18

Radiative corrections

Let us now calculate the observable QED modification of the electron scattering cross section following from the analysis in the previous chapter. For this purpose, assume a static (time-independent) external field, in which case

$$a_\mu^{\text{ext}}(q) = -2\pi i\delta(W_f - W_i)\tilde{a}_\mu^{\text{ext}}(\mathbf{q}) \tag{18.1}$$

One can identify the T-matrix and cross section corresponding to Eq. (17.32) from the general relations

$$\begin{aligned} S_{fi} &= -2\pi i\delta(W_f - W_i)T_{fi} \\ d\sigma &= 2\pi|T_{fi}|^2\delta(W_f - W_i)\frac{d\rho_f}{\text{Flux}} \end{aligned} \tag{18.2}$$

For illustration, we here confine the discussion to scattering where the target is left in its ground state. It follows that

$$\frac{d\rho_f}{\text{Flux}} = \frac{\Omega d^3k_2}{(2\pi)^3}\frac{1}{v_1/\Omega} \tag{18.3}$$

Let the superscript denote the order in e, then to $O(e^4)$ one has for the square of the T-matrix

$$|T_{fi}|^2 = |T_{fi}^{(1)} + T_{fi}^{(3)}|^2 = |T_{fi}^{(1)}|^2 + 2\text{Re}\, T_{fi}^{(1)*}T_{fi}^{(3)} \tag{18.4}$$

If the explicit magnetic moment contribution is suppressed for the time being, then, since the QED amplitude in Eq. (17.32) contains only a real modification of the coefficient of γ_μ, one finds to this order

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{el}} \doteq \left(\frac{d\sigma}{d\Omega}\right)_0 \{1 + 2[F_E(q^2) + q^2\Pi_f(q^2)] + \cdots\} \tag{18.5}$$

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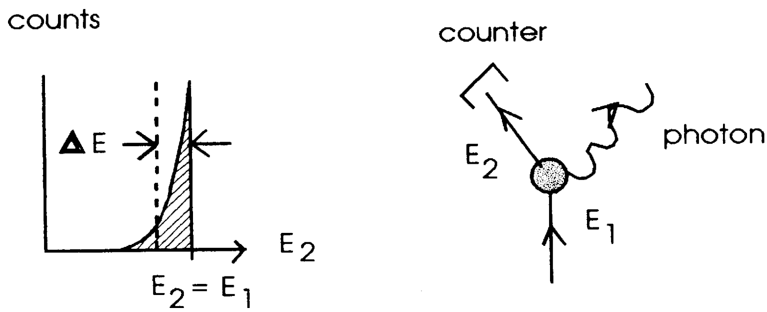

Fig. 18.1. Observation in an electron scattering experiment.

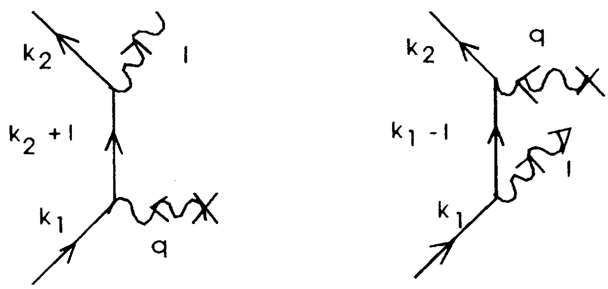

Fig. 18.2. Bremsstrahlung in an external field. The photon polarization is ε_ν.

Here $(d\sigma/d\Omega)_0$ is the lowest order cross section, and the dots denote the magnetic moment contribution.

One now has to think carefully about what is actually *observed* in the experiment. Since an electron can always radiate a photon of arbitrarily long wavelength (or low energy) during the scattering process, what one will observe in an electron scattering experiment is illustrated in Fig. 18.1. Experimentally, all one can observe is the *sum of these elastic and inelastic electromagnetic cross sections.*

$$d\sigma = d\sigma_{\text{el}} + d\sigma_{\text{in}} \tag{18.6}$$

One is therefore required to also calculate the cross section for radiation of a photon, the bremsstrahlung cross section. The two Feynman diagrams for bremsstrahlung in the same external field are shown in Fig. 18.2. The analytic expression follows from the previous Feynman rules as

$$\begin{aligned} S_{fi} \quad = \quad & -\frac{ie^2}{\Omega}\frac{1}{\sqrt{2\omega_l\Omega}}\bar{u}(k_2)\left[\gamma_\nu\varepsilon_\nu\frac{1}{i\gamma_\lambda(k_2+l)_\lambda+m_e}\gamma_\mu\right. \\ & \left.+\gamma_\mu\frac{1}{i\gamma_\sigma(k_1-l)_\sigma+m_e}\gamma_\nu\varepsilon_\nu\right]u(k_1)a_\mu^{\text{ext}}(q) \end{aligned} \tag{18.7}$$

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This amplitude will give the bremsstrahlung cross section to $O(e^4)$, which is of exactly the same order as the last term in Eq. (18.4).

First, rationalize the term in brackets in Eq. (18.7) and use the Dirac equation to the left and right

$$[\cdots] = -\left[\frac{2ik_2\cdot\varepsilon + i(\gamma_\lambda\varepsilon_\lambda)(\gamma_\sigma l_\sigma)}{2k_2\cdot l + l^2}\gamma_\mu + \gamma_\mu\frac{2ik_1\cdot\varepsilon - i(\gamma_\lambda l_\lambda)(\gamma_\sigma\varepsilon_\sigma)}{-2k_1\cdot l + l^2}\right] \tag{18.8}$$

Now let the photon energy become very small

$$|\mathbf{l}| \equiv \Delta E \to 0 \tag{18.9}$$

Then

$$S_{fi} \doteq -\frac{e^2}{\Omega}\bar{u}(k_2)\gamma_\mu u(k_1)a_\mu^{\text{ext}}(q)\left[\frac{1}{\sqrt{2\omega_l\Omega}}\left(\frac{k_2\cdot\varepsilon}{k_2\cdot l} - \frac{k_1\cdot\varepsilon}{k_1\cdot l}\right)\right] \tag{18.10}$$

Note that at this stage strict current conservation (gauge invariance) is still maintained since if one replaces $\varepsilon_\mu \to l_\mu$ this amplitude vanishes.

Assume again a static external field as in Eq. (18.1) and read off the T-matrix as in Eq. (18.2). The bremsstrahlung cross section is then

$$d\sigma_{\text{in}} = 2\pi\delta(W_f - W_i)|T_{fi}|^2\frac{\Omega d^3l}{(2\pi)^3}\frac{\Omega d^3k_2}{(2\pi)^3}\frac{1}{v_1/\Omega} \tag{18.11}$$

Under the condition in Eq. (18.9), one can replace

$$\begin{aligned} \tilde{a}_\mu^{\text{ext}}(\mathbf{q}) &\approx \tilde{a}_\mu^{\text{ext}}(\mathbf{k}_2 - \mathbf{k}_1) \\ E_2 + \omega_l = W_f &\approx E_2 \end{aligned} \tag{18.12}$$

Since these quantities are now the same as in elastic scattering, $d\sigma_{\text{in}}$ will again be proportional to $d\sigma_0$! It follows from the above that

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \left(\frac{d\sigma}{d\Omega}\right)_{\text{el}} + \left(\frac{d\sigma}{d\Omega}\right)_{\text{in}} \\ &= \left(\frac{d\sigma}{d\Omega}\right)_0 \left\{1 + 2[F_E(q^2) + q^2\Pi_f(q^2)]\right. \\ &\quad \left. + \frac{\alpha}{4\pi^2}\sum_{\text{pol}}\int_0^{\Delta E}\frac{d^3l}{\omega_l}\left(\frac{k_2\cdot\varepsilon}{k_2\cdot l} - \frac{k_1\cdot\varepsilon}{k_1\cdot l}\right)^2 + \cdots\right\} \end{aligned} \tag{18.13}$$

In this express ΔE is the resolution of the electron detector, and one must include all inelastic processes that give an electron in the detector within this resolution. A correct calculation of radiative corrections thus *depends on the geometry of the experiment*. The dots again denote the additional magnetic moment contribution.

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Since a, albeit tiny, mass has been assumed for the photon, the bremsstrahlung term must be evaluated consistently for this situation. The polarization sum for a massive vector meson yields[1]

$$\sum_{\sigma} \varepsilon_{\mu}^{(\sigma)} \varepsilon_{\nu}^{(\sigma)} = \delta_{\mu\nu} + \frac{l_{\mu} l_{\nu}}{\lambda^2} \tag{18.14}$$

Since the bremsstrahlung amplitude satisfies strict current conservation, the terms in $l_\mu l_\nu$ in this expression give a vanishing contribution. Hence

$$\sum_{\text{pol}} \left(\frac{k_2 \cdot \varepsilon}{k_2 \cdot l} - \frac{k_1 \cdot \varepsilon}{k_1 \cdot l} \right)^2 = -\frac{m_e^2}{(k_2 \cdot l)^2} - \frac{m_e^2}{(k_1 \cdot l)^2} - \frac{2k_1 \cdot k_2}{(k_1 \cdot l)(k_2 \cdot l)} \tag{18.15}$$

Here $l_\mu = (\mathbf{l}, i\omega_l)$ where $\omega_l = \sqrt{\mathbf{l}^2 + \lambda^2}$.

One must now do the remaining $\int d^3l/\omega_l$, with the limiting results

$$\begin{aligned} &\frac{\alpha}{4\pi^2} \sum_{\text{pol}} \int_0^{\Delta E} \frac{d^3 l}{\omega_l} \left(\frac{k_2 \cdot \varepsilon}{k_2 \cdot l} - \frac{k_1 \cdot \varepsilon}{k_1 \cdot l} \right)^2 \\ &\quad = \frac{2\alpha}{3\pi} \frac{q^2}{m_e^2} \left(\ln \frac{2\Delta E}{\lambda} - \frac{5}{6} \right) \qquad k^2 \ll m_e^2 \ ; \ q^2 \ll m_e^2 \\ &\quad = \frac{2\alpha}{\pi} \ln \frac{q^2}{m_e^2} \ln \frac{\Delta E}{\lambda} \qquad q^2 \gg m_e^2 \end{aligned} \tag{18.16}$$

Note that this bremsstrahlung cross section is also infrared divergent so that it, by itself, is *unobservable*; however, when adding the two results in Eq. (18.13), *the infrared divergent terms in* $\ln \lambda$ *cancel identically in the observable cross section*!

A combination of the above results then yields, for the scattering of an electron in a static Coulomb field

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \left(\frac{d\sigma}{d\Omega} \right)_{\text{Mott}} (1 - \delta) \\ \delta &\approx \frac{2\alpha}{3\pi} \frac{q^2}{m_e^2} \left(\ln \frac{m_e}{\Delta E} + \frac{5}{6} - \frac{1}{5} - \frac{3}{8} + \frac{3}{8} \right) \qquad k^2 \ll m_e^2 \\ \delta &\approx \frac{2\alpha}{\pi} \ln \frac{q^2}{m_e^2} \ln \frac{E}{\Delta E} \qquad q^2 \gg m_e^2 \ ; \quad E \gg \Delta E \end{aligned} \tag{18.17}$$

Here we have identified $(d\sigma/d\Omega)_0 = (d\sigma/d\Omega)_{\text{Mott}}$ for scattering in a static Coulomb field. The last $+3/8$ in the second line, canceling the term before it, is the hitherto suppressed magnetic moment contribution; the $-1/5$

[1] Use Lorentz covariance, $l \cdot e = 0$, and $l^2 = -\lambda^2$.

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comes from vacuum polarization. We have also written $E_1 \equiv E$, and ΔE is the experimental resolution.[2]

These results on the radiative corrections are originally due to Schwinger [Sc49], who argued that the correct result for the infrared divergent series, to all orders in α, is really

$$1 - \delta + \frac{\delta}{2!} + \cdots = e^{-\delta} \tag{18.18}$$

When $\Delta E \rightarrow 0$, one then has $e^{-\delta} \rightarrow 0$, and there is *no perfectly elastic scattering*.[3]

Note that while the ultraviolet divergences truly reflect a lack of knowledge of the physics at very short distances, the treatment of the infrared divergences is basically a technical problem. The emission of very long wavelength radiation (photons) is essentially governed by classical physics. This is a problem first treated in detail by Bloch and Nordsieck [Bl37]. The difficulty arises because analyzing the emission photon-by-photon (i.e. as a power series in e) is not an efficient way of attacking this problem.

[2] Note that $\ln q^2 \approx \ln E^2$ as $E \rightarrow \infty$.
[3] For applications of radiative corrections see [Ma69, Mo69a].

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