477–498.

[7] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales. An Introduction with Applications.* Birkhäuser Boston, Boston, MA, 2001.

[8] M. Bohner and A. Peterson, eds. *Advances in Dynamic Equations on Time Scales.* Birkhäuser Boston, Boston, MA, 2003.

[9] A. Cabada and D. R. Vivero, *Expression of the Lebesgue Δ-integral on time scales as a usual Lebesgue integral. Application to the calculus of $\Delta-$antiderivatives.* Math. Comput. Modelling **43**(2006), no. 1-2, 194–207.

[10] ———, *Criterions for absolutely continuity on time scales.* J. Difference Equ. Appl. **11**(2005), no. 11, 1013–1028.

[11] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology. Vol. 2. Functional and Variational Methods.* Springer-Verlag, Berlin, 1988.

[12] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities.* Second edition. Cambridge, at the University Press, 1952.
[13] R. Hilscher, *A time scales version of a Wirtinger-type inequality and applications. Dynamic equations on time scales.* J. Comput. Appl. Math. **141**(2002), no. 1-2, 219–226.
[14] S. Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten.* Ph.D. thesis, Universität Würzburg, 1988.
[15] B. Kaymakçalan, V. Lakshmikantham, and S. Sivasundaram, *Dynamical Systems on Measure Chains*. Kluwer, Dordrecht, 1996.
[16] C.-F. Lee, C.-C. Yeh, C.-H. Hong, and R. P. Agarwal, *Lyapunov and Wirtinger inequalities.* Appl. Math. Lett. **17**(2004), no. 7, 847–853.
[17] G. V. Milovanović and I. Z. Milovanović, *Discrete inequalities of Wirtinger's type.* In: Recent Progress in Inequalities. Math. Appl. 430, Kluwer, Dordrecht, 1998, pp. 289–308.
[18] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems.* Applied Mathematical Sciences 74. Springer-Verlag, New York, 1989.
[19] C. A. Swanson, *Wirtinger's inequality.* SIAM J. Math. Anal. **9**(1978), no. 3, 484–491.

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